MATH 53 DISCUSSION SECTION ANSWERS – 2/2/23

1. The cross product; geometry with vectors

(1) (a) False: the magnitude of the dot product involves a cosine, and the magnitude of the cross product involves a sine.
(b) False: $u \bullet v$ is a scalar, so it can't be crossed with anything.

(2) The cross product is:

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 1 & 1/t \\ t^2 & t^2 & 1 \end{vmatrix} &= (1 \cdot 1 - 1/t \cdot t^2)\mathbf{i} - (t \cdot 1 - 1/t \cdot t^2)\mathbf{j} + (t \cdot t^2 - 1 \cdot t^2)\mathbf{k} \\ &= (1-t)\mathbf{i} + (t^3 - t^2)\mathbf{k}. \end{aligned}$$

The dot products of this with the original vectors are:

$$\begin{aligned} &t \cdot (1-t) + 1 \cdot 0 + 1/t \cdot (t^3 - t^2) = (t - t^2) + (t^2 - t) = 0 \text{ and} \\ &t^2 \cdot (1-t) + t^2 \cdot 0 + 1 \cdot (t^3 - t^2) = (t^2 - t^3) + (t^3 - t^2) = 0. \end{aligned}$$

(3) Instead of using the general formula, let's just expand what we know about cross products of the standard basis vectors:

$$\begin{aligned} \mathbf{k} \times (\mathbf{i} - 2\mathbf{j}) &= \mathbf{k} \times \mathbf{i} - 2\mathbf{k} \times \mathbf{j} \\ &= \mathbf{j} - 2(-\mathbf{i}) = 2\mathbf{i} + \mathbf{j}. \end{aligned}$$

(4) For the first part, let's calculate $\langle 1, 0, 1 \rangle \times \langle v_1, v_2, v_3 \rangle$:

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ v_1 & v_2 & v_3 \end{vmatrix} &= (0v_3 - 1v_2)\mathbf{i} - (1v_3 - 1v_1)\mathbf{j} + (1v_2 - 0v_1)\mathbf{k} \\ &= -v_2\mathbf{i} + (v_1 - v_3)\mathbf{j} + v_2\mathbf{k}. \end{aligned}$$

This equals $\langle -2, 4, 2 \rangle$ if and only if $v_2 = 2$ and $v_1 - v_3 = 4$. So any vector of the form $\langle c+4, 2, c \rangle$ works.

On the other hand, there is no vector $\mathbf{v}$ such that $\langle 1, 0, 1 \rangle \times \mathbf{v} = \langle 2, 4, 2 \rangle$ because $\langle 2, 4, 2 \rangle$ is not orthogonal to $\langle 1, 0, 1 \rangle$. (Remember that the cross product of two vectors is always orthogonal to both of them.) This can also be seen from the equation above, since we can't simultaneously have $-v_2 = 2$ and $v_2 = 2$.

(5) This volume is the absolute value of the scalar triple product of the vectors $Q - P$, $R - P$, and $S - P$:

$$\begin{aligned} \begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} &= -4(1 \cdot 1 - (-2) \cdot 4) - 2(2 \cdot 1 - (-2) \cdot (-3)) + 4(2 \cdot 4 - 1 \cdot (-3)) \\ &= -4 \cdot 9 - 2 \cdot (-4) + 4 \cdot 11 \\ &= 16. \end{aligned}$$

(6) The scalar triple product is

$$\begin{aligned} \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} &= 1((-1) \cdot (-4) - 0 \cdot 9) - 5(3 \cdot (-4) - 0 \cdot 5) + (-2)(3 \cdot 9 - (-1) \cdot 5) \\ &= 1(4) - 5(-12) - 2(32) \\ &= 4 + 60 - 64 = 0, \end{aligned}$$

so the vectors are coplanar. This method works because the scalar triple product calculates the volume of the parallelepiped spanned by the three vectors, which is 0 if and only if the three vectors are coplanar.

(7) We have
$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = \sqrt{3}$$
and
$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta = |\langle 1,2,2\rangle| = 3,$$
so dividing the second equation by the first gives
$$\tan\theta = \frac{3}{\sqrt{3}} = \sqrt{3}$$
and thus $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$.

