HARVARD UNIVERSITY

SENIOR HONORS THESIS

Hodge-Tate Theory

$$(B_{\mathrm{HT}} \otimes H^n_{\textbf{ét}}(X_{\overline{K}}, \mathbf{Q}_p))^{G_K} \simeq \bigoplus_{p+q=n} H^p(X, \Omega^q)$$

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[In response to "Poetry is the art of giving different names to the same thing."]
"Mathematics is the art of giving the same name to different things."

Henri Poincaré

ABSTRACT

This thesis aims to expose the amazing sequence of ideas, concerning p-adic representations coming from geometry, that form the heart of what was called Hodge-Tate theory. This subject, initiated by Tate in the late '60s in analogy to classical Hodge theory, leads in to the now vast and highly fruitful program of p-adic Hodge Theory.

The central result of the theory is the Hodge-Tate decomposition for abelian varieties, which gives a comparison isomorphism relating the étale cohomology and the de Rham cohomology. The original proofs are by Tate's seminal analysis of p-divisible groups ([Tat67]) in the case of good reduction and Raynaud's generalization to bad reduction using the semistable reduction theorem ([Gro72]). Here we present two different approaches, with the goal of accessibility. The first, due to Fontaine, is an elegant proof in full generality without the deep algebraic geometry machinery from SGA 7. The second proof, by Coleman, is for the case of good reduction, and falls out of a more explicit analysis of the geometry of abelian varieties.

To build up to this result, we will develop requisite theory on the Galois cohomology and ramification theory of p-adic fields. After proving the decomposition, we discuss how it fits into Fontaine's general formalism of admissible representations and period rings, and undertake the construction of B_{dR} and the theory of de Rham representations. Finally, we give examples of computing periods for elliptic curves.

Acknowledgements

When I walked onto Harvard Yard less than four years ago, I could not have written a single page of this thesis. The support of many friends, the encouragement of many mentors, and the knowledge of many teachers has made this project possible. There are too many people to thank in the little space that I have here, so I am faced with the painful task of choosing a few.

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Learning mathematics is a challenging endeavor, and I could not have made this journey without companions. Levent Alpoge and Akhil Mathew deserve special mentions for proofreading a draft of this document and making many helpful comments and corrections.

When I was eleven years old, my father told me about $i = \sqrt{-1}$ and Euler's magical formula $e^{\pi i} = -1$, and gave me puzzles like "What is i^i?" I would like to say that this was how I fell in love with mathematics, but the truth is that I could barely divide numbers, and I had no clue what he was talking about. I remember, with comic nostalgia, the disappointment in his face as he watched my eyes glaze over at these then-impenetrable ideas. Everything that I have accomplished in the years since has only been possible because of my parents' unwavering support and infinite patience. I cannot imagine more loving parents. Thank you, mom and dad: these pages are for you.

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To my parents.

Chapter 1

Introduction

1.1 Classical and p-adic Hodge Theory

Classical Hodge theory has its roots in the work of mathematicians of the early 1800s, who studied the problem of computing integrals of the form

$$\int \frac{dx}{\sqrt{(x-z_1)\dots(x-z_n)}}$$

over contours in the complex plane ([PS08]). Such integrals of algebraic functions are called *period integrals*. When $n = 1$ or 2, they are easy to calculate through methods taught in any course on one-variable calculus. The case $n = 1$ is trivial and the case $n = 2$ boils down to an understanding of the standard periodic functions $\sin x$ and $\cos x$. For $n \geq 3$, however, the problem becomes much more subtle. Riemann realized that these expressions are naturally studied by passing to more exotic geometric structures, now called *Riemann surfaces*. Using this line of thinking, mathematicians translated the problem of computing period integrals into that of studying the geometry of complex manifolds, which gave birth to the field of Hodge theory.

A major early achievement in Hodge theory is the *Hodge decomposition* for a compact Kähler manifold. This result gives a comparison between the ordinary singular cohomology of such a manifold and its de Rham cohomology. If $\Omega^{p,q}$ denotes the sheaf of differential forms of type (p, q), then the theorem says:

Theorem 1.1. *If X is a compact Kähler manifold, then*

$$H^n(X, \mathbf{Z}) \otimes \mathbf{C} \simeq \bigoplus_{p+q=n} H^p(X, \Omega^q).$$

In the last half century, mathematicians began to notice amazing parallels between properties of classical complex manifolds and those of “p-adic projective varieties.” In the late 1960s, John Tate observed a similar comparison theorem for the étale cohomology of an abelian variety with good reduction over a local field K ([Tat67]), which we may phrase as:

$$(H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes B_{\text{HT}})^{G_K} \simeq \bigoplus_{p+q=n} H^p(X, \Omega^q).$$

The meaning of this statement will be explained over the course of this thesis, but for now we merely draw attention to its similarity to Theorem 1.1.

Tate went on to conjecture that such a comparison should hold more generally, for any smooth projective variety over a local field. This question has inspired a sequence of striking discoveries, mirroring the developments of clasical Hodge theory, in a rich and highly successful field now called p-adic Hodge theory. The “Hodge-Tate” decomposition for abelian varieties, the first piece of this wonderful analogy, forms the heart of our story.

1.2 Galois Representations and Number Theory

The p-adic Hodge theory provides a framework for analyzing and understanding p-adic Galois representations, and we take a brief digression to remark upon the essential role that Galois representations have played in modern number theory.

Experience has shown that many of the deepest questions in number theory are bound with understanding the *absolute Galois group* $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. From the Langlands program to the modularity of elliptic curves, the notion of *Galois representations*, or linear actions of Galois groups on vector spaces, has proved to be a fundamental framework for understanding the arithmetic of number-theoretic objects.

Above all, the richest Galois representations are those that “come from geometry.” For instance, let E be an elliptic curve defined over $\mathbf{Q}$. It is well-known that the m-torsion points of $E(\mathbb{C})$ form a group, which we denote by $E[m]$, isomorphic to $(\mathbf{Z}/m\mathbf{Z})^2$. The Galois group $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ commutes with the group operations of E and thus acts on $E[m]$, furnishing a Galois representation

$$\rho_m : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \text{GL}_2(\mathbf{Z}/m\mathbf{Z}).$$

As m ranges over the integral powers of a prime ℓ, the ℓ-power torsion $E[\ell^n]$ form a compatible system of groups, giving rise to the *ℓ-adic Tate module* $T_\ell(E)$, and a corresponding Galois representation

$$\rho_{\ell^\infty} : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \text{GL}_2(\mathbf{Z}_\ell).$$

These Galois representations encode important arithmetic information about E. For instance, the theorem of Néron-Ogg-Shafarevich tells us that ρ_{ℓ^∞} is unramified at $p \neq \ell$ (in other words, the prime p is unramified in the Galois extension corresponding to ρ_{ℓ^∞}) if and only if E has "good reduction" at p. For these primes of good reduction, we may define a Frobenius element $\mathrm{Frob}_p \in \rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}))$ up to conjugacy. Representation theory leads us to consider the *character* of ρ_{ℓ^∞}, and we define $a_p := \mathrm{Tr}\,\mathrm{Frob}_p$. The *local Euler factor* at p is then

$$L_p(s, E) = (1 - a_p p^{-s} + p^{1-2s})^{-1}.$$

The famous Birch and Swinnerton-Dyer conjecture predicts that the global L-function obtained by multiplying together all of the local factors $L_p(s, E)$ captures the key arithmetic invariants of E, including its rank.

As seen in the preceding example, there is an important principle in mathematics that "global" objects, such as the Galois groups of global fields, are fruitfully studied through "local" objects, such as the Galois groups of local fields. Fixing an embedding of algebraic closures $\overline{\mathbf{Q}} \subset \overline{\mathbf{Q}_p}$ gives an inclusion $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p) \subset \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. We may then analyze the latter object by separately considering the representations of the $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$, and then piecing together the results. This approach has seen remarkable success. As an early demonstration of its power, Serre used the idea of examining compatible systems of ℓ-adic representations to determine the Lie algebras of the Galois representations attached to an elliptic curve without complex multiplication. Later, Faltings employed p-adic Hodge theory to prove the Mordell Conjecture. As another application, Andrew Wiles' demonstration that all Galois representations of elliptic curves are "modular," which led to the resolution of Fermat's Last Theorem, depends critically on this local theory.

1.3 Comparison Isomorphisms

In recent years, there has been much work towards classifying and codifying the Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ that "come from geometry," culminating in Faltings' proof of a Hodge decomposition for smooth projective varieties and Kisin's breakthroughs on the Fontaine-Mazur conjecture. We focus on the story of Hodge-Tate representations, the first step in this vast program, which concern representations of absolute Galois groups of p-adic fields.

The most natural Galois representations coming from geometric objects are obtained from their cohomology groups. Let X be a variety defined over a p-adic field K.

Definition 1.2. A **p-adic field** is a field of characteristic 0 that is complete with respect to a discrete valuation, such that the residue field of its valuation ring is perfect of characteristic $p > 0$.

Then X has an action by the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$, which induces endomorphisms of its cohomology groups by functoriality. As an example, when X is an abelian variety, the étale cohomology group $H^1_{\text{ét}}(X, \mathbf{Z}_p)$ is dual to the Tate module of X, and we recover the Galois representation on torsion points discussed in the previous section for the special case of elliptic curves.

Tate's main result in [Tat67] is the following.

Theorem 1.3. *Let X be an abelian variety with good reduction over a local field K. Then there are canonical G_K-equivariant isomorphisms.*

$$(\mathbf{C}_p(j) \otimes_{\mathbf{Q}_p} H^i_{\acute{e}t}(X_{\overline{K}}, \mathbf{Q}_p))^{G_K} = \begin{cases} 0 & j < 0 \text{ or } j > i \\ H^{i-j}(X, \Omega^j_X) & \text{otherwise} \end{cases}.$$

The relation between étale cohomology and de Rham cohomology expressed in the theorem is called a *comparison isomorphism.* The decomposition involved in the comparison isomorphism suggests a general notion of "Hodge-Tate" representation, which we will later explain precisely.

There are many refinements and generalizations of this comparison isomorphism. The geometric condition of semistability of an abelian variety may be expressed through its Galois representation, which leads to the notion of *de Rham* representations. The notion of *crystalline representation* is inspired by considering the crystalline cohomology instead of the étale cohomology. Recently, comparison isomorphisms have been developed for rigid-analytic varieties, so they still constitute an active and exciting field of research.

1.4 Structure of the Paper

The thesis is organized as follows. In Part I, we establish results on the arithmetic of p-adic fields that will be used in the proof of the comparison isomorphism. Specifically, in §2 we compute the Galois cohomology of $\mathbf{C}_p$ and its "Tate twists," and explain some applications. In §3, we explore the ramification of $\overline{K}/K$ with the goal of understanding the module of Kähler differentials $\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ and its relation to $\mathbf{C}_p(1)$.

In Part II, we turn towards the proof of the comparison theorem for abelian varieties. After a brief review of necessary background on abelian varieties in §4, we will explain Fontaine's proof of the theorem in §5, following [Fon82]. In §6, we will actually specialize to the case of abelian varieties with good reduction, and give an alternative proof in this case, due to [Col84], which complements the earlier proof in making certain isomorphisms more explicit. In particular, this proof circumvents the dependence on the abstract cohomology theory of $\mathbf{C}_p$ established in §I (at the cost of generality).

In Part III, we give a glimpse of the refinements of the Hodge-Tate theory through Fontaine's formalism of *period rings.* In §7 we introduce the notions of regular rings, admissible representations, and the ring of Hodge-Tate periods B_{HT}. In §8, we undertake the construction of the ring of de Rham periods B_{dR}, formulate the notion of de Rham representation, and show how it can be used to analyze the periods of Tate curves.

