UNIFORM BOUNDEDNESS OF RATIONAL POINTS

TONY FENG

1. Introduction

1.1. **Faltings' Theorem.** This story begins with the following theorem proved by Faltings in the 80s, which is one of the crowning achievements of 20th century arithmetic geometry.

Theorem 1.1 (Faltings). *Let X be a smooth, projective curve defined over a number field K of genus $g \geq 2$. Then $X(K)$ is finite.*

Remark 1.2. When $g = 0$, we know that either $X(K) = \emptyset$ or $X \cong \mathbb{P}^K_1$ (and thus has "many" rational points).

When $g = 1$, the important and deep *Mordell-Weil Theorem* says that $X(K)$ is a *finitely generated abelian group*. Conjecturally, $X(K)$ is finite about half of the time and has rank one the other half ("$X(K) = \infty^1$"), whatever that means.

Example 1.3. A consequence of the theorem is that a typical polynomial equation in two variables $f(x, y) = 0$ admits only *finitely* many solutions with $x, y \in K$ if $\deg f \geq 4$. By contrast, $f(x, y)$ will often have infinitely many such solutions if $\deg f \leq 3$.

Similarly, a "typical" equation of the form $y^2 = f(x)$ will admit only finitely many solutions with $x, y \in K$ if $\deg f \geq 5$.

One of the remarkable things about this theorem is the way in which it suggests that geometry informs arithmetic. The geometric genus g is a manifestly *geometric* condition, yet it is controlling what seems to be an arithmetic property. Why should the number of *integral* solutions to $x^n + y^n = z^n$ have anything to do with the shape of the complex solutions?

You might argue that that the genus is essentially the same invariant as the degree in the the cases we discussed (plane conics and hyperelliptic curves). But smoothness, another morally geometric condition, is also crucial here. So both the "global" and "local" topological properties are reflected in the arithmetic behavior.

Of course, when you see a great theorem you should ask how it can be generalized. We are going to discuss two possible and seemingly unrelated directions of generalization. The focus of the talk will then be the connection between the two.

1.2. **Uniformity.** Faltings' Theorem is the epitome of mathematicians' sensibilities: aesthetically beautiful, and practically useless. It guarantees that for all (smooth) genus $g \geq 2$ curves there are only finitely many rational points, but for any given curve gives us no clue how to find them, or "how finite" they are. Analysts might call this an "ineffective" bound:

$$\#X(K) \leq C_{X,K}$$

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because we cannot exert any explicit control over the constant $C_{X,K}$. So one way to generalize it would be to try and make it *effective* - that is, to remove some of the dependence on X and K.

Of course, any true bound must have *some* dependence on K. Indeed, for a fixed X, any point of X defined over $\overline{K}$ (and there are infinitely many of them) is defined over some *finite* extension of K. Therefore, by choosing a sufficiently large number field K we can make $\#X(K)$ as large as we want.

Similarly, the bound must also have *some* dependence on X. Indeed, it is easy to see that we can construct polynomials $f(x, y)$ with arbitrarily many zeros over K by interpolation if we are allowed to make $d := \deg f$ arbitrarily large. Now, the genus of the corresponding curve is determined by $\binom{d-1}{2}$, so this suggests a dependence of the constant on X at least through its genus.

So any hypothetical effective bound on $\#X(K)$ should be expressible as function of something having to do with K and g (at least). In order for such a bound to be possible, it must be the case that $\#X(K)$ is uniformly bounded among all curves of a fixed genus g.

Conjecture 1.4 (Uniform Boundedness Conjecture)**.** *There exists a function* $C(g, K)$ *such that for any smooth projective curve* X *of genus* g,

$$\#X(K) \leq C(g, K).$$

There are other variants of this conjecture, which we will not discuss.

Remark 1.5. Actually, in the end we will not be able to say anything effective about $C(g, K)$. The best lower bound on $C(g, K)$ is due to Brumer: $C(g, \mathbb{Q}) \geq 8 \cdot g + 12$.

