MATH 53 DISCUSSION SECTION ANSWERS – 1/31/23

1. Vectors and the geometry of space

(1) (a) False (to the extent that it's even a meaningful statement): by convention, we draw the x, y, and z-axes according to the right-hand rule.
(b) False: $-\mathbf{v}$ is the vector obtained by negating all the coordinates of $\mathbf{v}$; for example, $-\langle a, b, c\rangle$ is the vector $\langle -a, -b, -c\rangle$.

(2) The distance between these two points is $\sqrt{(4-3)^2 + (3-8)^2 + (-1-1)^2} = \sqrt{30}$, so the sphere consists of all points whose distance from $(3, 8, 1)$ is $\sqrt{30}$. This is defined by the equation

$$\sqrt{(x-3)^2 + (y-8)^2 + (z-1)^2} = \sqrt{30},$$

or equivalently

$$(x-3)^2 + (y-8)^2 + (z-1)^2 = 30.$$

(3) The sum is $\langle 5, 2\rangle$; this can be illustrated by the parallelogram with vertices at $(0, 0)$, $(-1, 4)$, $(6, -2)$, and $(5, 2)$.

(4) The horizontal component is $50\cos(38^\circ) \approx 39.4$ N in the direction that the child is walking, and the vertical component is $50\sin(38^\circ) \approx 30.8$ N upwards.

(5) Treat every continuous function $f : [0, 1] \to \mathbb{R}$ as a vector. *Define* addition and scalar multiplication of these vectors by the formulas

$$(f+g)(x) = f(x) + g(x)$$

and

$$(cf)(x) = cf(x),$$

which are both continuous functions (given that f and g are). Then you can check that all of the properties on page 802 of your textbook hold true, where the zero vector is the constant function $f(x) = 0$ and $-f$ is the function $(-f)(x) = -f(x)$.

2. The dot product

(6) (a) True: in the equation $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$, the left-hand side is positive if and only if the angle is acute, and the right-hand side is positive if and only if the dot product is positive.

(7) We have

$$\begin{aligned}|\mathbf{a}||\mathbf{b}|\cos\theta = \mathbf{a}\cdot\mathbf{b} &= 7\cdot 4\cdot\cos(30^\circ)\\ &= 7\cdot 4\cdot\frac{\sqrt{3}}{2} = 14\sqrt{3}.\end{aligned}$$

(8) If $\mathbf{w}$ is a unit vector which makes an angle of 60° with $\mathbf{v}$, then this means that

$$\mathbf{w}\cdot\mathbf{w} = |\mathbf{w}|^2 = 1$$

and

$$\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta = 5\cos(60^\circ) = \frac{5}{2}.$$

Writing $\mathbf{w}$ in coordinates as $\langle w_1, w_2\rangle$, this means that

$$w_1^2 + w_2^2 = 1$$

and

$$3w_1 + 4w_2 = \frac{5}{2}.$$

The second equation implies $w_2 = \frac{5/2 - 3w_1}{4}$. Plugging this into the first equation gives

$$w_1^2 + \frac{(5/2 - 3w_1)^2}{16} = 1.$$

Solving the quadratic equation gives

$$w_1 = \frac{3}{10} \pm \frac{2\sqrt{3}}{5},$$

and the corresponding values of w_2 are

$$w_2 = \frac{2}{5} \mp \frac{3\sqrt{3}}{10}.$$

(9) The first line is parallel to the line $2x - y = 0$, which passes through the origin and the point $(1, 2)$, so it is parallel to the vector $\langle 1, 2\rangle$. The second line is parallel to the line $3x + y = 0$, which passes through the origin and the point $(-1, 3)$, so it is parallel to the vector $\langle -1, 3\rangle$. Thus the angle between the given lines is equal to the angle θ between the vectors $\langle 1, 2\rangle$ and $\langle -1, 3\rangle$. We have

$$\begin{aligned} \cos\theta &= \frac{\langle 1,2\rangle \cdot \langle -1,3\rangle}{|\langle 1,2\rangle||\langle -1,3\rangle|} \\ &= \frac{-1+6}{\sqrt{5}\sqrt{10}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}, \end{aligned}$$

so $\theta = \cos^{-1}(1/\sqrt{2}) = \pi/4$. (Note that if we had flipped the sign on one of our two vectors, the dot product would have been negative, so we would have gotten $\theta = \cos^{-1}(-1/\sqrt{2}) = 3\pi/4$. This is the other angle formed by the two given lines; the acute angle is obtained by subtracting it from π.)

(10) Let's call the two vectors $\mathbf{v} = \langle 2, 4, -1\rangle$ and $\mathbf{w} = \langle 3, -3, 1\rangle$ respectively. The scalar projection of $\mathbf{v}$ onto $\mathbf{w}$ is

$$\frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{w}|} = \frac{-7}{\sqrt{19}},$$

and the vector projection is

$$\frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{w}|^2}\mathbf{w} = \frac{-7}{19}\langle 3, -3, 1\rangle.$$

(11) It is commutative because

$$f \bullet g = \int_0^1 f(x)g(x)dx = \int_0^1 g(x)f(x)dx = g \bullet f.$$

Our dot product is distributive over addition because

$$f \cdot (g + h) = \int_0^1 f(x)(g(x) + h(x))dx = \int_0^1 f(x)g(x)dx + \int_0^1 f(x)h(x)dx = f \cdot g + f \cdot h.$$

Scalar multiplication is associative because

$$(cf) \bullet g = \int_0^1 (cf(x))g(x)dx = c\int_0^1 f(x)g(x)dx = c(f \bullet g),$$

and similarly

$$f \bullet (cg) = \int_0^1 f(x)(cg(x))dx = c\int_0^1 f(x)g(x)dx = c(f \bullet g).$$

The dot product of a function with itself, $f \bullet f$, is always nonnegative because it's the integral of the nonnegative function $f(x)^2$; it equals zero if and only if this nonnegative function is identically zero. (Technically this uses the fact that f is continuous.)

Consequently, we can define the length of a vector by

$$|f| = \sqrt{f \bullet f}$$

and the angle between two vectors by

$$\theta = \cos^{-1}\left(\frac{f \bullet g}{|f||g|}\right).$$

This is closely related to Fourier coefficients: taking the Fourier series of a function can be thought of as writing it in terms of an orthogonal basis consisting of sine and cosine functions; the Fourier coefficients are the scalar projections onto the individual basis functions.