Part I

Analysis of p-adic fields

Chapter 2

The Cohomology of $\mathbf{C}_p$

Let K be a p-adic field with absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ and denote by $\mathbf{C}_K$ the completion of $\overline{K}$. We also write $\mathbf{C}_p = \mathbf{C}_{\mathbf{Q}_p}$, and note that $\mathbf{C}_p \simeq \mathbf{C}_K$, so we will use these two notations interchangeably. The Galois action on $\overline{K}$ extends by continuity on $\mathbf{C}_K$, and in this section, we study the Galois theory of $\mathbf{C}_K$. The results here constitute the main analytical ingredients in our exploration of p-adic representations.

Let $T_p(\mathbb{G}_m)$ denote the Tate module of p-power roots of unity in $\overline{K}$, which is isomorphic to $\mathbf{Z}_p$ as a group but also possesses a Galois action $(g, x) \mapsto g \cdot x$ by the p-adic cyclotomic character $\chi \colon G_K \to \mathbf{Z}_p^\times$:

$$g \cdot x = \chi(g)x \qquad \text{for all } g \in G_K.$$

More generally, if M is any $\mathbf{Z}_p$-module with an action by G_K, denoted $(g, m) \mapsto g(m)$, then we may form a G_K-module $M(i)$ which is isomorphic to M as a group, but whose Galois action is twisted by the i^{th} power of the cyclotomic character: for all $m \in M(i)$,

$$g \cdot m = \chi(g)^i g(m) \qquad \text{for all } g \in G_K.$$

The module $M(i)$ is called the i^{th} *Tate twist* of M.

Example 2.1. $T_p(\mathbb{G}_m) \simeq \mathbf{Z}_p(1)$ as Galois modules, where $\mathbf{Z}_p$ is given the trivial G_K-action.

Consequently, $M(i)$ may be realized as

$$M(i) = M \otimes_{\mathbf{Z}_p} T_p(\mathbb{G}_m)^{\otimes i}.$$

Galois theory shows that $\overline{K}^{G_K} = K$, so it is clear that $K \subset \mathbf{C}_K^{G_K}$. It is not obvious whether or not there are any other invariant elements of $\mathbf{C}_K$, or any non-zero invariant elements of $\mathbf{C}_K(i)$. We seek to address questions such as these, which are best phrased in the language of Galois cohomology. Our goal is to prove the following theorem.

Theorem 2.1 (Tate [Tat67]). *The Galois cohomology groups of* $\mathbf{C}_K(i)$ *are*

$$H^0(G_K, \mathbf{C}_K(i)) = \begin{cases} K & i = 0 \\ 0 & i \neq 0 \end{cases}$$

and

$$H^1(G_K, \mathbf{C}_K(i)) \simeq \begin{cases} K & i = 0 \\ 0 & i \neq 0 \end{cases}$$

These results go back to Tate's original paper on p-divisible groups [Tat67]. Our presentation fills out the details in his arguments, proceeding along the lines of [Bin]. We have tried to convey the spirit and key ideas of the proofs, and we omit some of the more technical calculations concerning the ramification theory of p-adic fields. Nonetheless, the arguments are rather technical in nature, and the reader who is happy to accept Theorem 2.1 may safely skip the proofs.

2.1 Review of Galois Cohomology

Let G be a group and M a topological G-module. Then we define the group of continuous n-cochains of G into M to be

$$C^n_{\text{cont}}(G, M) = \{f : G^n \to M \text{ continuous}\}.$$

By convention, we set $C^0_{\text{cont}}(G, M) = M$. There is a boundary map $d_n : C^n_{\text{cont}}(G, M) \to C^{n+1}_{\text{cont}}(G, M)$ given by

$$d_n f(g_1, \dots, g_{n+1}) = g_1 f(g_2, \dots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + f(g_1, \dots, g_n).$$

It is easy to check that $\operatorname{Im} d_{n-1} \subset \ker d_n$, so there is a chain complex

$$C^\bullet_{\text{cont}}(G, M) : 0 \to C^0_{\text{cont}}(G, M) \overset{d_0}{\to} C^1_{\text{cont}}(G, M) \overset{d_1}{\to} \dots$$

and we define the cohomology groups

$$H^n(G, M) = \frac{\ker d_n : C^n_{\text{cont}}(G, M) \to C^{n+1}_{\text{cont}}(G, M)}{\operatorname{Im} d_{n-1} : C^{n-1}_{\text{cont}}(G, M) \to C^n_{\text{cont}}(G, M)}.$$

If $H \subset G$ is a subgroup, then the natural restriction map on cochains induces a map

$$\text{res} : H^1(G, M) \to H^1(H, M).$$

If H is a normal subgroup, then M^H has a well-defined G/H-action, inducing a map

$$\inf : H^1(G/H, M^H) \to H^1(G, M)$$

by composition with the quotient $G \to G/H$.

Example 2.2. $H^0(G, M) = M^G$.

Example 2.3. $H^1(G, M)$ consists of *cocycles*, which are continuous maps $f : G \to M$ such that $f(g_1 g_2) = g_1 f(g_2) + f(g_1)$, modulo *coboundaries*, which are cocycles of the form $f(g) = g(m) - m$ for some $m \in M$.

Proposition 2.2 (Inflation-Restriction exact sequence)**.** *The sequence*

$$0 \to H^1(G/H, M^H) \to H^1(G, M) \to H^1(H, M)$$

is exact.

Proof. See [Sil09] B.1.3. ☐

Let $\chi : G_K \to \mathbf{Z}_p^\times$ be the cyclotomic character. Then $\chi(G_K)$ is an open subgroup of $\mathbf{Z}_p^\times$, and its image under the p-adic logarithm map $\log : \mathbf{Z}_p^\times \to \mathbf{Z}_p$ is an open subgroup of $\mathbf{Z}_p$, which is isomorphic to $\mathbf{Z}_p$. Let $H_K = \ker(\log \circ \chi)$ and $K_\infty = \mathbf{C}_K^{H_K}$. By construction, K_∞ is a totally ramified $\mathbf{Z}_p$-extension of K, and we let $\Gamma_K = \text{Gal}(K_\infty/K) \simeq \mathbf{Z}_p$.

Using the inflation-restriction sequence (2.2), we have an exact sequence

$$0 \to H^1(\Gamma_K, \mathbf{C}_K^{H_K}) \to H^1(G_K, \mathbf{C}_K) \to H^1(H_K, \mathbf{C}_K)$$

This allows us to break up the study of G_K into separate analyses of Γ_K and H_K, which we now undertake.

2.2 Galois Cohomology of H_K

Our first step is to compute $\mathbf{C}_K^{H_K}$. By definition, H_K fixes K_∞, and by continuity it also fixes $L := \widehat{K_\infty}$. The Theorem below asserts that nothing else is fixed.

Theorem 2.3. *We have* $H^0(H_K, C_K) \simeq L$.

In preparation for the proof, we build up some results about the density of K_∞ in its finite extensions. If $\mathcal{O}_L/\mathcal{O}_K$ is étale (i.e. unramified), then the induced trace map on the ring of integers is surjective. The following fundamental fact expresses the "almost étaleness" of totally ramified $\mathbf{Z}_p$-extensions: the trace map from a finite extension is not necessarily surjective, but its image contains the entire maximal ideal. We draw attention to this notion of "almost étaleness" because it plays a critical role in Faltings' theory ([Fal88]).

Theorem 2.4. *Let* K_∞ *be a totally ramified* $\mathbf{Z}_p$*-extension of* K *and* M *a finite extension of* K_∞*. Then* $\mathrm{Tr}_{M/K_\infty} \mathcal{O}_M \supset \mathfrak{m}_{K_\infty}$

Proof. See [Tat67], Proposition 9. □

By definition of the trace, $\mathrm{Tr}_{M/K_\infty} \mathcal{O}_M \subset \mathcal{O}_{K_\infty}$. Theorem 2.4 says that the image is large: large enough to contain the maximal ideal of $\mathcal{O}_{K_\infty}$, which includes all elements of positive valuation. All we will need is that the image of the trace map contains elements of arbitrarily small valuation.

Let M be any finite Galois extension of K_∞ and $J = \mathrm{Gal}(M/K_\infty)$. The following Lemma allows us to approximate elements of M by elements of K_∞.

Lemma 2.5. *Let* $c > 1$ *be a constant. For all* $m \in M$*, there exists some* $a \in K_\infty$ *such that*

$$|m - a| < c \sup_{g \in J} |gm - m|.$$

Proof. Theorem 2.4 implies that there exist elements of $\mathcal{O}_M$ with trace having arbitrarily small valuation, hence arbitrarily large norm. Therefore, we may choose some $x \in \mathcal{O}_M$ such that

$$|y = \mathrm{Tr}_{M/K_\infty}(x)| > \frac{1}{c}.$$

Set $a = \frac{\mathrm{Tr}_{M/K_\infty}(mx)}{y}$, so that

$$a = \frac{1}{y} \sum_{g \in J} g(mx) = m + \frac{1}{y} \sum_{g \in J} [g(mx) - mg(x)] = m + \frac{1}{y} \sum_{g \in J} g(x)(g(m) - m).$$

Applying the ultrametric inequality, the choice of $|y| > \frac{1}{c}$, and the fact that $|g(x)| = |x| \leq 1$ to the preceding equation, we find that

$$|a - m| = \left| \frac{1}{y} \sum_{g \in J} g(x)(g(m) - m) \right| < c \sup_{g \in J} |g(m) - m|$$

□

Proof of Theorem 2.3. Let $x \in H^0(H_K, \mathbf{C}_K)$ and $\{x_n\}_{n=1}^{\infty}$ be any Cauchy sequence in $\overline{K}$ converging to x. By passing to a subsequence, we may assume that

$|x - x_n| < p^{-n}$. Since each x_n is algebraic over K, we may choose a finite Galois extension M_n/K_∞ containing x_n and let $J_n = \mathrm{Gal}(M_n/K_\infty)$.

Now we use Lemma 2.5 to construct a sequence of elements of K_∞ approximating $\{x_n\}$ closely enough to converge to the same limit x, which would show that $x \in L$. Indeed, the Lemma implies that for any $c > 1$, there exist $\{a_n\}_{n=1}^{\infty} \subset K_\infty$ such that

$$|x_n - a_n| < c \sup_{g \in J_n} |g(x_n) - x_n|.$$

Noting that $g(x) = x$ by hypothesis, we compute

$$|g(x_n) - x_n| = |g(x - x_n) - (x - x_n)| \leq |x - x_n| \leq p^{-n}$$

by the ultrametric inequality and the fact that g acts by isometries. Therefore, $|x_n - a_n| < cp^{-n}$ for any $c > 1$, which easily implies that $\{a_n\}$ is also a Cauchy sequence converging to the same limit x. ☐

Now we turn towards the higher cohomologies. We need to prove an analogue of Lemma 2.5 for cochains. For $f \in C^r(G, M)$, we define

$$|f| = \sup_{g_i \in G} |f(g_1, \ldots, g_r)|$$

where the absolute value is normalized with respect to the base field K_∞.