1.3. **Lang's Conjecture.** Another direction of generalization of Theorem 1.1 is to higher-dimensional varieties. For curves, we had

$$\#X(K) < \infty \text{ if } g \geq 2.$$

There are then two questions: what is the generalization of ($\#X(K) < \infty$), and what is the generalization of ($g \geq 2$)?

There is an accepted answer to the second question: the analogue of $g \geq 2$ for higher-dimensional X is "general type."

Definition 1.6. Let X be a smooth, projective variety and K_X its canonical bundle. We say that X is of *general type* if

$$\dim H^0(X, K_X^{\otimes n}) \sim n^{\dim X}.$$

Intuitively, X is general if it has "lots of holomorphic (top) forms."

Example 1.7. If K_X is ample, then X is general type, since the dimension of X coincides with the degree of its Hilbert polynomial under an embedding.

Let X be a hypersurface in $\mathbb{P}^n$ of degree d. By the Adjunction formula,

$$\begin{aligned} K_X &\cong K_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(X)|_X \\ &= \mathcal{O}_{\mathbb{P}^n}(-n - 1 + d)|_X. \end{aligned}$$

Therefore, X has general type if and only if $\mathcal{O}(-n - 1 + d)$ is ample, i.e. $d > n + 1$.

Example 1.8. Rational varieties, which has no holomorphic one forms at all (the canonical bundles of projective space are anti-ample), are in some sense the "opposite" of being general type.

Example 1.9. Let X be a curve. Then $\deg K_X = 2g-2$, and by Riemann-Roch we see that X is of general type if and only if $g \geq 2$.

Since this is a birational invariant of smooth projective varieties, we can and do extend it to singular varieties by birationality. In particular, "general type" is on the opposite end of the spectrum from "rational" since (as we used above) the canonical bundle of $\mathbb{P}^n$ is anti-ample.

Now what's the analogue $\#X(K) < \infty$? This is way too strong to demand literally, because even if X is of general it may contain rational subvarieties (for instance, if X is a hypersurface of high degree in projective space it could still contain lines). There are many possible salvages, but we adopt the following.

Conjecture 1.10 (Weak Lang Conjecture)**.** *If X is of general type, then $X(K)$ is not Zariski-dense in X.*

Example 1.11. If X is a curve, then "not Zariski-dense" is the same as "finite" for $X(K)$ (for a set of closed points).

The focus of this talk is the following (surprising!) implication:

Theorem 1.12 (Caporaso-Harris-Mazur)**.** *Lang's Conjecture 1.10 implies the Uniform Boundedness Conjecture 1.4.*

By the way, the truth of either of these conjectures is by no means intuitive. Rumor has it that Caporaso-Harris-Mazur originally viewed this theorem as evidence *against* Lang's conjecture, because they didn't believe the Uniform Boundedness Conjecture.

2. Outline of the Proof

2.1. **Motivation.** The key idea is to consider the geometry of a family $f\colon X \to B$ of smooth, projective curves of genus g. (Technically, f should be a proper, flat morphism of integral varieties with generic fiber a smooth curve of genus g.) Imagine for a moment that we could prove that any such X were of general type. (Intuitively, X is fiberwise of general type.) Then Lang's conjecture would assure us that the rational points of X lie in some Zariski closed subset, so there is a large open subset of B such that the fibers have no rational points.

Applying this to some family whose fibers exhaust the isomorphism classes of genus g curves (e.g. an appropriate Hilbert scheme) we would deduce that "most" of them have no rational points. By an inductive argument, we could hope to prove a uniform bound.

2.2. **The Correlation Theorem.** Unfortunately, one cannot hope that such a "fiberwise general type" family is of general type (as we'll see in the examples later). We need a salvage.