(8) First calculate the cross product $\overrightarrow{OA}\times\overrightarrow{OB}$:
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -2 & 2 & -2 \end{vmatrix} = -4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} = \langle -4,-2,2\rangle.$$
Let's call this vector $\mathbf{v}$. This is perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$, so the answer should be a scalar multiple of it. Now let's take the cross product of $\mathbf{v}$ with $\overrightarrow{OP}$:
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -2 & 2 \\ -1 & 0 & 1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} = \langle -2,2,-2\rangle.$$
So the area of the parallelogram spanned by $\mathbf{v}$ and $\overrightarrow{OP}$ is $\left|\mathbf{v}\times\overrightarrow{OP}\right| = \sqrt{12}$. We want the area to be 12, so we multiply by $\sqrt{12}$; that is, our answer is
$$X = \sqrt{12}(-4,-2,2) = 2\sqrt{3}(-4,-2,2) = (-8\sqrt{3},-4\sqrt{3},4\sqrt{3}).$$
(We could have also taken the negative of this.)

(9) We should define
$$\langle a_1,a_2\rangle\wedge\langle b_1,b_2\rangle = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{i}\wedge\mathbf{j}$$
and
$$\begin{aligned}\langle a_1,a_2,a_3,a_4\rangle\wedge\langle b_1,b_2,b_3,b_4\rangle = {}& \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{i}\wedge\mathbf{j} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{i}\wedge\mathbf{k} + \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix}\mathbf{i}\wedge\mathbf{l} \\ &+ \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{j}\wedge\mathbf{k} + \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix}\mathbf{j}\wedge\mathbf{l} + \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix}\mathbf{k}\wedge\mathbf{l}.\end{aligned}$$
In general, if e_i denotes the i-th standard basis vector in $\mathbb{R}^n$, we define
$$\begin{aligned}\langle a_1,\ldots,a_n\rangle\wedge\langle b_1,\ldots,b_n\rangle &= \sum_{1\le i<j\le n}\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}\mathbf{e}_i\wedge\mathbf{e}_j \\ &= \sum_{1\le i<j\le n}(a_ib_j - b_ia_j)\mathbf{e}_i\wedge\mathbf{e}_j.\end{aligned}$$
If we declare that the symbol $\wedge$ obeys the antisymmetry rule
$$\mathrm{v}\wedge\mathrm{w} = -\mathrm{w}\wedge\mathrm{v}$$
for all vectors v,w (and in particular $\mathrm{v}\wedge\mathrm{v} = -\mathrm{v}\wedge\mathrm{v} = 0$), then this can be written in an even simpler form: we define
$$\langle a_1,\ldots,a_n\rangle\wedge\langle b_1,\ldots,b_n\rangle = \sum_{i,j} a_ib_j(\mathbf{e}_i\wedge\mathbf{e}_j),$$
where we no longer require that $i<j$. At this point, it's not much of a definition at all; we're just saying that linear combinations like
$$\langle a_1,\ldots,a_n\rangle = a_1\mathrm{e}_1 + \cdots + a_n\mathrm{e}_n$$
distribute over the symbol $\wedge$. Of course, we're hiding all of the real work by not saying what the symbols $\mathrm{v}\wedge\mathrm{w}$ actually *are*, and why $\wedge$ is antisymmetric—that requires some multilinear algebra.

If you're interested, the symbol $\wedge$ is commonly known as "wedge product" or "exterior product". It is similar to the cross product in that they're both bilinear (i.e. it satisfies the distributivity property mentioned above) and antisymmetric. There are two main differences: first, the cross product only makes sense in three dimensions, while the wedge product makes sense in any number of dimensions; second, the cross product produces ordinary vectors as outputs, while the exterior product produces "bivectors" (symbols like v $\wedge$ w) as outputs. If we're working in $\mathbb{R}^n$, bivector turn out to form a vector space of dimension $\frac{n(n-1)}{2}$, which is usually much larger than n. But if $n = 3$, then we have n(n $-$ 1)2 $=$ 3 $=$ n, and in this case we can identify the space of bivectors with the space of vectors (namely, let i $\wedge$ j, j $\wedge$ k, and k $\wedge$ i correspond to k, i, and j respectively). This can be seen as an explanation of why the cross product only makes sense in three dimensions.