Lemma 2.6. *Let f be an r-cochain of J with coefficients in M. If $r \geq 1$ and $c > 1$, then there exists an $(r-1)$-cochain f' of J such that*

$$|f - df'| \leq c|df| \qquad \textit{and} \qquad |f'| \leq c|f|.$$

Proof. Again, Theorem 2.5 implies that we may choose some $x \in \mathcal{O}_M$ such that

$$|y = \mathrm{Tr}_{M/K_\infty}(x)| > \frac{1}{c}.$$

For $x \in M$ and $f \in C^r(G, M)$, let $x \smile f$ be the $(r-1)$-cochain defined by

$$x \smile f(g_1, \ldots, g_{r-1}) = (-1)^r \sum_{g \in J} (g_1 \ldots g_{r-1} g x) f(g_1, \ldots, g_{r-1}, g).$$

Then one checks by explicit computation that

$$d(x \smile f) + x \smile df = yf,$$

We take $f' = \frac{x \smile f}{y}$ to be our $(r-1)$-cocycle, so that

$$|df' - f| = |\frac{x \smile df}{y}| < c|df|$$

by the ultrametric inequality and the fact that $|y| > \frac{1}{c}$. Similarly,

$$|f'| = |\frac{x \smile f}{y}| < c|f|.$$

$\square$

Remark 2.7. If one sets $C^{-1}(J, M) = M$ and defines $d_{-1} : C^{-1}(J, M) \to C^0(J, M)$ by the trace map, then the result here can be extended to $r = 0$, which is Lemma 2.5, and we recover the same proof as for Lemma 2.5.

If $\overline{K}_{\text{disc}}$ denotes the field $\overline{K}$ endowed with the *discrete* topology, then [Ser02] §2.2 says that

$$C^r(H_K, \overline{K}_{\text{disc}}) = \varinjlim_{M/K_\infty} C^r(\mathrm{Gal}(M/K_\infty), L)$$

where the limit is taken over finite Galois extensions. As a consequence, we deduce the same conclusion for coefficients in $\overline{K}_{\text{disc}}$.

Corollary 2.8. *Let f be an r-cochain of H_K with coefficients in $\overline{K}_{disc}$. If $r \geq 1$ and $c > 1$, then there exists an $(r-1)$-cochain f' of J such that*

$$|f - df'| \leq c|df| \qquad \textit{and} \qquad |f'| \leq c|f|.$$

Remark 2.9. Note that it is *harder* to find cochains into $\overline{K}_{\text{disc}}$ than $\overline{K}$ with the usual topology, since it is harder to find continuous maps into a space with the discrete topology. In particular, cochains for $\overline{K}_{\text{disc}}$ are cochains for $\overline{K}$.

Theorem 2.10. *For $r \geq 1$, we have $H^r(H_K, \mathbf{C}_K) = 0$.*

Proof. The idea is to approximate any cochain in $\mathbf{C}_K$ with cochains in $\overline{K}$, and then to apply Corollary 2.8 to construct a sequence of converging coboundaries.

Note that $p^\nu \mathcal{O}_{\mathbf{C}_K}$, for $\nu \in \mathbf{Q}$, form a basis of open neighborhoods of $\mathbf{C}_K$. Let $\pi_\nu : \mathbf{C}_K \to \mathbf{C}_K / p^\nu \mathcal{O}_{\mathbf{C}_K}$ be the natural quotient map. Since $\overline{K}$ is dense in $\mathbf{C}_K$ and $\mathbf{C}_K / p^\nu \mathcal{O}_{\mathbf{C}_K}$ is discrete, $\pi_\nu(\overline{K}) = \mathcal{O}_{\mathbf{C}_K} / p^\nu \mathcal{O}_{\mathbf{C}_K}$. Therefore, we may choose a section $\phi_\nu : \mathbf{C}_K / p^\nu \mathcal{O}_{\mathbf{C}_K} \to \overline{K}$, which is automatically continuous because the domain is discrete. For any $f \in C^r(H_K, \mathbf{C}_K)$, the composition $f_\nu := \phi_\nu \circ \pi_\nu \circ f$ is an element of $C^r(H_K, \overline{K})$ that agrees with f modulo p^ν, so that

$$|f - f_\nu| \leq |p|^\nu.$$

Choosing $\nu = 1, 2, \ldots$, we have a sequence of r-cochains $\{f_n\}_{n=1}^{\infty}$ such that $|f - f_n| \leq |p|^n$. By Lemma 2.8 or Lemma 2.5 (if $r = 1$), we find that there are $(r-1)$-cochains $\{f'_n\}_{n=1}^{\infty}$ such that (recalling that $df = 0$ by hypothesis)

$$|f_n - df'_n| \leq c|df_n| = c|d(f - f_n)| \leq c|f - f_n|,$$

which tends to 0 as $n \to \infty$. Therefore, $\{df'_n\}_{n=1}^{\infty}$ forms a Cauchy sequence converging to f. Unfortunately, we cannot yet say that $\{f'_n\}_{n=1}^{\infty}$ forms a Cauchy sequence, but we can achieve this after modifying by coboundaries.

If $r \geq 2$, then applying Lemma 2.8 or Lemma 2.5 again, we find that there exist $(r-2)$-cochains f''_n such that

$$|f'_{n+1} - f'_n - df''_n| \leq c|df'_{n+1} - df'_n| \to 0.$$

Therefore,

$$f' = f'_1 + \sum_{n=1}^{\infty}(f'_{n+1} - f'_n - df''_n)$$

is a well-defined $(r-1)$-cochain such that

$$df' = \lim_{n\to\infty} df'_n = f.$$

If $r = 1$, then we use the a_n furnished by Lemma 2.5 instead of df''_n, and the rest of the proof goes through as before. □

2.3 Galois Cohomology of Γ_K

We keep the notation $L = \widehat{K_\infty}$ from the previous section. The action of Γ_K on K_∞ extends to L in a unique manner by continuity. Since K_∞/K is a $\mathbf{Z}_p$-extension, it has a unique subextension K_r of degree p^r for each $r \in \mathrm{N}$. We begin by defining a normalized trace map as follows: if $x \in K_r$, then

$$T_K(x) = \frac{1}{p^r}\operatorname{Tr}_{K_r/K}(x).$$

Observe that this is well defined because if $x \in K_{r'}$ and $r' \geq r$, then

$$\frac{1}{p^{r'}}\operatorname{Tr}_{K_{r'}/K}(x) = \frac{1}{p^r}\left(\frac{1}{p^{r'-r}}\operatorname{Tr}_{K_{r'}/K}(x)\right) = \frac{1}{p^r}\operatorname{Tr}_{K_r/K}(x).$$

Let $\gamma_0 \in \Gamma_K$ be a topological generator. The following proposition is analogous to Lemma 2.5 from the previous section: it allows us to approximate $x \in K_\infty$ by elements of K.

Proposition 2.11. *For all* $x \in K_\infty$*, there is a constant* $c > 0$ *such that*

$$|T_K(x) - x| \le c|\gamma_0 x - x|$$

Proof. See [Fon04], Proposition 1.13. □

Corollary 2.12. *The map* $T_K : K_\infty \to K$ *is continuous.*

Proof. Indeed, we compute that

$$|T_K(x) - T_K(y)| = |T_K(x-y) - (x-y)| \le c|\gamma_0(x-y) - (x-y)| \le c|x-y|.$$

□

We extend T_K to a map $\widehat{T_K} : L \to K$ by continuity. From the fact that $\widehat{T}_K(K) = K$, we see that there is a decomposition $L \simeq K \oplus L_0$, where $L_0 = \ker T_K$. Since $T_K(x) = T_K(\gamma_0(x))$, $\gamma_0 - 1$ is an operator on L_0.

Proposition 2.13. *The operator* $\gamma_0 - 1$ *is bijective on* L_0 *with continuous inverse.*

Proof. Let $K_{r,0} = K_r \cap L_0$. Then $K_{\infty,0} = \bigcup_{r=1}^{\infty} K_{r,0}$ and L_0 is the closure of $K_{\infty,0}$. Since $\gamma_0 - 1$ is injective and hence bijective on each $K_{r,0}$, it is bijective on $K_{\infty,0}$. Let ρ denote the set-theoretic inverse.

Proposition 2.11 applied to $\rho(x)$ says that there exists a constant $c > 0$ such that $|\rho(x)| \le c|x|$ (since T_K vanishes on $K_{\infty,0}$), so ρ is continuous, and extends to a continuous inverse to $\gamma_0 - 1$ on L_0.

□

Proposition 2.14. *Let* $\lambda \in K_\infty$ *be a unit such that* $|\lambda - 1| < 1$ *and* λ *is not a root of unity. Then* $\gamma_0 - \lambda$ *is bijective on* L.

Proof. Note that $(\gamma_0 - \lambda)|_K$ is just multiplication by $(1 - \lambda)$, which is obviously bijective. From the splitting $L \simeq K \oplus L_0$, we are reduced to showing that $(\gamma_0 - \lambda)|_{L_0}$ is bijective.

Let ρ be the inverse of $(\gamma_0 - 1)|_{L_0}$. Then

$$\rho(\gamma_0 - \lambda) = \rho(\gamma_0 - 1 + 1 - \lambda) = 1 - (\lambda - 1)\rho.$$

Let c be the constant in Proposition 2.11. If $|\lambda - 1|c < 1$, then Proposition 2.11 implies that

$$|(\lambda - 1)\rho(x)| < |x| \text{ for all } x \in L_0.$$

Therefore,

$$(\rho(\gamma_0 - \lambda))^{-1} = (1 - (\lambda - 1)\rho)^{-1} = \sum_{i=1}^{\infty} [(\lambda - 1)\rho]^i$$

converges to a well-defined operator on L_0, showing that $(\gamma_0 - \lambda)$ is invertible.

Since $|\lambda - 1| < 1$ by assumption, there exists some integer r such that $|\lambda^{p^r} - 1|c < 1$. A topological generator for K_∞/K_r is $\gamma_r = \gamma_0^{p^r}$, so we may apply the argument above to λ^{p^r} to conclude that $\gamma_r - \lambda^{p^r}$ is invertible on L. Since $\gamma_0 - \lambda \mid \gamma_0^{p^r} - \lambda^{p^r}$, the operator $\gamma_0 - \lambda$ is also invertible on L. □

If $\eta : \Gamma_K \to L^\times$ is any character, then we denote by $L(\eta)$ the K-module isomorphic to L, but with Galois action twisted by η: for all $\gamma \in \Gamma_K$ and $x \in L(\eta)$, we have

$$\gamma \cdot x = \eta(\gamma)\gamma(x).$$

Note that if η is χ^i, then $L(\chi) \simeq L(i)$.

Theorem 2.15. *We have Galois cohomology groups*

(i)

$$H^0(\Gamma_K, L) = K \quad \text{and} \quad H^1(\Gamma_K, L) \simeq K.$$

(ii) If $\eta(\Gamma_K)$ *is infinite, then*

$$H^0(\Gamma_K, L(\eta)) = H^1(\Gamma_K, L(\eta)) = 0.$$

Proof. (i) First, since cohomology commutes with direct sums,

$$H^0(\Gamma_K, L) \simeq H^0(\Gamma_K, K) \oplus H^0(\Gamma_K, L_0).$$

Clearly $H^0(\Gamma_K, K) = K$, and $H^0(\Gamma_K, L_0) \subset \ker(\gamma_0 - 1)|_{L_0}$. But we showed in Proposition 2.13 that $(\gamma_0 - 1)|_{L_0}$ is bijective, so $H^0(\Gamma_K, L_0) = 0$.

Similarly,

$$H^1(\Gamma_K, L) \simeq H^1(\Gamma_K, K) \oplus H^1(\Gamma_K, L_0)$$

and $H^1(\Gamma_K, K)$ is the space of continuous homomorphisms from Γ_K to K, since the action of Γ_K on K is trivial, hence is a one-dimensional K-vector space because any continuous map $\mathbf{Z}_p \to L_0$ is determined by the image of a topological generator. This last remark also implies that we have an injection

$$C^1_{\text{cont}}(\Gamma_K, L_0) \to L_0.$$

Under this map, the coboundaries are the image of $\gamma_0 - 1$, so $H^1(\Gamma_K, L_0) \subset \text{coker}(\gamma_0 - 1)|_{L_0} = 0$.

