Discussion Session on p-divisible Groups

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These are notes from a discussion session of p-divisible groups. Some questions were posed by Dennis Gaitsgory, and then their answers were discussed by Jared Weinstein.

1 Questions

1.1 Question 1

We have a p-divisible group over $\mathcal{O}_{\mathbb{C}_p}$. There was a "universal cover" $\widetilde{\mathcal{G}}$. What is this? Also, please explain the short exact sequence

$$0 \to T_p(\widetilde{\mathcal{G}}^{\mathrm{an}})[1/p] \to \widetilde{\mathcal{G}}^{\mathrm{an}} \to \mathrm{Lie}(G) \otimes \mathbb{G}_a \to 0.$$

1.2 Question 2

How do you associate isocrystals to p-divisible groups? What is the period map?

1.3 Question 3

How do you describe modifications of bundles on X in terms of p-divisible groups?

1.4 Question 4

What is the analogue of this stuff in equal characteristic?

2 Discussion of Question 1

Let me first write down the sequence without taking universal covers. The exact sequence is basically coming from the logarithm:

$$0 \to G[p^\infty](\mathcal{O}_C) \to G(\mathcal{O}_C) \xrightarrow{\log} \mathrm{Lie}(G) \otimes C \to 0. \tag{1}$$

2.1 Basics on p-divisible groups

There are often implicit identifications made in talking about p-divisible groups. If G a p-divisible group, then it is "represented" by a formal scheme which usually is also denoted G. What do we mean by this? (What is the associated formal scheme?) By definition a p-divisible group G is an inductive system

$$G = \varinjlim G_n$$

where G_n are group schemes. We could view G as a sheaf on the category Nilp_p of rings in which p is nilpotent. Then it turns out to be representable by a formal scheme.

Example 2.1. For the p-divisible group

$$\mu_{p^\infty} = \varprojlim \mu_{p^n}$$

the formal scheme is $\widehat{\mathbb{G}}_m$. In general it can be difficult to describe.

So if G is a p-divisible group, we denote by $G(\mathcal{O}_C)$ the points of the formal scheme. This is a $\mathbb{Z}_p$-module. We have a logarithm map

$$G(\mathcal{O}_C) \xrightarrow{\log} \mathrm{Lie}\, G.$$

Before discussing what this is technically, we give some examples.

Example 2.2. For $G = \mathbb{Q}_p/\mathbb{Z}_p$ the constant group scheme, this map is

$$\mathbb{Q}_p/\mathbb{Z}_p \to 0$$

so in this case the sequence (1) is

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0 \to 0.$$

Example 2.3. For $G = \mu_{p^\infty}$, we have $G(\mathcal{O}_C) = 1 + \mathfrak{m}_C$ (considered as a multiplicative group). Why? You might think at first that $G(\mathcal{O}_C)$ should be the p-power roots of unity, but we cannot evaluate it directly on $\mathcal{O}_C$ because $\mathcal{O}_C$ is not an object of Nilp_p.

Instead, we have to view $\mathcal{O}_C$ as an inverse limit of $\mathcal{O}_C/p^n$. Then

$$\mu_{p^\infty}(\mathcal{O}_C) = \varprojlim_n \mu_{p^\infty}(\mathcal{O}_C/p^n)$$

Now on $\mathcal{O}_C/p^n$, which really is an object of Nilp_p, we can apply the definition of μ_{p^∞} literally. But in this ring there will be many p-power roots of unity - anything close to 1 works. So

$$\mu_{p^\infty}(\mathcal{O}_C) = \varprojlim_n \mu_{p^\infty}(\mathcal{O}_C/p^n) = \varprojlim_n 1 + \mathfrak{m}_C/p^n = 1 + \mathfrak{m}_C.$$

Alright, let's finally start building the short exact sequence. Again, the right map is the p-adic logarithm:

$$1 + \mathfrak{m}_C \xrightarrow{\log} C$$

The kernel of the logarithm is the torsion subgroup, so in this case (1) is

$$0 \to (1 + \mathfrak{m}_{C_p})[p^\infty] \to 1 + \mathfrak{m}_C \xrightarrow{\log_p} C \to 0.$$

The two examples just discussed, $\mathbb{Q}_p/\mathbb{Z}_p$ and μ_{p^∞}, can be thought of as the "building blocks" of p-divisible groups. Everything else "looks like" a mix between them. For instance, if G is a general p-divisible group then $G(\mathcal{O}_C)$ will look like a product of disks (as in $1 + \mathfrak{m}_{\mathcal{O}_{C_p}}$ for μ_{p^∞}) times a product of $\mathbb{Q}_p/\mathbb{Z}_p$ factors.

