MATH 53 DISCUSSION SECTION ANSWERS – 2/16/23

1. Limits of multivariable functions

(1) (a) False: a function can approach L along some paths (e.g. the three coordinate axes) but not others. For example, consider the function $g(x, y, z)$ that equals 0 for points lying on the three coordinate axes and 1 everywhere else. This does not have a limit at $(0, 0, 0)$, because its limit along the path $x = y = z = t$ is different from its limit along the path $x = y = 0, z = t$.

(2) Polynomial functions are always continuous, so the limit exists and equals the value of the function at $(3, 2)$: $3^2 2^3 - 4 \cdot 2^2 = 56$.

(3) In polar coordinates, this function is given by $\frac{r^2}{\sqrt{r^2+1}-1}$, which has the form $\frac{0}{0}$ as $r \to 0^+$. So we use l'Hospital's rule:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} &= \lim_{r\to 0^+} \frac{r^2}{\sqrt{r^2+1}-1} \\ &= \lim_{r\to 0^+} \frac{2r}{2r \cdot \frac{1}{2}(r^2+1)^{-1/2}} \\ &= \lim_{r\to 0^+} 2\sqrt{r^2+1} \\ &= 2.\end{aligned}$$

(4) (Done in lecture.) If we take the limit along $x = t$, $y = z = 0$, we get

$$\lim_{t\to 0} \frac{0}{t^2} = 0.$$

But the limit along $x = y = t, z = 0$ is

$$\lim_{t\to 0} \frac{t^2}{2t^2} = \frac{1}{2}.$$

These limits don't equal each other, so the overall limit does not exist.

(5) (a) The domain is the set of points $(x, y) \in \mathbb{R}^2$ such that the denominator $x^3 + y^3$ is nonzero; that is, everywhere in $\mathbb{R}^2$ except for the line $y = -x$. The function is continuous everywhere it is defined—this is true of all rational functions.

(b) Since the function is continuous here, the limit is just the value of the function at $(2, 1)$, namely

$$\frac{2^3 - 1^3}{2^3 + 1^3} = \frac{7}{9}.$$

(c) If we take the limit along the x-axis ($y = 0$), we get

$$\lim_{x\to 0} \frac{x^3}{x^3} = 1.$$

If we take the limit along the y-axis ($x = 0$), we get

$$\lim_{y\to 0} \frac{-y^3}{y^3} = -1.$$

These limits don't agree, so the overall limit does not exist.

(6) **(**)** The statement is false. Here's a somewhat cheap counterexample: let $D = \mathbb{Q}^2$, the set of points in the $\mathbb{R}^2$ whose coordinates are both rational numbers. Then there are *no* nonconstant paths lying in the domain D: you can't move around in the plane at all without stepping on a point with an irrational coordinate. So *any* function defined on D has a limit at $(0, 0)$ along all paths in D (simply

because there are no such paths). But such functions need not have limits at $(0, 0)$; for example, consider the function $f : D \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

2. Notes

Original author: James Rowan.

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked (*) are challenge problems, with problems marked (**) especially challenging problems.