(ii) If η has infinite image, then we may apply the same argument except that Proposition 2.14 shows that $H^0(\Gamma_K, K) = H^1(\Gamma_K, K) = 0$ as well. ☐

2.4 Consequences

We are now able to finish off the proof of Theorem 2.1.

Proof of Theorem 2.1. For the statement concerning H^0, note that for

$$(\mathbf{C}_K(i))^{G_K} = (\mathbf{C}_K(i)^{H_K})^{\Gamma_K} = L(i)^{\Gamma_K} = \begin{cases} K & i = 0 \\ 0 & i \neq 0 \end{cases}.$$

by Theorems 2.3 and 2.15 (since H_K is the kernel of the cyclotomic character, its Galois cohomology is unaffected by the Tate twist).

For the statement concerning H^1, we use the Inflation-Restriction sequence (2.2):

$$0 \to H^1(\Gamma_K, \mathbf{C}_K(i)^{H_K}) \to H^1(G_K, \mathbf{C}_K(i)) \to H^1(H_K, \mathbf{C}_K(i)).$$

By Theorem 2.10, $H^1(H_K, \mathbf{C}_K(i)) \simeq 0$. By Theorem 2.15,

$$H^1(\Gamma_K, \mathbf{C}_K(i)^{H_K}) \simeq \begin{cases} K & i = 0 \\ 0 & i \neq 0 \end{cases}.$$

☐

Corollary 2.16. *Let $H \subset G_K$ be any open normal subgroup. Then $\mathbf{C}_K(i)^H = 0$ if $i \neq 0$.*

Proof. Indeed, we used no special property of G_K except that $\chi(G_K)$ was open in $\mathbf{Z}_p^\times$, which is true for any open subgroup of G_K. ☐

Let us point out some consequences of these cohomological results.

Corollary 2.17. *There are no non-zero $\mathbf{C}_K[G_K]$-module homomorphisms $\mathbf{C}_K(i) \to \mathbf{C}_K(j)$ if $i \neq j$.*

Proof. After twisting by $-i$, we may reduce to the case where $i = 0$. In any such homomorphism, the image of $1 \in \mathbf{C}_K$ would be a G_K-invariant element of $\mathbf{C}_K(j)$, but the only such element is 0. ☐

Recall that if M' and M'' are modules over a ring R, then an *extension* of M' by M'' is an R-module M equipped with maps $f : M' \to M$ and $g : M \to M''$

such that the sequence

$$0 \to M' \to M \to M'' \to 0$$

is exact. The set of extensions forms a group, denoted $\mathrm{Ext}_R(M'', M')$. The following is a classical group-theoretic fact.

Proposition 2.18. *If G is a group and M a G-module, then*

$$\mathrm{Ext}^1_{\mathbf{Z}[G]}(\mathbf{Z}, M) \simeq H^1(G, M).$$

It does not (immediately) apply in our situation, since we are considering *continuous* cohomology of G_K, but it motivates us to find the following result.

Corollary 2.19. *If $i \neq j$, then* $\mathrm{Ext}^1_{\mathbf{C}_K[G]}(\mathbf{C}_K(i), \mathbf{C}_K(j))) = 0$.

Proof. After twisting by $-j$, we may reduce to the case where $j = 0$.

$$0 \to \mathbf{C}_K(i) \xrightarrow{f} M \xrightarrow{g} \mathbf{C}_K \to 0$$

be any extension of $\mathbf{C}_K(i)$ by $\mathbf{C}_K$ as $\mathbf{C}_K[G_K]$ modules. Pick any element $y \in M$ mapping to $1 \in \mathbf{C}_K$, which induces a $\mathbf{C}_K$-vector space section $\mathbf{C}_K \to M$. We define a cocycle $\xi : G_K \to M$ by $\xi(\sigma) = \sigma(y) - y$. It is easy to see that ξ is continuous, since the G_K-action on M is continuous.

By construction, $g(\xi(\sigma)) = 0$, so ξ can be viewed as a continuous cocycle $G_K \to \mathbf{C}_K(i)$. By Theorem 2.1, it is a coboundary, so there exists some $x \in \mathbf{C}_K(i)$ such that $\xi(\sigma) = \sigma(x) - x$. Then for all $\sigma \in G_K$, we have

$$\sigma(y) - y = \sigma(x) - x \implies \sigma(y - x) = y - x$$

so the map $1 \mapsto y - x$ defines a G_K-equivariant section $\mathbf{C}_K \to M$. □

Chapter 3

The Kähler Differentials of $\mathcal{O}_{\overline{K}}$

If L/K is any finite extension of p-adic fields, then $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is a torsion $\mathcal{O}_L$-module that measures the ramification of the map $\operatorname{Spec}\mathcal{O}_L \to \operatorname{Spec}\mathcal{O}_K$. In this chapter, we consider the module of differentials $\Omega := \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$. This somewhat more subtle, and the key result of this section is a relation between Ω and $\mathbf{C}_p(1)$.

If M is a $\mathbf{Z}_p$-module, then we define the *p-Tate module* (or just Tate module when the context is clear) to be

$$T_p(M) := \operatorname{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p/\mathbf{Z}_p, M).$$

Since $\mathbf{Q}_p = \varinjlim p^{-n}\mathbf{Z}_p$,

$$\operatorname{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p/\mathbf{Z}_p, M) \simeq \varprojlim \operatorname{Hom}_{\mathbf{Z}_p}(\mathbf{Z}_p/p^n\mathbf{Z}_p, M) \simeq \varprojlim M[p^n]$$

from which we recognize the usual definition in terms of compatible system of p-power torsion elements.

Definition 3.1. For any $\mathbf{Z}_p$-module M, the **rational Tate module** $V_p(M)$ is

$$V_p(M) := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p(M).$$

Definition 3.2. For any $\mathbf{Z}_p$-module M, we define

$$W_p(M) := \operatorname{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p, M) \simeq \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}[p^{-1}], M)$$

It is clear that both V_p and W_p are functors. $W_p(M)$ consists of "sequences of p-divisible elements in M," i.e. sequences $(x_0, x_1, \ldots)$ with $x_i \in M$ such that $px_i = x_{i-1}$.

The main goal of this section is to provide the following characterization of $\mathbf{C}_p(1)$:

Theorem 3.3 (Fontaine [Fon82])**.** *There is a canonical* G_K*-equivariant isomorphism*

$$W_p(\Omega) \simeq \mathbf{C}_p(1).$$

This isomorphism is deduced from a pairing that we now discuss.

3.1 A pairing

Let $\mu_{p^\infty}(\overline{K})$ be the group of p-power roots of unity in $\overline{K}$. There is a pairing

$$\mathcal{O}_{\overline{K}} \times \mu_{p^\infty}(\overline{K}) \to \Omega$$

defined by $\langle a, \epsilon \rangle = a\frac{d\epsilon}{\epsilon}$. It is easy to check that this pairing is $\mathbf{Z}_p$-bilinear and satisfies $\langle gu, \omega \rangle = g\langle u, \omega \rangle$ for all $g \in G_K$, $u \in \mathcal{O}_{\overline{K}}$, and $\omega \in \mu_{p^\infty}(\overline{K})$, and hence descends to a Galois-equivariant map on the tensor product, where the Galois action on $\mu_{p^\infty}(\overline{K})$ is trivial.

Definition 3.4. The map

$$\xi : \mathcal{O}_{\overline{K}} \otimes \mu_{p^\infty}(\overline{K}) \to \Omega \tag{3.1}$$

is defined by sending pure tensors

$$u \otimes \epsilon \mapsto u\frac{d\epsilon}{\epsilon}.$$

Our goal is to prove that ξ surjective and characterize its kernel. This will lead to the description of $W_p(\Omega)$ in Theorem 3.3.

There is another way of phrasing this pairing that generalizes to abelian varieties in a crucial way, as we will see later. Note that $\omega := \frac{dT}{1+T}$ is the invariant differential for the formal group of $\mathbb{G}_m$, which is the Lubin-Tate formal group of $\mathbf{Q}_p$. The map ξ sends $\epsilon \in \mu_{p^\infty}(\overline{K})$ to the pullback of ω via the corresponding $\mathcal{O}_{\overline{K}}$-point of the formal group. This perspective may seem somewhat contrived now, but it is important for both generalizing the result here to arbitrary Lubin-Tate groups (see Remark 3.15), and for defining the pairing that will play an important role in the Hodge-Tate Theorem later.

3.2 Review of Kähler Differentials

For a homomorphism of commutative rings $\phi : A \to B$, recall that a *derivation* of B over A is an A-module homomorphism $d : B \to M$ satisfying

(i) $d(bb') = bdb' + b'db$ for all $b, b' \in B$, and

(ii) $d(\phi(a)) = 0$ for all $a \in A$.

There is a B-module $\Omega_A(B) := \Omega_{B/A}$ called the *Kähler differentials* of B over A, which is equipped with a derivation $d : B \to \Omega_{B/A}$ over A such that, for any other B-module M equipped with a derivation $d_M : B \to M$ over A, there exists a unique B-module homomorphism f making the following diagram commute

$$\begin{tikzcd}
B \arrow[r, "d"] \arrow[dr, "d_M"'] & \Omega_{B/A} \arrow[d, "f"] \\
 & M
\end{tikzcd}$$

The module $\Omega_{B/A}$ may be explicitly described as the B-module generated by the symbols db, for $b \in B$, subject to the relations (i) and (ii) above.

Lemma 3.5. *If $A \to B \to C$ are ring homomorphisms, then there is a natural exact sequence*

$$C \otimes \Omega_{B/A} \xrightarrow{c \otimes db \mapsto cdb} \Omega_{C/A} \xrightarrow{dc \mapsto dc} \Omega_{C/B} \to 0.$$

Proof. See [Har77], II.8.3A. □

It also follows from the universal property that forming Kähler differentials commutes with direct limits: that is, if $B = \varinjlim B_i$ with each B_i an A-algebra, then

$$\Omega_{B/A} \simeq \varinjlim \Omega_{B_i/A}.$$

Indeed, if M is any B-module, then

$$\mathrm{Hom}_B(\Omega_{B/A}, M) \simeq \varprojlim \mathrm{Hom}_{B_i}(\Omega_{B_i/A}, M) \simeq \varprojlim \mathrm{Der}_A(B_i, M) \simeq \mathrm{Der}_A(B, M).$$

For a scheme X over a base S, we define the sheaf of *relative Kähler differentials* (or just "sheaf of differentials") $\Omega_{X/S}$ to be $\Delta^*(\mathscr{I}/\mathscr{I}^2)$, where $\Delta : X \to X \times_S X$ is the diagonal map and $\mathscr{I}$ is the ideal sheaf of $\Delta(X)$. As suggested by the notation, $\Omega_{X/S}$ is related to the Kähler differentials construction: over any affine open subset $\operatorname{Spec} B \simeq U \subset X$ lying over an affine open $\operatorname{Spec} A \simeq V \subset S$ is the module $\Omega_{B/A}$. In our applications, $S = \operatorname{Spec} A$, so the theory is especially simple. For further discussion, see [Har77] §II.8.