2.2 Analytification

Let's now return to the discussion of the short exact sequence

$$0 \to G[p^\infty](\mathcal{O}_C) \to G(\mathcal{O}_C) \to \mathrm{Lie}(G) \otimes C \to 0.$$

We now construct the analogous sequence at the level of analytic spaces. To a p-divisible group G there is an associated adic space G^{an} over C. The construction passes through the associated formal scheme over $\mathcal{O}_C$. We know that there is a fully faithful embedding from formal schemes over $\mathcal{O}_C$ to adic spaces over $\mathrm{Spa}(C, \mathcal{O}_C)$; then we form the generic fiber over $\mathrm{Spa}(C)$. (This was denoted G_η^{ad} in Arthur's talk.)

The claim is that there is a short exact sequence

$$0 \to G[p^\infty] \to G^{\mathrm{an}} \to \mathrm{Lie}(G) \otimes \mathbb{G}_a^{\mathrm{an}} \to 0.$$

What is $\mathbb{G}_a^{\mathrm{an}}$ and why did we tensor with it? We tensored with it because we want an exact sequence of objects in the category of adic spaces, so we have to turn the vector space $\mathrm{Lie}(G)$ into an "adic vector space". (We are viewing $G[p^\infty]$ as a discrete adic space. Strictly speaking, maybe we should underline it) Now, $\mathbb{G}_a$ is what you expect in terms of the functor of points:

$$\mathbb{G}_a^{\mathrm{an}}(R, R^+) = R.$$

However, it is slightly subtle to present this as an adic space. This is *not* represented by an affinoid adic space. It is something like the whole affine line, which is not quasi-compact as an analytic space: it should be presented as a rising union of infinitely many disks of increasing radius.

Now what is the logarithm actually? Let's go to the very basics. If G is a formal group of dimension 1, then by definition there is some power series

$$X +_G Y = X + Y + \dots.$$

What's the Lie algebra? You just choose a coordinate X, and the addition law is given by this power series. Multiplication by p should be finite, so we should have

$$[p]_G(X) = uX^{p^h} + \dots \quad u \in \mathcal{O}_C^*.$$

Now we need to give a map

$$G(\mathcal{O}_C) \to C.$$

As a set $G(\mathcal{O}_C)$ is $\mathfrak{m}_C$, but with group law given by that power series. So

$$\log_G(X) = \lim_{n\to\infty} \frac{[p^n]_G(X)}{p^n}.$$

Exercise 2.4. Do this explicitly for $\widehat{\mathbb{G}}_m$.

Why is this valued in Lie(G)? The Lie algebra is dual to differentials. So if ω is an invariant differential on G, there should be a natural way to evaluate

$$\langle \log(x), \omega \rangle = \text{``} \int_0^x \omega \text{''} \in C.$$

What does this mean precisely? We can write the Kähler differential as $\omega = df$, and there is a unique normalization of f so that $f(0) = 0$. We set

$$\int_0^x \omega := f(x)$$

for this f.

2.3 Passing to the universal cover

Now, what is the universal cover? We could describe as a formal scheme whose functor of points is

$$\widetilde{G}(R) = \varprojlim_p G(R).$$

This should be a $\mathbb{Q}_p$-vector space. Indeed, applying $\varprojlim_p$ to any $\mathbb{Z}_p$-module gives a $\mathbb{Q}_p$-vector space.

Example 2.5. If $G = \mathbb{Q}_p/Z_p$ then $\widetilde{G} = \mathbb{Q}_p$.

Example 2.6. If $G = \mu_{p^\infty}$ then

$$\widetilde{G}(R) = \varprojlim 1 + R^{00}$$

where R^{00} is the ideal of topologically nilpotent elements. Why? Again we need to express R as a limit of rings in Nilp_p in order to compute:

$$\mu_{p^\infty}(R) = \varprojlim_n \mu_{p^\infty}(R/p^n) = \varprojlim_n 1 + R^{00}/p^n$$

since at the finite levels anything topologically nilpotent is a p^nth root of unity.

