MATH 53 DISCUSSION SECTION PROBLEMS – 2/2/23

1. The cross product; geometry with vectors

(1) True/false practice:
  (a) The magnitude of the dot product of two vectors is equal to the magnitude of the cross product of those two vectors.
  (b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^3$, the expression
$$(\mathbf{u} \bullet \mathbf{v}) \times \mathbf{w}$$
  makes sense.

(2) **(textbook 12.4.7)** Find the cross product of $\langle t, 1, \frac{1}{t} \rangle$ and $\langle t^2, t^2, 1 \rangle$. Verify that it is orthogonal to both vectors.

(3) **(textbook 12.4.10)** Find $\mathbf{k} \times (\mathbf{i} - 2\mathbf{j})$.

(4) **(similar to textbook 12.4.44)** Find all vectors $\mathbf{v}$ such that $\langle 1, 0, 1 \rangle \times \mathbf{v} = \langle -2, 4, 2 \rangle$. Why are there no vectors $\mathbf{v}$ such that $\langle 1, 0, 1 \rangle \times \mathbf{v} = \langle 2, 4, 2 \rangle$?

(5) **(textbook 12.4.36)** Find the volume of a parallelepiped with adjacent edges PQ, PR, and PS, where
$$P = (3, 0, 1), \quad Q = (-1, 2, 5), \quad R = (5, 1, -1), \quad S = (0, 4, 2).$$

(6) **(textbook 12.4.37)** Use the scalar triple product to verify that the vectors $\mathbf{u} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$, and $\mathbf{w} = 5\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$ are coplanar. Why does this method work?

(7) **(textbook 12.4.43)** If $\mathbf{a} \cdot \mathbf{b} = \sqrt{3}$ and $\mathbf{a} \times \mathbf{b} = \langle 1, 2, 2 \rangle$, find the angle between $\mathbf{a}$ and $\mathbf{b}$.

(8) **(from an old practice exam)** Consider the following points in 3-dimensional space:
$$O = (0, 0, 0) \qquad A = (0, 1, 1) \qquad B = (-2, 2, -2) \qquad P = (-1, 0, 1)$$
  Find a point X such that both of the following statements are true
  (a) $\overrightarrow{OX}$ is perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$, and
  (b) The area of a parallelogram with adjacent edges given by $\overrightarrow{OX}$ and $\overrightarrow{OP}$ is 12.

(9) **(*)** Rather than choosing to write the cross product of $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ as
$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k},$$
  we could write it as
$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{j} \times \mathbf{k} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{i} \times \mathbf{k} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{i} \times \mathbf{j}.$$
  Let's replace the symbol $\times$ with the symbol $\wedge$. We now have
$$\langle a_1, a_2, a_3 \rangle \wedge \langle b_1, b_2, b_3 \rangle = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{j} \wedge \mathbf{k} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{i} \wedge \mathbf{k} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{i} \wedge \mathbf{j}.$$
  What should $\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle$ be? What about $\langle a_1, a_2, a_3, a_4 \rangle \wedge \langle b_1, b_2, b_3, b_4 \rangle$? What about for n-dimensional vectors? How is this new operation $\wedge$ like the cross product? How is it different? (Spoiler alert: this operation might become relevant in a future challenge problem in chapter 16; it appears in a more general framework describing vector-field-like derivatives like the curl.)

2. Notes

Original author: James Rowan.

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked **(*)** are challenge problems, with problems marked **(**)** especially challenging problems.