3.3 The structure of Ω

For a field extension L/K, let $\mathfrak{D}_{L/K}$ denote the different of L/K, which is the inverse ideal of the dual lattice to $\mathcal{O}_L$ under the trace pairing to K. The relative

Kähler differentials $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ are a torsion $\mathcal{O}_L$-module whose annihilator is $\mathfrak{D}_{L/K}$ ([Ser79], III.7.14). We choose a valuation v_K on K such that $v_K(p) = 1$ for all finite extensions $K/\mathbf{Q}_p$. Since this respects the inclusions $K \subset L$, we may let $v = v_K$ be the valuation on $\overline{K}$ extending all v_K.

Lemma 3.6. *Let $K \subset M \subset L$ be a tower of finite, separable extensions of local fields and $\iota : \Omega_{\mathcal{O}_M/\mathcal{O}_K} \to \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ the natural map. For all $\omega \in \Omega_{\mathcal{O}_M/\mathcal{O}_K}$, we have*

$$Ann_{\mathcal{O}_L}(\iota(\omega)) = \mathcal{O}_L Ann_{\mathcal{O}_M}(\omega).$$

Proof. The inclusion $\supseteq$ is obvious. For the other direction, we may reduce to the case where L/M is unramified or totally ramified. If it is unramified, then $\mathfrak{D}_{L/M}$ is the unit ideal, and by [Ser79] Proposition III.8 and Lemma 3.5, we see that

$$\mathcal{O}_L \otimes \Omega_{\mathcal{O}_M/\mathcal{O}_K} = \Omega_{\mathcal{O}_L/\mathcal{O}_K}.$$

If L/M is totally ramified, then let b' be a uniformizer of $\mathcal{O}_L$, so that b' generates $\Omega_{\mathcal{O}_L/\mathcal{O}_M}$ over $\mathcal{O}_M$, and let

$$F(X) = \sum a_i X^i$$

be the minimal polynomial of b'. Since L/K is totally ramified, $F(X)$ is an Eisenstein polynomial, and $b := -a_0$ is a uniformizer for $\mathcal{O}_M$.

Now, let $\omega = a\ d_{M/K}b$, so that $\iota(\omega) = a\ d_{L/K}b \in \Omega_{\mathcal{O}_L/\mathcal{O}_M}$. Since

$$b = a_n(b')^n + \ldots + a_1 b',$$

we have $d_{L/K}b = F'(b')\ d_{L/K}b'$, and $F'(b')$ generates $\mathfrak{D}_{L/M}$. Then $c \in \mathcal{O}_L$ annihilates $\iota(\omega)$ if and only if

$$v(c) \geq v(\mathfrak{D}_{L/K}) - v(a) - v(F'(b')) = v(\mathfrak{D}_{L/K}) - v(a) - v(\mathfrak{D}_{L/M}) = v(\mathfrak{D}_{M/K}) - v(a).$$

This precisely tells us that c is in $\mathcal{O}_L \mathrm{Ann}_{\mathcal{O}_M}(\omega)$.

□

Recall that we defined

$$\Omega = \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} = \varinjlim \Omega_{\mathcal{O}_L/\mathcal{O}_K}, \tag{3.2}$$

the limit taken over finite Galois extensions L/K. Since each $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is a torsion $\mathcal{O}_L$-module, we see that Ω is a torsion $\mathcal{O}_{\overline{K}}$-module. An immediate consequence of Lemma 3.6 is the following.

Corollary 3.7. *The canonical map $\Omega_{\mathcal{O}_L/\mathcal{O}_K} \to \Omega$ is injective.*

In particular, if $\omega \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ and has annihilator $\mathfrak{a} \subset \mathcal{O}_K$, then its annihilator in $\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ is simply $\mathcal{O}_{\overline{K}}\mathfrak{a}$.

Lemma 3.8. *Let* ω, ω' *be two elements of* Ω. *Then*

$$Ann_{\mathcal{O}_{\overline{K}}}(\omega) \subset Ann_{\mathcal{O}_{\overline{K}}}(\omega') \iff \omega' = c\omega \textit{ for some } c \in \mathcal{O}_{\overline{K}}.$$

Proof. The implication $\Leftarrow$ is obvious. For the other direction, (3.2) and Corollary 3.7 imply that we may choose some finite field extension L/K such that ω, ω' are already in the image of $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$, and identify $\omega = a\ d_{L/K}b$ and $\omega' = a'\ d_{L/K}b$, where $d_{L/K}b$ is a generator of $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$. By the remarks preceding the lemma,

$$\begin{aligned}\mathrm{Ann}_{\mathcal{O}_{\overline{K}}}(\omega) \subset \mathrm{Ann}_{\mathcal{O}_{\overline{K}}}(\omega') &\iff \mathrm{Ann}_{\mathcal{O}_L}(\omega) \subset \mathrm{Ann}_{\mathcal{O}_L}(\omega') \\ &\iff v(\mathfrak{D}_{L/K}) - v(a) \geq v(\mathfrak{D}_{L/K}) - v(a') \\ &\iff v(a') \geq v(a) \\ &\iff a = ca' \text{ for some } c \in \mathcal{O}_{\overline{K}}.\end{aligned}$$

□

3.4 Ramification calculations

The results of the previous section reduce much of the study of Ω to an analysis of the valuations of annihilators, which we now undertake. The first step is to compute the different for cyclotomic extensions; the result is well known but we include it for completeness.

Lemma 3.9. *Let* $E_r = \mathbf{Q}_p(\mu_{p^r})$, *where* μ_{p^r} *is the group of* p^r *roots of unity. Then*

$$v(\mathfrak{D}_{E_r/\mathbf{Q}_p}) = r - \frac{1}{p-1}.$$

Proof. The field extension $E_r/\mathbf{Q}_p$ is defined by the cyclotomic polynomial $\Phi(X) = \frac{X^{p^r}-1}{X^{p^{r-1}}-1}$, which is irreducible. Letting ζ be any primitive p^r root of unity, we know that $\mathfrak{D}_{E_r/\mathbf{Q}_p}$ is generated by $\Phi'(\zeta)$, which is

$$p^r \frac{\zeta^{p^r-1}}{\zeta^{p^{r-1}}-1} + \frac{\zeta^{p^r}-1}{(\zeta^{p^{r-1}}-1)^2} p^{r-1} = p^r \frac{\zeta^{p^r-1}}{\zeta^{p^{r-1}}-1}.$$

We note that the minimal polynomial for $\zeta - 1$ is

$$\Phi(X+1) = 1 + Y + \ldots + Y^{p-1}$$

where $Y = (X+1)^{p^{r-1}}$, which is an Eisenstein polynomial of degree $p^{r-1}(p-1)$. Therefore, $\zeta - 1$ is a uniformizer of valuation $\frac{1}{p^{r-1}(p-1)}$. Since $\zeta - 1$ is an associate of $\zeta^i - 1$ for $i = 2, \ldots, p^r - 1$, we see that $\zeta^{p^{r-1}} - 1$ has valuation $\frac{1}{p-1}$. □

The next Lemma allows us to extend this computation to K.

Lemma 3.10. *Let $K \subset M \subset L$ be a tower of finite, separable extensions of local fields and $f : \Omega_{\mathcal{O}_L/\mathcal{O}_K} \to \Omega_{\mathcal{O}_L/\mathcal{O}_M}$ be the natural map. For any $\omega \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$, we have*

$$v(Ann(f(\omega)) = \max\{0, v(Ann(\omega)) - v(\mathfrak{D}_{M/K})\}.$$

Proof. Let b be a generator of $\mathcal{O}_L/\mathcal{O}_K$, so that $\mathcal{O}_L = \mathcal{O}_K[b]$ and $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is generated by db. Then we may write $\omega = adb$ for some $a \in \mathcal{O}_L$, so that

$$v(\mathrm{Ann}(\omega)) = v(\mathfrak{D}_{L/K}) - v(a).$$

Similarly

$$v(\mathrm{Ann}(f(\omega))) = v(\mathfrak{D}_{L/M}) - v(a).$$

By [Ser79] Proposition III.8,

$$v(\mathfrak{D}_{L/K}) = v(\mathfrak{D}_{L/M}) + v(\mathfrak{D}_{M/K}).$$

The result is then concluded by subtracting the two preceding equations. □

Let $K_r = E_r K = K(\zeta_r)$ for ζ_r a primitive p^r root of unity.

Corollary 3.11. *We have*

$$v(Ann_{\mathcal{O}_{K_r}}(d\zeta_r)) = \max\left\{0, r - \frac{1}{p-1} - v(\mathfrak{D}_{K/\mathbf{Q}_p})\right\}.$$

Theorem 3.12. *The map*

$$\xi : \mathcal{O}_{\overline{K}} \otimes_{\mathbf{Z}_p} \mu_{p^\infty}(\overline{K}) \to \Omega$$

as defined in (3.4) *is surjective. Let ζ_1 be a primitive p^{th} root of unity and b be a generator of the absolute different $\mathfrak{D}_{K/\mathbf{Q}_p}$. Then the kernel of ξ consists of those elements annihilated by $(\zeta_1 - 1)b$.*

Proof. Let $\omega \in \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$. By Lemma 3.8, $\omega = c\omega'$ for some $c \in \mathcal{O}_K$ if and only if $\mathrm{Ann}_{\mathcal{O}_{\overline{K}}}(\omega) \supset \mathrm{Ann}_{\mathcal{O}_{\overline{K}}}(\omega')$, which occurs if and only if

$$v(\mathrm{Ann}_{\mathcal{O}_{\overline{K}}}(\omega)) \leq v(\mathrm{Ann}_{\mathcal{O}_{\overline{K}}}(\omega')).$$

We have just seen that for ζ_r a primitive p^r root of unity, $v(\mathrm{Ann}_{\mathcal{O}_{K_R}}(\xi(1 \otimes \zeta_r))) = r - \frac{1}{p-1}$, so by choosing r sufficiently large we may ensure that $\omega = \xi(c \otimes \zeta_r)$ for some $c \in \mathcal{O}_{\overline{K}}$. Now let $a \otimes \zeta_r \in \mathcal{O}_{\overline{K}} \otimes_{\mathbf{Z}_p} \mu_{p^\infty}(\overline{K})$. By Corollary 3.11,

$$v(\mathrm{Ann}_{\mathcal{O}_{K_r}}(a\ d\zeta_r)) = \max\left\{0, r - \frac{1}{p-1} - v(b) - v(a)\right\}.$$

Therefore,

$$\begin{aligned}\xi(a \otimes \zeta_r) = 0 &\iff r - \frac{1}{p-1} - v(b) - v(a) \leq 0 \\ &\iff v(a) + \frac{1}{p-1} + v(b) \geq r \\ &\iff ab(\zeta_1 - 1) \otimes \zeta_r = 0,\end{aligned}$$

which concludes the proof. □

3.5 Consequences

By restriction of ξ to $\mu_{p^\infty}(\overline{K}) \subset \mathcal{O}_{\overline{K}} \otimes \mu_{p^\infty}(\overline{K})$, we obtain a map

$$\begin{aligned}\mu_{p^\infty}(\overline{K}) &\to \Omega \\ \epsilon &\mapsto \frac{d\epsilon}{\epsilon}\end{aligned}$$

which is surjective, and whose kernel consists of elements annihilated by sufficiently high valuation. By Lemma 3.11, the annihilator of $d\zeta_r$ is an ideal with arbitrarily high valuation for arbitrarily large r. Therefore, the induced map on Tate modules

$$T_p(\mathbb{G}_m) \to T_p(\Omega)$$

is both surjective and injective. Tensoring with $\mathbf{C}_p$, we obtain the isomorphism

$$\mathbf{C}_p(1) \to T_p(\Omega) \otimes \mathbf{C}_p.$$

We have thus proved:

Corollary 3.13. *The map $\mathbf{C}_p(1) \to T_p(\Omega) \otimes \mathbf{C}_p$ sending*

$$a \otimes \{\epsilon_n\}_{n=0}^{\infty} \mapsto a \otimes \left\{\frac{d\epsilon_n}{\epsilon_n}\right\}_{n=0}^{\infty}.$$

is an isomorphism.