Note that this limit looks like $R^\flat$ if R is perfectoid. In fact, recall that there are two parallel constructions of the tilt: one in characteristic 0, and one after modding out by p. Indeed we have here

$$\begin{aligned}\widetilde{G}(R) &= \mu_{p^\infty}(R) \\ &= \varprojlim_n 1 + R^{00}/p^n \\ &= \varprojlim_{x\mapsto x^p} 1 + R^{00}/p \\ &= \widetilde{G}(R/p)\end{aligned}$$

The preceding example reflects the general phenomenon that

$$\widetilde{G}(R) \to \widetilde{G}(R/p)$$

is always an isomorphism. We might say that $\widetilde{G}$ is a "crystalline" construction because it is insensitive to infinitesimal extensions.

Now what about the exact sequence? There is a map

$$\begin{array}{c}\widetilde{G}(O_C) \\ \downarrow \\ G(O_C)\end{array}$$

which is projection onto the 0th coordinate. Let's compare the logarithm maps for G and its analytification.

$$\begin{array}{ccccccccc} & & \widetilde{G}(O_C) & \longrightarrow & \mathrm{Lie}(G)\otimes C & \longrightarrow & 0 & & \\ & & \downarrow & & \| & & & & \\ 0 & \longrightarrow & G[p^\infty](O_C) & \longrightarrow & G(O_C) & \longrightarrow & \mathrm{Lie}(G)\otimes C & \longrightarrow & 0 \end{array}$$

The map $\widetilde{G}(O_C) \to \mathrm{Lie}(G)\otimes C$ can therefore be thought of as the composition with projection and the logarithm map for $G(O_C)$. In particular, the kernel consists of elements whose 0th part is killed by the classical logarithm, i.e. consists of elements whose 0th part is torsion.

$$\begin{array}{ccccccccc} & & T_p(G) & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & T_pG\otimes_{\mathbb{Z}_p}\mathbb{Q}_p & \longrightarrow & \widetilde{G}(O_C) & \longrightarrow & \mathrm{Lie}(G)\otimes C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & G[p^\infty](O_C) & \longrightarrow & G(O_C) & \longrightarrow & \mathrm{Lie}(G)\otimes C & \longrightarrow & 0 \end{array}$$

Theorem 2.7. *$\widetilde{G}$ is a perfectoid space.*

Example 2.8. For $G = \mu_{p^\infty}$, what is $\widetilde{G}$ as a perfectoid space? It turns out to be the "perfectoid open ball of radius 1" (since these are precisely the topologically nilpotent elements for multiplication). This is easiest to describe at the level of points:

$$\widetilde{\mu}_{p^\infty}^{\mathrm{an}}(R) = \varprojlim_{x \mapsto x^p} 1 + R^{00}.$$

How do we describe this as a perfectoid space in terms of affinoid charts? Again, the description is a little complicated: it is certainly not affinoid since the open ball is not quasicompact. We can exhaust it from inside by closed balls.

Note that $\mathrm{Spa}(C\langle x^{1/p^\infty}\rangle)$ is a "perfectoid closed ball of radius 1". First of all, what does $C\langle x^{1/p^\infty}\rangle$ even mean? $C\langle X\rangle$ is the Tate algebra, with elements being convergent power series. Then $C\langle x^{1/p^\infty}\rangle$ is obtained by adjoining all p-power roots of X and completing. Now to exhaust the open ball from within, we need to taking a rising union of rescaled closed disks $|X| \le |p^\epsilon|$ as $\epsilon \to 0$:

$$\varinjlim_{\epsilon \to 0} \mathrm{Spa}(C\langle \left(\frac{X}{p^\epsilon}\right)^{1/p^\infty} \rangle).$$

Perhaps a slicker way to describe this is as $(\mathrm{Spf}\, \mathcal{O}_C[[X^{1/p^\infty}]])^{\mathrm{an}}$.

3 Discussion of Question 2

3.1 Dieudonné modules

Let k be a perfect field of characteristic p. There is an equivalence of categories

$$M\colon \{p\text{-div groups}/k\} \xrightarrow{\sim} \{\text{Dieudonné modules}/W(k)\}. \tag{2}$$

What are Dieudonné modules?

Definition 3.1. A *Dieudonné module* is a finite free $W(k)$-modules M, together with maps

$$F, V\colon M \to M$$

where F is σ-linear and V is σ^{-1}-linear and $FV = p$.

Now, what we actually discussed were not Dieudonné modules but *isocrystals*, which looked similar but were defined over fields. We can get that from a Dieudonné module by inverting p. But then what happens to the equivalence (2)?

