# MATH 53 DISCUSSION SECTION ANSWERS – 1/26/23

## 1. Areas and lengths in polar coordinates

(1) Since we want only the area in the right half of the plane, we integrate from $\theta = -\pi/2$ to $\pi/2$:

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \frac{r^2}{2} d\theta &= \int_{-\pi/2}^{\pi/2} \frac{(4+3\sin\theta)^2}{2} d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (16 + 24\sin\theta + 9\sin^2\theta) d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(16 + 24\sin\theta + \frac{9(1-\cos 2\theta)}{2}\right) d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(16 + \frac{9}{2} + 24\sin\theta - \frac{9\cos 2\theta}{2}\right) d\theta \\
&= \frac{1}{2}\left(16\pi + \frac{9\pi}{2} + 24\int_{-\pi/2}^{\pi/2} \sin\theta d\theta - \frac{9}{2}\int_{-\pi/2}^{\pi/2} \cos 2\theta\right) d\theta \\
&= \frac{1}{2}\left(16\pi + \frac{9\pi}{2} + 24[-\cos\theta]_{-\pi/2}^{\pi/2} - \frac{9}{2}\left[\frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}\right) \\
&= \frac{1}{2}\left(16\pi + \frac{9\pi}{2}\right) \\
&= \frac{1}{2}\left(\frac{41\pi}{2}\right) = \frac{41\pi}{4}.
\end{aligned}$$

(2) The curve passes through the origin whenever $\theta$ is a multiple of $\pi/4$, so one loop goes from $\theta = 0$ to $\theta = \pi/4$. We use the integral:

$$\begin{aligned}
\int_0^{\pi/4} \frac{r^2}{2} d\theta &= \int_0^{\pi/4} \frac{\sin^2 4\theta}{2} d\theta \\
&= \int_0^{\pi/4} \frac{1-\cos 8\theta}{4} d\theta \\
&= \left[\theta - \frac{\sin 8\theta}{8}\right]_0^{\pi/4} \\
&= \frac{\pi/4}{4} = \frac{\pi}{16}.
\end{aligned}$$

(3) If we graph the two regions, we see that they are two cloverleaf shapes with four lobes each, offset from each other by a 45-degree rotation. By symmetry, we can find the area between $\theta = 0$ and

$\theta = \pi/8$ (where $r = \sin 2\theta$ is the inner curve), and multiply the area by 16:

$$\begin{aligned} 16 \int_0^{\pi/8} \frac{r^2}{2} d\theta &= 8 \int_0^{\pi/8} \sin^2 2\theta d\theta \\ &= 8 \int_0^{\pi/8} \frac{1 - \cos 4\theta}{2} d\theta \\ &= 4 \left[ \theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/8} \\ &= 4 \cdot \frac{\pi}{8} - \sin \frac{\pi}{2} = \frac{\pi}{2} - 1. \end{aligned}$$

(4) Since $\frac{dr}{d\theta} = -2\sin\theta$, the arclength is:

$$\begin{aligned} \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta &= \int_0^{\pi} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2} d\theta \\ &= \int_0^{\pi} \sqrt{4(\cos^2\theta + \sin^2\theta)} d\theta \\ &= \int_0^{\pi} 2 d\theta = 2\pi. \end{aligned}$$

Alternatively, this can be done (with more work) by converting the curve into the parametric form $x = 2\cos^2\theta$, $y = 2\sin\theta\cos\theta$. We have:

$$\begin{aligned} \frac{dx}{d\theta} &= -4\sin\theta\cos\theta, \\ \frac{dy}{d\theta} &= 2\cos^2\theta - 2\sin^2\theta, \end{aligned}$$

and the arclength is

$$\begin{aligned} \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta &= \int_0^{\pi} \sqrt{\left(16\sin^2\theta\cos^2\theta + 4\cos^4\theta - 8\sin^2\theta\cos^2\theta + 4\sin^4\theta\right)} d\theta \\ &= \int_0^{\pi} \sqrt{\left(4\cos^4\theta + 8\sin^2\theta\cos^2\theta + 4\sin^4\theta\right)} d\theta \\ &= \int_0^{\pi} \left(2\cos^2\theta + 2\sin^2\theta\right) d\theta \\ &= \int_0^{\pi} 2 d\theta = 2\pi. \end{aligned}$$