Definition 2.1. For a proper morphism $f\colon X \to B$ of integral varieties whose generic fiber is of general type, we denote by X_B^n the nth fiber product of X over B. ♠♠♠ TONY: [may be skirting some technical issues of reducibility here?]

We say that the family $f\colon X \to B$ has *correlation* if there exists an n such that X_B^n admits a dominant morphism to a variety of general type.

Here is an explanation of the terminology. For motivation, imagine that $f\colon X \to B$ has generic fiber a smooth projective curve of genus $g \geq 2$. For $b \in B$, let $X_b := f^{-1}(b)$ denote the fiber of f over b. By Faltings' theorem, we know that the rational points of X_b are finite. However, we don't know anything a priori about their distribution in X - they could be dense, for instance. If the rational points conspired to lie in a proper Zariski-closed subset of X, then the algebraic equations defining that closed subset could be interpeted as relations describing a "correlation" between the rational points of *different* fibers. Lang's theorem predicts this to be the case if X is of general type.

More generally, as long as X_B^2 admits a dominant rational map to a variety of general type, then Lang's theorem implies that $X_B^2(K)$ lands in a proper Zariski-closed subset. In particular, $X_B^2(K)$ lies in a proper Zariski-closed subset of X_B^2, which suggests that there are algebraic relations governing the distribution of *pairs* of rational points in the fibers. We can interpret this as an algebraic "correlation." Similarly, if X_B^n dominates a variety of general type, then there is an "n-point correlation."

The technical heart of Theorem 1.12 is:

Theorem 2.2 (Correlation)**.** *Let $f\colon X \to B$ be a proper morphism of integral varieties whose general fiber is a smooth curve of genus at least 2. Then $X \xrightarrow{f} B$ has correlation. Moreover if X is defined over K then we can take the correlation morphism (and in particular the target) to be so as well.*

We will not prove this in our talk, but we will give some illustrative examples. First, however, we will show why Theorem 1.12 follows from this.

Proposition 2.3. *Let $f\colon X \to B$ be as above. Then there exists N and an open subset $U_0 \subset B$ such that every $b \in U_0(k)$ has $\#X_b(K) < N$.*

Proof. By localizing on the target, as may assume that f is flat, henced fibered in (arithmetic) genus g curves everywhere.

By assumption, some X_B^n admits a dominant map to a positive-dimensional variety of general type, say W:

$$\varphi\colon X_B^n \to W.$$

By Lang's conjecture, there is a Zariski-dense open subset $U \subset W$ with no rational points, so $\varphi^{-1}(U) =: U_n \subset X_B^n$ has no rational points.

Let $\pi_j : X_B^j \to X_B^{j-1}$ denote the projection to the last factor. We then inductively construct $U_{j-1} = \pi_j(U_j)$ and Z_{j-1} to be the complement of U_j. By our flatness assumption U_{j-1} is open, so Z_{j-1} is closed. Noting that π_j is fibered in curves

$$\begin{array}{ccc} X_B^j & \longrightarrow & X \\ \downarrow{\scriptstyle \pi_j} & & \downarrow{\scriptstyle f} \\ X_B^{j-1} & \longrightarrow & B \end{array}$$

we see that $Z_j \cap \pi_j^{-1}(U_{j-1}) \to U_{j-1}$ is *finite*. Therefore, there exists some d_j such that for $u \in U_{j-1}$ has at most d_j pre-images in Z_j. (We can take d_j to be the sum of the degrees from the irreducible components of Z_j, for example.)

Letting $d = \max_j d_j$, we see that any $u \in U_{j-1}$ has at most d pre-images in Z_j for all j. This is the key property of d, and we claim that we can take $N = d$.