On the left, we get p-divisible groups up to isogeny. On the right, we don't need to specify the V because it is determined by F once p is invertible, but there is still a condition because V had to preserve a lattice. So the right side becomes the category of isocrystals over k (which by definition modules over $W(k)[1/p]$) with slopes in $[0, 1]$.

$$\{p\text{-div groups}/k\}/\text{isogeny} \xrightarrow{\sim} \{\text{isocrystals}/k \text{ with slopes in } [0,1]\}.$$

Example 3.2. $M(\mathbb{Q}_p/\mathbb{Z}_p) = W(k)$ with $F = p\sigma$.

Example 3.3. $M(\mu_{p^\infty}) = W(k)$ with $F = \sigma$.

In general, we have

$$\begin{aligned} \operatorname{ht}(G) &= \operatorname{rank}\, M(G) \\ \dim G &= \dim_k M(G)/VM(G) \end{aligned}$$

Note that the latter is a module over $W(k)/p$ since V divides p.

So what is this equivalence M? Given G/k, lift to $G'/W(k)$ arbitrarily. Then it is a fact G' has a *universal vector extension*. What does this mean? A vector extension is an extension of G' by a sheaf of $W(k)$-algebras isomorphic to $\mathbb{G}_a^n$.

$$0 \to V \cong \mathbb{G}_a^n \to EG' \to G' \to 0.$$

They form a category with morphisms required to be linear over $W(k)$ on the vector parts; the universal vector extension is the initial object. This turns out not to depend on G'. The reason is basically that the difference between different lifts is divisible by p, so the logarithm converges.

A good analogy to keep in mind is the following. Given a curve, its Picard scheme depends on the complex structure, but the universal vector extension is the stack of local systems on the curve, which is *independent of complex structure*. Then the Dieudonné module is

$$M(G) := \operatorname{Lie} EG'.$$

Why is this actually a Dieduonné module? Our original G has a Frobenius morphism

$$F\colon G \to G^{(p)}$$

inducing

$$V\colon M(G) \to M(G^{(p)}) = M(G) \otimes_{W(k),\sigma} W(k).$$

because of our conventions (note that this is σ^{-1}-linear). Since $F\colon G \to G^{(p)}$ divides p, $V\colon M(G) \to M(G^{(p)})$ also divides p, so we can define F as well.

Remark 3.4. There is also a contravariant version of the Dieudonné module in which "F actually induces F".

3.2 The period map

Now what's the period map? It's usually attributed to Gross-Hopkins or Grothendieck-Messing. Fix a p-divisible group G. The target of the period morphism is $\operatorname{Gr}(d, M(G))^{\mathrm{an}}$ (d-dimensional quotients of the Dieudonné module) where $d = \dim G$ and $n = \operatorname{ht} G$. The source is a deformation space for p-divisible groups, denoted $\mathcal{M}_G$. This is an adic space.

What are its points? Roughly speaking, an (R, R^+)-point is a deformation of G to R^+. (Here (R, R^+) is an affinoid algebra over $(W(k)[1/p], W(k))$.)

$$\mathcal{M}_G(R, R^+) \approx \left\{ \begin{matrix} G'/R = \ p\text{-div group} \\ \iota = \text{quasi-isogeny}\colon G \otimes_k R^+/p \to G' \otimes_{R^+} R^+/p \end{matrix} \right\}$$

(Caveat: we have to sheafify, and this only applies to bounded rings. If R is not bounded, then we need to first express it as a rising union of bounded subrings.)

Now we can finally define the period morphism. Given (G', ι) we have

$$0 \to V \to EG' \to G' \to 0.$$

We have a rigidification

$$\operatorname{Lie} EG'[1/p] \overset{\iota}{\cong} M(G) \otimes_{W(k)} R.$$

This map is induced by the "crystalline property" of the Dieudonné module because ι is only defined modulo p. Then the map $\operatorname{Lie} EG'[1/p] \to \operatorname{Lie} G'[1/p]$ defines a point in the classifying space of d-dimensional quotients of $M(G)$, which is $\operatorname{Gr}(d, M(G))$.