In a different direction, we define a map

$$\tilde{\xi} : \overline{K}(1) \to \Omega$$

by

$$\frac{a}{p^r} \otimes \{\epsilon_n\}_n \mapsto a\frac{d\epsilon_r}{\epsilon_r}.$$

To see that this is well-defined, observe that since $\epsilon_{r+1}^p = \epsilon_r$,

$$a\frac{d\epsilon_r}{\epsilon_r} = ap\epsilon^{p-1}\frac{d\epsilon_{r+1}}{\epsilon_{r+1}^p} = ap\frac{d\epsilon_{r+1}}{\epsilon_{r+1}}.$$

Let

$$\mathfrak{a} = \{a \in \overline{K} \mid v(a) \geq -v(\mathfrak{D}_{K/\mathbf{Q}_p}) - \frac{1}{p-1}\}.$$

By Theorem 3.12, the kernel of $\tilde{\xi}$ consist of $\mathfrak{a}(1)$. Therefore, we have the following characterization of Ω.

Corollary 3.14. *Let $\widehat{\mathfrak{a}}$ denote the completion of $\mathfrak{a}$ in $\mathbf{C}_p(1)$. We have the following identifications.*

(i) $\Omega \simeq (\overline{K}/\mathfrak{a})(1)$.

(ii) $T_p(\Omega) \simeq \widehat{\mathfrak{a}}(1)$.

(iii) $W_p(\Omega) \simeq \mathbf{C}_p(1)$.

Proof. Assertion (i) is immediate from Theorem 3.12. For (ii), we have

$$T_p(\Omega) \simeq \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p/\mathbf{Z}_p, \Omega) \simeq \varprojlim \mathrm{Hom}_{\mathbf{Z}_p}(p^{-n}\mathbf{Z}_p/\mathbf{Z}_p, \Omega).$$

Using (i),

$$\mathrm{Hom}_{\mathbf{Z}_p}(p^{-n}\mathbf{Z}_p/\mathbf{Z}_p, \Omega) \simeq \varprojlim \left(\frac{\mathfrak{a}}{p^n\mathfrak{a}}\right)(1) \simeq \widehat{\mathfrak{a}}(1).$$

which establishes (ii).

Finally, for (iii) we have by definition

$$W_p(\Omega) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p, \Omega) \simeq \varprojlim \mathrm{Hom}_{\mathbf{Z}_p}\left(p^{-n}\mathbf{Z}_p, \Omega\right)$$
$$= \varprojlim \left(\frac{\overline{K}}{p^n\mathfrak{a}}\right)(1) \simeq \mathbf{C}_p(1).$$

□

Remark 3.15. These results are consequences of more general results for Lubin-Tate groups. It is not difficult to write down the pairing constructed here in a more general form that applies to Lubin-Tate groups. Essentially the same argument may be applied after one establishes analogous results to Corollary 3.11 for general Lubin-Tate extensions. The details may be found in [Fon82].

Part II

Comparison Theorems for Abelian Varieties

Chapter 4

Review of Abelian Varieties

Here we summarize basic properties and constructions concerning abelian varieties and abelian schemes that we will use later. For a comprehensive treatment, the standard references are [Mum08] and [Mil].

4.1 Definitions and Basic Properties

Recall that an abelian variety over a field k is a complete, connected group object X in the category of varieties over k. In other words, there are morphisms

$$m : X \times X \to X \qquad e : \operatorname{Spec} k \to X \qquad i : X \to X$$

satisfying the usual relations between the group multiplication, the identity, and the inverse. (In what follows, we will also use the notation $e : X \to X$ to denote the composition $X \to \operatorname{Spec} k \to X$.)

- Associativity

$$\begin{tikzcd}
X \times X \times X \arrow[r, "{id,m}"] \arrow[d, "{m,id}"] & X \times X \arrow[d, "m"] \\
X \times X \arrow[r, "m"'] & X
\end{tikzcd}$$

- Identity

$$\begin{tikzcd}
X \times X \arrow[r, "{id,e}"] \arrow[rr, bend right, "id"'] & X \times X \arrow[r, "m"] & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \times X \arrow[r, "{e,id}"] \arrow[rr, bend right, "id"'] & X \times X \arrow[r, "m"] & X
\end{tikzcd}$$

- Inverse

$$\begin{tikzcd} X \arrow[r, "{id,i}"] \arrow[rr, bend right, "e"'] & X \times X \arrow[r, "m"] & X \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "{i,id}"] \arrow[rr, bend right, "e"'] & X \times X \arrow[r, "m"] & X \end{tikzcd}$$

Although commutativity is not built into the definition, the group law of an abelian variety is abelian. This is a consequence of some simple rigidity results concerning maps of abelian varieties.

The existence of a group law implies that any abelian variety X is "homogeneous," as there is a transitive group action on X. For any two points x_1, x_2 in X there is an isomorphism $X \to X$ taking x_1 to x_2, namely translation by $x_2 x_1^{-1}$. This has immediate consequences for the geometry: for instance, an abelian variety is smooth. Indeed, all varieties are smooth on an open dense subset, and the group law shows that all the tangent spaces must have the same dimension.

An abelian variety over $\mathbf{C}$ is isomorphic to $\mathbf{C}^g/\Lambda$, where $\Lambda \simeq \mathbf{Z}^{2g}$ is a lattice. This picture is familiar from the theory of elliptic curves, but in general not all lattices Λ give rise to abelian varieties.

Torsion

Let $X[m]$ denote the group m-torsion points of $X(\overline{K})$. If $\dim X = g$, then

$$X[m] \simeq (\mathbf{Z}/m)^{2g}.$$

Taking the inverse limit over all p-power torsion, we obtain the p-*Tate module*

$$T_p(X) = \varprojlim X[p^n] \simeq \mathbf{Z}_p^{2g}.$$

Definition 4.1. An **isogeny** $X \to Y$ of algebraic groups is a finite, surjective homomorphism of algebraic groups.

An isogeny of abelian varieties $X \to Y$ induces a map of Tate modules $T_p(X) \to T_p(Y)$. If the isogeny is defined over K, then this map is a homomorphism of G_K-modules with finite kernel, hence induces an isomorphism of rational Tate modules.

Theorem 4.2. *If $X_{\overline{K}}$ denotes the base-change of X to $\overline{K}$, then there is a canonical G_K-equivariant isomorphism*

$$Hom(T_p(X), \mathbf{Z}_p) \simeq H^1_{\acute{e}t}(X_{\overline{K}}, \mathbf{Z}_p)$$

and the cup product defines isomorphisms

$$\bigwedge^r H^1_{\acute{e}t}(X, \mathbf{Z}_p) \simeq H^r_{\acute{e}t}(X, \mathbf{Z}_p).$$

Proof. See [Mil], I.12.1. □

4.2 The Dual Abelian Variety

The dual abelian variety $\widehat{X}$ parametrizes the elements of $\mathrm{Pic}^0(X)$ (or more formally, represents the Picard functor). In the case of elliptic curves, there is a canonical isomorphism between these two varieties, but this is not so for general abelian varieties.

The dual abelian variety can be characterized as follows. There is a line bundle $\mathscr{P}$ on $X \times \widehat{X}$ such that:

(i) $\mathscr{P}|_{X\times\{x\}} \in \mathrm{Pic}^0(X \times \{x\})$ for all $x \in \widehat{X}$, and

(ii) $\mathscr{P}_{\{0\}\times\widehat{X}}$ is trivial.

Moreover, the pair $(\widehat{X}, \mathscr{P})$ is universal with respect to this property, in the sense that for any pair $(Z, \mathscr{L})$ satisfying

(i) $\mathscr{L}|_{X\times\{x\}} \in \mathrm{Pic}^0(X \times \{x\})$ for all $x \in Z$, and

(ii) $\mathscr{L}_{\{0\}\times Z}$ is trivial,

there is a unique morphism $f : T \to \widehat{A}$ such that $(1 \times f)^*\mathscr{P} = \mathscr{L}$.

Definition 4.3. A line bundle satisfying these properties is called the **Poincaré line bundle**. The corresponding divisor is called the **Poincaré divisor.**

It is easy to check that the dual of the dual abelian variety is the original, i.e. there is a canonical isomorphism $\widehat{\widehat{X}} \simeq X$.

Definition 4.4. A **polarization** is an isogeny $X \to \widehat{X}$.

Given any ample line bundle $\mathscr{N}$ on X, we may construct a polarization as follows. We define the "Mumford sheaf" on $X \times X$ by

$$\mathscr{L} = m^*\mathscr{N} \otimes (\pi_1^*\mathscr{N})^{-1} \otimes (\pi_2^*\mathscr{N})^{-1}.$$

This is easily checked to satisfy the conditions (4.2), and therefore induces a morphism $\phi_{\mathscr{N}} \colon X \to \widehat{X}$ ([Con]).

In particular, since any abelian variety is projective, it has a very ample line bundle and hence a polarization. By earlier comments, we deduce:

Corollary 4.5. *If X is an abelian variety, then there is a (non-canonical) isomorphism of Galois modules*

$$V_p(X) \simeq V_p(\widehat{X}).$$

4.3 The Sheaf of Differentials

Topologically, a complex abelian variety is homeomorphic to $(S^1)^{2g}$, so its cohomology ring is the exterior algebra on the first cohomology group. The following theorem posits a similar structure on the cohomology of the differentials of an abelian variety.

Theorem 4.6. *Let X/K be an abelian variety and Ω^p_X its sheaf of p-forms. Then there is an isomorphism of K-algebras*

$$\bigoplus_{p,q} H^q(X, \Omega^p_X) \simeq \bigwedge \left(H^0(X, \Omega_X) \bigoplus H^1(X, \mathscr{O}_X) \right).$$

Just as a Lie group is parallelizable because the group law allows one to construct a full space of invariant vector fields from the tangent space at the identity, the sheaf of differentials on an abelian variety is trivial.

Proposition 4.7. *X be an abelian variety and Ω_e the cotangent space to X at the identity e. Then there is a canonical isomorphism*

$$\Omega_e \otimes \mathscr{O}_X \simeq \Omega_X.$$

Proof. (See [Mum08].) For any $\omega_0 \in \Omega_e$, we define $\omega \in H^0(X, \Omega_X)$ on fibers

$$\omega_x = T^*_{-x}(\omega_0)$$

where $T_{-x} : X \to X$ is translation by $-x$. This map induces an isomorphism of fibers at each $x \in X$, hence an isomorphism of stalks by Nakayama's Lemma. ☐

Theorem 4.8. *There is a canonical isomorphism*

$$H^1(X, \mathscr{O}_X) \simeq H^0(\widehat{X}, \Omega_{\widehat{X}})^*.$$

Proof. See [Mil], §I.8.7. ☐

4.4 The Weil Pairing

There is a canonical pairing between the Tate module of an abelian variety and that of its dual, called the Weil pairing, which is a cornerstone of the theory of abelian varieties.

Theorem 4.9 (Weil pairing)**.** *There is a canonical bilinear, nondegenerate, Galois-equivariant pairing*

$$T_p(X) \times T_p(\widehat{X}) \to T_p(\mathbb{G}_m) \simeq \mathbf{Z}_p(1).$$

From the non-degeneracy, we immediately see:

Corollary 4.10. *There is a canonical isomorphism of Galois modules*

$$V_p(X) \simeq Hom_{\mathbf{Z}_p}(V_p(\widehat{X}), \mathbf{Q}_p(1)) \simeq V_p(\widehat{X})^*(1).$$

4.5 Abelian schemes

More generally, let S be a base scheme. We can define an *abelian scheme* over S, which is a family of abelian varieties parametrized by the points of S.