For $b \in U_0$, either $\pi_1^{-1}(b)$ has a rational point in U_1 or it does not. In the latter case, all the rational points lie in $Z_2 \cap \pi_1^{-1}(U_1)$, and there at most d such points by construction, so a fortiori at most d rational points. In the former case, let b_1 be such a rational point. Then

$\pi_2^{-1}(b_1) \cong X_b$:

$$\begin{array}{ccccc} \pi_2^{-1}(b_1) & \longrightarrow & X_B^2 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle f} \\ b_1 & \longrightarrow & X_B & \longrightarrow & B \end{array}$$

If $\pi_2^{-1}(b_1)$ has does not have a rational point in U_2, then it has at most d rational points, by the same reasoning. If it does, let b_2 be such a rational point, and continue the argument. We know that U_n has *no* rational points, so we will win eventually. $\square$

To finish the proof, we basically want a family of curves where every smooth, projective, genus g curve appears as a fiber. Fortunately, such families are supplied by the theory of Hilbert schemes, which provide a *fine moduli space* for the subvarieties of projective space with a given Hilbert polynomial. Since every smooth, projective curve X of genus $g \geq 2$ is embedded by $K_X^{\otimes 2}$ (by an easy exercise with Riemann-Roch), the Hilbert scheme of curves with Hilbert polynomial $2(2g-2)n + 1 - g$ in $\mathbb{P}^{3g-g}$ features all the curves we are interested in.

It is a fact that the locus of pluricanonical smooth curves corresopnds to a locally closed, irreducible, smooth subscheme of the Hilbert scheme. We take B to be the closure of this locus, and X to be the restriction of the universal family to B.

By the proposition, there is a dense subset $U_0 \subset B$ and N_0 such that all fibers of the family have at most N_0 rational points. Let B_1 be the closure of the irreducible components of $B \setminus U_0$ whose generic fiber is smooth. Applying Proposition 2.3 to each component, we obtain a dense open subset $U_1 \subset B_1$ and upper bound N_1 on the number of rational points lying on any rational fiber over U_1. Since $\dim B_i < \dim B_{i-1}$, this process ends at a finite step. (Note that all smooth fibers will be exhausted, since smoothness is open.)

Remark 2.4. In order to produce an *effective* uniform bound from this argument, we would need to have *very* detailed knowledge about the Hilbert schemes in questions, which seems impossible to achieve.

3. Examples of Correlation

3.1. **Pencils of plane curves.** Let $f(x, y)$ and $g(x, y)$ be general polynomials of d. Then the pencil

$$f(x, y) + tg(x, y) = 0$$

has total space $X \subset \mathbb{P}^2 \times \mathbb{P}^1$. (We view this as a family over $B = \mathbb{P}^1$, of course.) Note that X is smooth as long as the base points of the pencil are distinct, which they will be for a general choice. The second projection map $X \to \mathbb{P}^1 =: B$ is fibered in curves of genus $g = \binom{d-1}{2}$, so $g \geq 2$ if $d \geq 4$. The first projection map $X \to \mathbb{P}^2$ presents X as the blowup of $\mathbb{P}^2$ along the d^2 points of intersection of f and g.

(Intuitively, as t varies $f(x_0, y_0) + tg(x_0, y_0)$ vanishes for a unique value of t, unless $g(x_0, y_0) = f(x_0, y_0) = 0$. Moreover, this locus is Cartier in the blowup, cut out by $g(x, y) = 0$. More conceptually, X is the blowup of the linear system generated by f and g in $O_{\mathbb{P}^2}(4)$ at the basepoints, which resolves the morphism $\mathbb{P}^2 \to \mathbb{P}^1$).