4 Discussion of Question 3

Let $G/\mathcal{O}_C$ be a p-divisible group. Then we have an exact sequence

$$0 \to V \to EG \to G \to 0.$$

The universal cover fits as

$$\begin{array}{ccccccccc} & & & & & & \widetilde{G} & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & EG & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

In fact the covering map factors through EG.

$$\begin{array}{ccccccccc} & & & & & & \widetilde{G} & & \\ & & & & & \swarrow & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & EG & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

Why? Informally speaking, $\widetilde{G} = \{(x_0, x_1, \ldots)\}$ so

$$\lim_{i \to \infty} p^i \widetilde{x_i} \in EG$$

defines a lift. The point is that choices were made in lifting x_i to $\widetilde{x_i}$ but they will be killed in the limit, because the ambiguity is measured by V and this is multiplied by higher and higher powers of p.

So we get a map

$$\widetilde{G}(\mathcal{O}_C) \to EG(\mathcal{O}_C) \xrightarrow{\log_{EG}} \operatorname{Lie} EG \otimes C.$$

The composition is called the *quasi-logarithm* $q\log_G$. We can geometrize it to a morphism

$$q\log_G \colon \widetilde{G}^{\mathrm{an}} \to M(G) \otimes \mathbb{G}_a.$$

Both source and target only depend on G modulo p; it is a theorem that the map itself also only depends on G modulo p.

Theorem 4.1. *Let (R, R^+) be a perfectoid C-algebra. There exists an isomorphism*

$$\widetilde{G}(R^+) \to (M(G) \otimes B^+_{\mathrm{cris}}(R^+/p))^{\varphi=1} = H^0(X_{(R,R^+)}, \mathcal{E}_{M(G)}).$$

Proof sketch. We have

$$\widetilde{G}(R^+) = \widetilde{G}(R^+/p) = \operatorname{Hom}_{R^+/p}(\underline{\mathbb{Q}_p}, G).$$

Since we're over R^+/p we know that some power of p dies in G, so this is a *forward* limit

$$\lim_{n\to\infty} \operatorname{Hom}(\mathbb{Q}_p/p^n\mathbb{Z}_p, G) = \operatorname{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G)[1/p].$$

Now passing to Dieudonné modules, we conclude that

$$\widetilde{G}(R^+) = \operatorname{Hom}_R(\underline{M}(\mathbb{Q}_p/\mathbb{Z}_p), \underline{M}(G))$$

in the category of Dieudonné crystals. This means that whenever you have a PD thickening of R^+/p, we can evaluate this on that thickening. We choose to evaluate it on $A_{\mathrm{cris}}(R^+/p) \to R^+/p$, which is the *universal* PD thickening. Then we get

$$\widetilde{G}(R^+) = \operatorname{Hom}(B^+_{\mathrm{cris}}, M(G) \otimes B^+_{\mathrm{cris}})$$

with the crystal structure on B^+_{cris} being $V = 1$, so

$$\widetilde{G}(R^+) = \operatorname{Hom}(B^+_{\mathrm{cris}}, M(G) \otimes B^+_{\mathrm{cris}}) = (M(G) \otimes B^+_{\mathrm{cris}})^{V=1,\text{ i.e. } F=p}$$

as desired. □

Go back to the exact sequence

$$0 \to T_p(G) \otimes \mathbb{Q}_p \to \widetilde{G} \to \operatorname{Lie}(G) \otimes \mathbb{G}_a \to 0.$$

We now see how to interpret the (R, R^+)-points of the middle term as global sections of a vector bundle $\mathcal{E}_{M(G)}$. Geometrizing to the actual vector bundles, we can interpret our sequence as a modification

$$0 \to T_p(G) \otimes \mathcal{O}_X \to \mathcal{E}_{M(G)} \to ? \to 0$$

Call $i\colon \infty \to X$ the inclusion of the point with residue field C. We can view $\mathrm{Lie}(G) \otimes C$ as $i_* \mathrm{Lie}(G) \otimes C$, getting

$$0 \to T_p(G) \otimes \mathcal{O}_X \to \mathcal{E}_{M(G)} \to i_* \mathrm{Lie}(G) \otimes C \to 0$$

The theorem also goes in the other direction: given any modification with trivial kernel, there is a corresponding p-divisible over $\mathcal{O}_C$ which induces it. Thus, there is a bijection

$$\{p\text{-divisible groups}/\mathcal{O}_C\}/\text{isogeny} \cong \{\text{modifications } 0 \to T \to \mathcal{E} \to i_\infty W \to 0\}$$

where T is trivial and W is miniscule, which means that it is a module over B_{dR}^+/t (i.e. killed by the uniformizer).