MATH 53 DISCUSSION SECTION ANSWERS – 1/24/23

1. Calculus with parametric curves

(1) (a) This is true if $\frac{dy}{dt} = \frac{dx}{dt}$ and they are nonzero. (Reason: in this case we have $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt} = 1$.) But it can be false when $\frac{dy}{dt} = \frac{dx}{dt} = 0$. For example, for the curve given by $x = t^3, y = 0$, both derivatives are 0 at $t = 0$, but the curve is horizontal.

(b) True: these are just two different ways to parametrize the unit circle (with area π).

(2) The curve passes through $(0, 3)$ at $t = 1$. At this time we have $\frac{dx}{dt} = 2t - 1 = 1$ and $\frac{dy}{dt} = 2t + 1 = 3$, so $\frac{dy}{dx} = \frac{3}{1} = 3$. The line with slope 3 passing through $(0, 3)$ is $y = 3x + 3$.

(3) We have $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 2t + 1$, so $\frac{dy}{dx} = \frac{2t+1}{2t} = 1 + \frac{1}{2t}$. For the second derivative, we have:

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \\ &= \frac{-t^{-2}/2}{2t} = -\frac{1}{4t^3}. \end{aligned}$$

This is positive (i.e. the curve is concave up) when $t < 0$.

(4) We have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{d\sin\theta}{r - d\cos\theta}. \end{aligned}$$

A vertical tangent line is only possible if the denominator is 0 for some value of θ, but this is impossible if $d < r$, because $r > d \geq d\cos\theta$.

(5) The curve intersects the x-axis at $t = 0$ and $t = 2$, at the points $(1, 0)$ and $(9, 0)$ respectively, so we use the integral:

$$\begin{aligned} \int_0^2 y\frac{dx}{dt}dt &= \int_0^2 (2t - t^2) \cdot 3t^2 dt \\ &= \int_0^2 (6t^3 - 3t^4)dt \\ &= [3t^4/2 - 3t^5/5]_0^2 \\ &= 24 - 96/5 = 24/5. \end{aligned}$$

(6)

$$\begin{aligned} \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt &= \int_0^2 \sqrt{(1 - e^{-t})^2 + (1 + e^{-t})^2} dt \\ &= \int_0^2 \sqrt{2 + 2e^{-2t}} dt \end{aligned}$$

(7) **(*)** This ellipse can be obtained by stretching the unit circle $x^2 + y^2 = 1$ by a factor of a horizontally and a factor of b vertically. This multiplies the area (π) of the unit circle by a factor of ab.

2. Polar coordinates

(8) (a) False: $(2, \frac{\pi}{2})$ and $(2, \frac{5\pi}{2})$ are the same, and $(-2, \frac{5\pi}{2})$ is opposite them.
(b) True: the equation $r = 6\cos\theta$ can be converted to Cartesian coordinates as $(6\cos^2\theta, 6\sin\theta\cos\theta)$, which is equivalent by trig identities to $(3\cos 2\theta + 3, 3\sin 2\theta)$; that is, a circle of radius 3 centered on $(3, 0)$.

(9) This point also has polar coordinates $(1, 9\pi/4)$ and $(-1, 5\pi/4)$; it is one unit northeast of the origin.

(10) This region is the quadrant of the plane which lies above the two diagonal lines through the origin; it's equivalent to the region defined by $y \geq |x|$.

(11) (a) In polar coordinates, this is $\theta = \pi/6$. In Cartesian coordinates, it's $y = \tan(\pi/6) \cdot x$, where $\tan(\pi/6) = 1/\sqrt{3}$.
(b) In Cartesian coordinates, this is $x = 3$.