Definition 4.11. An **abelian scheme** over S of relative dimension g is a proper, smooth group scheme over S whose geometric fibers are connected (hence abelian varieties) of dimension g.

Much of the theory of abelian varieties goes through: for instance, abelian schemes have abelian group laws. For our purposes, only two simple properties will be required. First, there is a construction of the dual abelian scheme in this more general setting, which is realized as representing a subfunctor of the Picard functor corresponding to line bundles that are "algebraically equivalent" to zero. See [FC90] §I.1, [MFK94] §6.1, or [Kle05] §9.6 for details.

Second, we need the fact that the relative cotangent sheaf (or sheaf of relative differentials) $\Omega_{A/S}$ for an abelian scheme A/S is generated by global sections. This follows from the same proof as that of Proposition 4.7, since the value at any stalk can be propagated around via the group law. In fact, we can describe this sheaf more explicitly: if $f : A \to S$ is the structure map and $s : S \to A$ is the identity section (sending each point to the identity point on its fiber), then $\Omega_{A/S} = f^* s^* \Omega_{A/S}$.

In our considerations, A will be an abelian scheme over a discrete discrete valuation ring R. Then the generic fiber $X = A_K$ is an abelian variety over K, and $\widehat{X} \simeq \widehat{A}_K$.

Definition 4.12. Let K be a discrete valuation field with valuation ring R. An abelian variety X/K has **good reduction** if there is an abelian scheme A/R such that $X = A_K$.

Chapter 5

The Tate-Raynaud Theorem

5.1 The Hodge Decomposition

The following theorem of Faltings [Fal88] describes a "Hodge-like" decomposition for the étale cohomology of a non-singular projective variety over discrete valuation fields.

Theorem 5.1 (Faltings)**.** *Let X be a projective, non-singular variety over K. Then we have canonical isomorphisms*

$$(\mathbf{C}_p(j) \otimes_{\mathbf{Q}_p} H^i_{\acute{e}t}(X_{\overline{K}}, \mathbf{Q}_p))^{G_K} = \begin{cases} 0 & j < 0 \textit{ or } j > i \\ H^{i-j}(X, \Omega^j_X) & \textit{otherwise} \end{cases}.$$

The theorem was proved for abelian varieties with good reduction by Tate in [Tat67] and for general abelian varieties by Raynaud in [Gro72], using heavy machinery from algebraic geometry. In this section we will present a simpler argument, due to Fontaine, that proves the theorem for all abelian varieties without recourse to p-divisible groups, Néron models, or the semistable reduction theorem.

For X an abelian variety, Theorem 4.2 implies that the étale cohomology is simply an exterior algebra generated in degree one, and Theorem 4.6 implies a similar result for the cohomology of differentials. In particular, $H^1_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \simeq \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p(X)^*$, so that

$$H^*_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \simeq \bigwedge (\mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p(X))^*$$

and

$$H^*_{\text{Hodge}} \simeq \bigwedge (H^1(X, \mathscr{O}_X) \oplus H^0(X, \Omega_X)).$$

Therefore, in the case of an abelian variety, the Tate-Raynaud theorem reduces to the two isomorphisms

$$(\mathbf{C}_p \otimes_{\mathbf{Z}_p} T_p(X)^*)^{G_K} \simeq H^1(X, \mathscr{O}_X) \tag{5.1}$$

$$(\mathbf{C}_p(1) \otimes_{\mathbf{Z}_p} T_p(X)^*)^{G_K} \simeq H^0(X, \Omega_X). \tag{5.2}$$

Theorem 4.8 says that there is a canonical isomorphism

$$H^1(X, \mathscr{O}_X) \simeq H^0(\widehat{X}, \Omega_{\widehat{X}})^*$$

and by the isomorphism $T_p(X)^* \simeq T_p(\widehat{X})(1)$ of Corollary 4.10, we see that (5.1) is equivalent to the existence of a G_K-invariant map

$$(\mathbf{C}_p(1) \otimes T_p(\widehat{X}))^{G_K} \simeq H^0(\widehat{X}, \Omega_{\widehat{X}})^* := \mathrm{Hom}_K(H^0(\widehat{X}, \Omega_{\widehat{X}}), K).$$

This looks very similar to (5.2) applied to the dual abelian variety $\widehat{X}$, so we might suspect that the pair of isomorphism is equivalent to the existence of a single one. The question is whether or not we can "commute" the operations of taking G_K-invariants and duals.

It is not true in general that if V is a $\mathbf{C}_p$-representation of G_K, then the natural restriction map $\mathrm{Hom}(V, \mathbf{C}_p)^{G_K} \to \mathrm{Hom}(V^{G_K}, K)$ is an isomorphism.

Example 5.1. Suppose V is a non-trivial extension of $\mathbf{C}_p$ by $\mathbf{C}_p$:

$$0 \to \mathbf{C}_p \xrightarrow{f} V \xrightarrow{g} \mathbf{C}_p \to 0$$

Such a representation may be obtained by defining the action of Galois to be given by $\begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}$, where χ is the cyclotomic character. Then $V^{G_K} = (\ker f)^{G_K} \simeq K$, and the homomorphism $V \to \mathbf{C}_p$ given by g restricts to the zero map on V^{G_K}.

However, it is true if V is a direct sum of $\mathbf{C}_p(j)$, since $(\mathbf{C}_p(j))^{G_K} = 0$ for $j \neq 0$ and $\mathrm{Hom}(\mathbf{C}_p(j), \mathbf{C}_p) = 0$ for $j \neq 0$ (by the results of §2.4). The next proposition shows that, in our case, (5.2) is sufficient to obtain this result, and hence the Hodge-Tate Theorem.

Proposition 5.2. *Let X be an abelian variety. If $(\mathbf{C}_p(1) \otimes_{\mathbf{Z}_p} T_p(X)^*)^{G_K} \simeq H^0(X, \Omega_X)$, then*

$$(\mathbf{C}_p \otimes_{\mathbf{Z}_p} T_p(X)^*)^{G_K} \simeq H^1(X, \mathscr{O}_X).$$

Proof. Let $V_{\mathbf{C}_p}(X) = W_p(X) \otimes \mathbf{C}_p$, where $W_p(X)$ is the rational Tate module of X. In these terms, the hypothesis says that $(V_{\mathbf{C}_p}(X)^*(1))^{G_K} \simeq H^0(X, \Omega_X)$.

By the preceding remarks, it suffices to show that $V_{\mathbf{C}_p}(X)^*(1)$ is a direct sum of $\mathbf{C}_p(i)$. The point is that the inclusion of the g-dimensional subspace $(V_{\mathbf{C}_p}(X)^*(1))^{G_K} \otimes_K \mathbf{C}_p$ automatically yields a large $\mathbf{C}_p(1)$-quotient of $V_{\mathbf{C}_p}(X)^*(1)$ by duality, and then these two subspaces must completely fill $V_{\mathbf{C}_p}(X)^*(1)$, ruling out the kind of behavior witnessed in Example 5.1.

To carry out this idea, apply $\mathrm{Hom}_{\mathbf{C}_p}(-, \mathbf{C}_p(1))$ to the injection

$$H^0(X, \Omega_X) \otimes_K \mathbf{C}_p \hookrightarrow V_{\mathbf{C}_p}(X)^*(1)$$

to obtain the surjection

$$\mathrm{Hom}(V_{\mathbf{C}_p}(X)^*(1), \mathbf{C}_p(1)) \simeq V_{\mathbf{C}_p}(X) \twoheadrightarrow H^0(X, \Omega_X)^* \otimes_K \mathbf{C}_p(1). \tag{5.3}$$

Since $V_p(X) \simeq V_p(\widehat{X}) \simeq V_p(X)^*(1)$ by Corollaries 4.10 and 4.5, we may rewrite the surjection (5.3) as

$$V_{\mathbf{C}_p}(X)^*(1) \twoheadrightarrow H^0(X, \Omega_X)^* \otimes_K \mathbf{C}_p(1). \tag{5.4}$$

Now we know that $\dim_{\mathbf{C}_p} V_{\mathbf{C}_p}(\widehat{X})^*(1) = 2g$ and $\dim H^0(X, \Omega_X) = g$. The composition

$$H^0(X, \Omega_X) \otimes \mathbf{C}_p \to V_{\mathbf{C}_p}(X)^*(1) \to H^0(X, \Omega_X)^* \otimes \mathbf{C}_p(1)$$

is homomorphism of G_K-modules, hence must be 0 by Corollary 2.19. So $H^0(X, \Omega_X) \otimes \mathbf{C}_p$ is contained in the kernel of (5.4) and is equal to it by counting dimensions, giving the short exact sequence

$$0 \to H^0(X, \Omega_X) \otimes \mathbf{C}_p \to V_{\mathbf{C}_p}(\widehat{X})^*(1) \to H^0(X, \Omega_X)^* \otimes_K \mathbf{C}_p(1) \to 0.$$

Corollary 2.19 shows that there are no nontrivial extensions of $\mathbf{C}_p$ by $\mathbf{C}_p(1)$. Therefore, the sequence splits, and $V_{\mathbf{C}_p}(X)^*(1) \simeq \mathbf{C}_p^g \oplus \mathbf{C}_p(1)^g$.

□

The main goal of this section is to prove the following theorem.

Theorem 5.3. *Let X/K be an abelian variety. There exists a K-linear injection*

$$\phi_X \colon H^0(X, \Omega_X) \to \mathit{Hom}_{\mathbf{Z}_p[G]}(T_p(X), \mathbf{C}_p(1))$$

which is canonical and functorial in X.

The Hodge Decomposition for abelian varieties follows from this seemingly weaker statement:

Corollary 5.4 (Tate-Raynaud)**.** *Theorem 5.1 holds for X an abelian variety.*

Before giving the proof, we record an important general fact about invariants of representations.

Proposition 5.5. *Let F be a field extension of K and $G = \mathrm{Gal}(F/K)$. For any F-representation V of G, the F-linear map*

$$F \otimes_K V^G \to V$$

induced by the inclusion $V^G \subset V$ is an injection.

Proof. Suppose otherwise for the sake of contradiction. Then there exists a minimal set of linearly independent vectors $\{v_i\}_{i=1}^m \subset V^G$ such that

$$\sum_{i=1}^m a_i \otimes v_i = 0 \qquad a_i \in F.$$

By dividing through by a_1, we may assume that $a_1 = 1$, so

$$v_1 + \sum_{i=2}^m a_i \otimes v_i = 0. \tag{5.5}$$

Applying any $g \in G$, we also have

$$v_1 + \sum_{i=2}^m g(a_i) \otimes v_i = 0. \tag{5.6}$$

But subtracting (5.6) from (5.5), we obtain

$$\sum_{i=2}^m (a_i - g(a_i)) \otimes v_i = 0,$$

so the minimality assumption implies that $a_i = g(a_i)$ for all i. This holds for any $g \in G$, so $a_i \in F^G = K$ for all i, which contradicts the assumption that the v_i were linearly independent over K.