In particular, X is birational to $\mathbb{P}^2$, and thus is certainly not of general type. We also could have just calculated using the adjunction formula:

$$\begin{aligned} K_X &\cong \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(-3,-2) \otimes \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(d,1)|_X \\ &\cong \mathcal{O}_X(d-3,-1) \end{aligned}$$

none of whose powers have sections. We remark that it is the negativity of K_B that is problematic here, and we can overcome this by taking the fiber square: $Y = X \times_B X$ is cut out in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ by the equations

$$\begin{aligned} f(x,y) + tg(x,y) &= 0 \\ f(u,v) + tg(u,v) &= 0 \end{aligned}$$

In particular, it is a complete intersection of two hypersurfaces of type $(d,d,1)$, hence Cohen-Macaulay (so there is a dualizing sheaf). By the adjunction formula, the dualizing sheaf is

$$\begin{aligned} \omega_Y &\cong \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^1}(-3,-3,-2) \otimes \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^1}(d,d,2)|_Y \\ &\cong \mathcal{O}_Y(d-3,d-3,0). \end{aligned}$$

The space of sections now grows like a cubic if $d \geq 4$, i.e. $g \geq 2$, as can be witnessed from the fact that this is basically projection away from the t coordinate.

♠♠♠ TONY: [can one see this directly using the Hilbert polynomial?]

You can see that this fails precisely when $d \leq 3$.

Actually, we are not quite done! The problem is that Y may be singular. So we have to show that there is a desingularization $\widetilde{Y} \to Y$ such that $\widetilde{Y}$ is of general type.

In this case, we are lucky because the singularities of Y are very mild. First note that the singular points of Y "come from" the singular fibers of $X \to B$. Indeed, if the fibers are smooth near some $x \in X$ then (as the family is flat) $X \to B$ is smooth near x, hence $X_B \to X$ is smooth (being the pullback of a smooth morphism).

In a general pencil, the singular fibers will have nodes at worst, and hence the singularities look étale locally (or "analytically locally," if you prefer) like $xy - t$. The base change then looks étale locally like

$$k[x,y,u,v,t]/(xy-t, uv-t) \cong k[x,y,u,v]/(xy-uv).$$

Therefore, the singularities of $Y = X_B^2$ are double points in a smooth fourfold Z, which can resolved after blowing up Z and then taking the proper transform of Y.

We have $K_{\widetilde{Z}} = \pi^*(K_Z)(3E)$ since we blew up a smooth fourfold - in local coordinates $(x,y,z,w) \mapsto (x', y'x', z'x', w'x')$ so

$$dx \wedge dy \wedge dz \wedge dw \mapsto (x')^3 dx' \wedge dy' \wedge dz' \wedge dw'.$$

The divisor of $\widetilde{Y}$ in $\widetilde{Z}$ is the pullback of the divisor of Y minus *twice* the exceptional divisor, since Y had a double point:

$$\mathcal{O}_{\widetilde{Z}}(\widetilde{Y}) \cong \pi^*\mathcal{O}_Z(Y)(-2E).$$

Therefore, by the adjunction formula we have

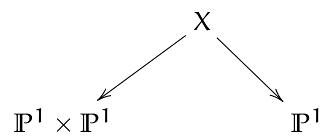
$$K_{\widetilde{Y}} \cong \pi^*(K_Y)(E).$$

This shows that holomorphic differentials pull back to holomorphic differentials, so the space of sections of the pluricanonical bundles of $K_{\widetilde{Y}}$ still grow cubically.

3.2. **Isotrivial families.** We now consider an *isotrivial* family which in some sense is opposite to the example we have just considered. Let $f(x)$ be a polynomial of degree $2g+2$ with no repeated roots. Consider the pencil of smooth genus g curves

$$ty^2 = f(x).$$

These curves are all isomorphic over $\overline{\mathbb{Q}}$ (thus "isotrivial") but not over K. Viewing this as a pencil of curves of bidegree $(2, 2g+2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, we see that the total space of the family is a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ along the basepoints, and is therefore rational.



What about the fiber square? $Y := X_B^2$ is cut out in $(\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{P}^1$ by the equations $ty^2 = f(x), tv^2 = f(u)$, i.e. equations of bidegree $(2, 2g+2, 0, 0, 1)$ and $(0, 0, 2, 2g+2, 1)$. Therefore, by adjunction we have

$$K_Y \cong \mathcal{O}_Y(0, 2g, 0, 2g, 0).$$

This basically crushes everything except the x and u coordinates. Specifying x and u, we see that there is still (generally) a 1-parameter family of solutions for t and y, so the Kodaira dimension of Y is only 2. Moreover, this *won't* be corrected by taking further powers over B.

