MATH 53 DISCUSSION SECTION ANSWERS – 2/21/23

1. Partial derivatives

(1) (a) True: f_{xyz} is the sum of the corresponding mixed third partial derivatives of $g(x,y)$, $h(y,z)$, and $k(x,z)$. But since each term depends on only two of the variables, we have

$$g(x,y)_{xyz} = \frac{d}{dx}\frac{d}{dy}\frac{d}{dz}(g(x,y)) = \frac{d}{dx}\frac{d}{dy}(0) = 0,$$

and similarly for the other two terms, by taking the derivatives in a different order. (This is possible by Clairaut's theorem, assuming that g, h, and k have continuous mixed partial derivatives.)

(b) True: adding $ax + by + cxy + d$ doesn't affect the value of $u_{xx} + u_{yy}$, since the pure second partial derivatives of $ax + by + cxy + d$ are both zero.

(2) We have

$$f_x(x,t) = -t^2 e^{-x}$$

and

$$f_t(x,t) = 2te^{-x}.$$

(3) Since

$$\begin{aligned} f_x &= 4x^3y - 6x^2y^2 \text{ and} \\ f_y &= x^4 - 4x^3y, \end{aligned}$$

we have

$$\begin{aligned} f_{xx} &= 12x^2y - 12xy^2, \\ f_{xy} = f_{yx} &= 4x^3 - 12x^2y, \text{ and} \\ f_{yy} &= -4x^3. \end{aligned}$$

(4) We first differentiate with respect to y to get rid of the arcsin term:

$$f_y = 2xyz^3.$$

Then we differentiate with respect to x and z:

$$\begin{aligned} f_{xy} &= 2yz^3, \\ f_{xyz} &= 6yz^2. \end{aligned}$$

(5) We calculate the pure second partial derivatives of u:

$$\begin{aligned} u_x &= f'(x+at) + g'(x-at), \text{ so} \\ u_{xx} &= f''(x+at) + g''(x-at); \\ u_t &= af'(x+at) - ag'(x-at), \text{ so} \\ u_{tt} &= a^2f''(x+at) + a^2g''(x-at) \\ &= a^2u_{xx}. \end{aligned}$$

(6) Many solutions are possible. For example, if $u(x,y)$ has the form $ax + by + cxy + d$ for some real numbers a, b, c, d (cf. problem 1b), then

$$u_{xx} + u_{yy} = 0 + 0 = 0.$$

Two other solutions are $e^x \sin(y)$ (cf. Example 9 in the textbook) and $e^x \cos(y)$; these functions don't change when differentiated twice with respect to x, but they are negated when we differentiate twice with respect to y.