However, Y is birational to

$$\{(x, y, u, v) : v^2 f(x) = y^2 f(u)\} \subset \mathbb{A}^4$$

which admits a dominant rational map to the surface

$$W = \{(x, u, w) : w^2 = f(u) f(x)\}$$

sending $(x, u, y, v) \mapsto (x, u, vf(x)/y)$. By the way, this surface can be viewed as a quotient of C^2 by the group of hyperelliptic involutions ($v \mapsto -v$ and $y \mapsto -y$), a fact which foreshadows the general situation!

Now, this surface is birationally a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ (via the u, x coordinates) ramified over the locus $f(x)f(u) = 0$, i.e. the union of $2g+2$ lines from each ruling. By Riemann-Hurwitz, the canonical divisor of Y is the pullback of the canonical divisor $\mathcal{O}(-2, -2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ twisted by the divisor of w. As $\mathrm{Div}(w^2) = \mathrm{Div}\, f(x)\, \mathrm{Div}\, f(u)$, we have

$$\begin{aligned} K_Y &= \pi^*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2) \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(g+1, g+1)) \\ &= \pi^*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(g-1, g-1)) \end{aligned}$$

Now this is ample (again notice the necessity of $g \geq 2$!).

By the way, this can all be done completely explicitly. Let $\theta = \frac{du \wedge dx}{w}$. Then you can check that

$$\theta_{k,l} := u^k x^\ell \theta$$

extend to regular differentials for $0 \leq k, \ell \leq g-1$. (This is very similar the computation of the holomorphic differentials on a hyperelliptic curve.) They define a finite (in fact, degree 2) map to projective space (with image the g-uple Veronese embedding of $\mathbb{P}^2$).

What about the singular points? Étale locally these have the form $w^2 = ux$, and the double points are desingularized after blowing up once. In the local coordinates of the blowup

$$w' = w, \quad u' = \frac{u}{w}, \quad x' = \frac{x}{w}$$

we see that θ pulls back to

$$\begin{aligned}\widetilde{\theta} &= \frac{d(u'w') \wedge d(x'w')}{w'} \\ &= w'(du' \wedge dx') + x'(du' \wedge dw') + u'(dw' \wedge dx')\end{aligned}$$

which is regular.

3.3. **Some words on the proof in general.** The general proof of the Correlation Theorem proceeds by separately studying the case of *maximal variation of moduli* - essentially, that the corresponding map to $\mathcal{M}_g$ (which may only after base change) is *finite* and the case of isotrivial families. These are clearly opposite ends of the spectrum, and the proof works inwards from them.

In the case where the moduli varies maximally, it turns out that X_B^n itself becomes general type. Morally, think of this as follows. The canonical bundle of X_B^n is (morally) composed of two pieces: the canonical bundle of X, and the canonical bundle of B. Now, general type is similar to a positivity condition on the canonical bundle (it's slightly weaker than ampleness, but it says that it has a lot of sections). The problem is that the canonical bundle of B may be quite "negative." But if we amplify the positive of K_X enough, then it should overcome that.

In the case of isotrivial families, it turns out that X_B^n will map to C^n modulo a finite group of automorphisms (necessary because we had to make a base change to get a trivial family). A quotient of a curve of general type may not be general type, but a higher enough power will have this property.

A key technical point, which we have witnessed in these examples, is that the singularities of the fibers squares are *canonical*. Essentially, their resolution does not kills the regularity of global canonical differentials.

References

[1] L. Caporaso, J. Harris, and B. Mazur. *Uniformity of Rational Points*. J. A. M. S. 10, No. 1 (1997), p. 1-35.