□

Proof of Corollary 5.4. By Proposition 5.2, it suffices to show that an injection ϕ_X as in Theorem 5.3 is necessarily an isomorphism. Let $\widehat{X}$ be the dual abelian variety to X, and define, as before $V_{\mathbf{C}_p}(X) := W_p(X) \otimes \mathbf{C}_p$. By Theorem 5.3 applied to X and $\widehat{X}$, we have

$$\dim_K(V_{\mathbf{C}_p}(X)^*(1))^{G_K} \geq g \quad \text{and} \quad \dim_K(V_{\mathbf{C}_p}(\widehat{X})^*(1))^{G_K} \geq g. \tag{5.7}$$

The Weil pairing is a non-degenerate, G_K-equivariant, bilinear pairing

$$T_p(X) \times T_p(\widehat{X}) \to \mathbf{Z}_p(1) \tag{5.8}$$

which induces the isomorphism $T_p(X) \simeq \mathrm{Hom}_{\mathbf{Z}_p}(T_p(\widehat{X}), \mathbf{Z}_p(1))$ as G_K-modules, and hence $V_{\mathbf{C}_p}(X)^*(1) \simeq V_{\mathbf{C}_p}(\widehat{X})$. Therefore, tensoring (5.8) with $\mathbf{C}_p$ induces a perfect, G_K-equivariant pairing

$$V_{\mathbf{C}_p}(\widehat{X})^*(1) \times V_{\mathbf{C}_p}(X)^*(1) \to \mathbf{C}_p(1). \tag{5.9}$$

By Proposition 5.5, we have natural inclusions $(V_{\mathbf{C}_p}(X)^*(1))^{G_K} \otimes_K \mathbf{C}_p \subset V_{\mathbf{C}_p}(X)^*(1)$, so we may restrict this to a pairing

$$((V_{\mathbf{C}_p}(X)^*(1))^{G_K} \otimes_K \mathbf{C}_p) \times ((V_{\mathbf{C}_p}(X)^*(1))^{G_K} \otimes_K \mathbf{C}_p) \to H^0(G_K, \mathbf{C}_p(1)) \otimes_K \mathbf{C}_p.$$

Since $H^0(G_K, \mathbf{C}_p(1)) \simeq 0$ by Theorem 2.1, these two subspaces must be orthogonal, so the sum of their dimensions is at most $\dim_{\mathbf{C}_p}(T_p(X) \otimes_{Z_p} \mathbf{C}_p) = 2g$. Combining this with (5.7), we find that we must have equality: $\dim(V_{\mathbf{C}_p}(X)^*(1))^{G_K} = \dim(V_{\mathbf{C}_p}(\widehat{X})^*(1))^{G_K} = g$. Therefore, ϕ_X is an isomorphism.

□

The rest of the section is devoted to the proof of Theorem 5.3.

5.2 Fontaine's pairing

We wish to construct a natural map

$$H^0(X, \Omega_X) \to \mathrm{Hom}_{\mathbf{Z}_p[G]}(T_p(X), \mathbf{C}_p(1))$$

or, equivalently, a G_K-equivariant pairing between global regular differentials on X and the Tate module:

$$H^0(X, \Omega_X) \times T_p(X) \to \mathbf{C}_p(1).$$

This is reminiscent of the pairing between the invariant differentials of $\mathbb{G}_m$ and its $\mathcal{O}_{\overline{K}}$ points that we considered in Section 3.12, and we might try to carry out a similar idea here. To do so, we need an $\mathcal{O}_K$-model of X. If X has good reduction over K, there is natural choice for such a model, but in general we need to work with a model that may not necessarily be smooth.

Let $\mathcal{X}/\mathcal{O}_K$ be any proper, flat, finite type extension of X, i.e. such that X is the generic fiber of $\mathcal{X}$.

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[d] & \mathcal{X} \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} \mathcal{O}_K
\end{tikzcd}$$

Such a model may be constructed, for instance, by composing an embedding of X into projective space with the map $\mathbb{P}^n_K \to \mathbb{P}^n_{\mathcal{O}_K}$, and then taking the closure. Now, $\mathcal{X}$ will not generally be an abelian scheme, but its $\mathcal{O}_{\overline{K}}$-points have a group structure due to the following lemma.

Lemma 5.6. *With the notation above, we have an isomorphism*

$$\mathcal{X}(\mathcal{O}_{\overline{K}}) \simeq X(\overline{K}).$$

Proof. Suppose that $p \in \mathcal{X}(\mathcal{O}_{\overline{K}})$ is represented by a morphism $\operatorname{Spec} \mathcal{O}_{\overline{K}} \to \mathcal{X}$. Composing with the map $\operatorname{Spec} \overline{K} \to \operatorname{Spec} \mathcal{O}_{\overline{K}}$, we obtain a diagram

$$\begin{tikzcd}
\operatorname{Spec} \overline{K} \arrow[r] \arrow[dr, dotted, "\exists!"] \arrow[ddr] & \operatorname{Spec} \mathcal{O}_{\overline{K}} \arrow[dr] & \\
& X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
& \operatorname{Spec} \overline{K} \arrow[r] & \operatorname{Spec} \mathcal{O}_{\overline{K}}
\end{tikzcd}$$

By the universal property, this induces a unique map $\operatorname{Spec} \overline{K} \to X$. Conversely, given such a map, we obtain a map $\operatorname{Spec} \overline{K} \to \mathcal{X}$ by composition, and by the valuative criterion of properness ([Har77] §II.4.7) there is a unique map $\operatorname{Spec} \mathcal{O}_{\overline{K}} \to \mathcal{X}$ making the diagram commute

$$\begin{tikzcd}
\operatorname{Spec} \overline{K} \arrow[r] \arrow[d] & X \arrow[r] & \mathcal{X} \arrow[d] \\
\operatorname{Spec} \mathcal{O}_{\overline{K}} \arrow[rr] \arrow[urr, dotted, "\exists!"] & & \operatorname{Spec} \mathcal{O}_{\overline{K}}
\end{tikzcd}$$

It is easy to see that the uniqueness of these maps makes the two associations into mutual inverses. □

For a morphism $X \to S$, let $\Omega_{X/S}$ denote the sheaf of relative Kähler differentials of $X \to S$. If $S \simeq \operatorname{Spec} R$ is affine, we abuse notation by writing $\Omega_{X/R} := \Omega_{X/S}$. When the context is clear, we will omit the base scheme. Since forming differentials commutes with base change ([Har77] §II.8.10), we have

$$\Omega_{X/K} \simeq \iota^* \Omega_{\mathcal{X}/\mathcal{O}_K}.$$

Since taking cohomology commutes with flat base change ([Har77], §III.9.3) we have

$$H^0(X, \Omega_{X/K}) \simeq K \otimes H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K}). \tag{5.10}$$

If $x : \operatorname{Spec} \mathcal{O}_{\overline{K}} \to X$ represents any point of $\mathcal{X}(\mathcal{O}_{\overline{K}})$, then there is a pairing

$$H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_{\overline{K}}}) \times \mathcal{X}(\mathcal{O}_{\overline{K}}) \to \Omega$$

defined by

$$\langle \omega, x \rangle = x^* \omega.$$

This pairing is clearly linear in the first variable, and we also have for any $g \in G_K$,

$$\langle \omega, g(x) \rangle = (gx)^* \omega = g(x^* \omega) = g(\langle \omega, x \rangle).$$

We might wish to define a map

$$H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_{\overline{K}}}) \to \operatorname{Hom}(\mathcal{X}(\mathcal{O}_{\overline{K}}), \Omega),$$

from this pairing but, a priori, we do not have linearity in the second variable of our pairing. The following proposition shows that we *can* guarantee linearity after passing to a submodule.

Proposition 5.7. *There exists an integer $r \geq 0$ such that:*

(i) the map

$$\begin{gathered} H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K}) \to H^0(X, \Omega_{X/K}) \\ \omega \mapsto 1 \otimes \omega \end{gathered}$$

is injective upon restriction to $p^r H^0(\mathcal{X}, \mathcal{O}_K)$, and

(ii) If $\omega \in p^r H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K})$ and x_1, x_2 represent points of $\mathcal{X}(\mathcal{O}_{\overline{K}}) \simeq X(\overline{K})$, then

$$\langle \omega, x_1 + x_2 \rangle = \langle \omega, x_1 \rangle + \langle \omega, x_2 \rangle.$$

Proof. For (i), note that the kernel of the map

$$H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K}) \to H^0(X, \Omega_{X/K}) \simeq K \otimes H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K})$$

is precisely the torsion submodule. But $\Omega_{\mathcal{X}/\mathcal{O}_K}$ is coherent since $\mathcal{X}$ is projective, hence $H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K})$ is finite type over $\mathcal{O}_K$. Therefore, the torsion submodule is killed by some *finite* power p^r. To see (ii), let $\mathcal{Y}/\mathcal{O}_K$ be a proper, flat, finite type scheme extending $X \times X$, i.e. $X \times X$ is the generic fiber

$$\begin{tikzcd}
X \times X \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} \mathcal{O}_K
\end{tikzcd}$$

and such that $\mathcal{Y}$ is equipped with morphisms $\pi_{1,\mathcal{X}}, \pi_{2,\mathcal{X}}$, and $m_{\mathcal{X}}$ extending the projection and multiplication maps $X \times X \to X$:

$$\begin{tikzcd}
X \times X \arrow[r] \arrow[d, "{\pi_1, \pi_2, m}"'] & \mathcal{Y} \arrow[d, "{\pi_{1,\mathcal{X}}, \pi_{2,\mathcal{X}}, m_{\mathcal{X}}}"] \\
X \arrow[r] & \mathcal{X}
\end{tikzcd}$$

For instance, given an embedding $\psi : X \to \mathbb{P}^n_K$, we could take $\mathcal{Y}$ to be the closure of the image of $X \times X$ under the composition

$$X \times X \overset{\psi, \pi_1, \pi_2, m}{\longrightarrow} \mathbb{P}^n_K \times X \times X \times X \longrightarrow \mathbb{P}^n_{\mathcal{O}_K} \times \mathcal{X} \times \mathcal{X} \times \mathcal{X}.$$

Since X is an abelian variety, $H^0(X, \Omega_X)$ consists precisely of the invariant forms (Proposition 4.7), so if $\omega \in H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, then

$$m_{\mathcal{X}}^*(\omega) - \pi_{1,\mathcal{X}}^*(\omega) - \pi_{2,\mathcal{X}}^*(\omega) \in \ker : H^0(\mathcal{Y}, \Omega_{\mathcal{Y}}) \to H^0(X \times X, \Omega_{X \times X}).$$

Since $\Omega_{\mathcal{Y}}$ is a coherent sheaf on a projective scheme, $H^0(\mathcal{Y}, \Omega_{\mathcal{Y}})$ is again finite type over $\mathcal{O}_{\overline{K}}$, so the kernel is annihilated by some finite power p^r. Now, let $x_1, x_2 : \operatorname{Spec} \mathcal{O}_{\overline{K}} \to \mathcal{X}$ represent two points of $\mathcal{X}(\mathcal{O}_{\overline{K}})$. Let $q \in X(\overline{K})$ be the point corresponding to $x_1 + x_2$, so we have a diagram

$$\begin{tikzcd}
\operatorname{Spec} \mathcal{O}_{\overline{K}} \arrow[r, "m"] \arrow[rr, bend right, "q"'] & \operatorname{Spec} \mathcal{O}_{\overline{K}} \times \operatorname{Spec} \mathcal{O}_{\overline{K}} \arrow[r, "{x_1, x_2}"] & \mathcal{Y}
\end{tikzcd}$$

Since $x_1 = \pi_{1,\mathcal{X}} \circ q$ and $x_2 = \pi_{2,\mathcal{X}} \circ q$, we have $x_1 + x_2 = m_{\mathcal{X}} \circ q$. Therefore,

$$(x_1 + x_2)^* \omega = q^* m_{\mathcal{X}}^*(\omega) = q^*(\pi_{1,\mathcal{X}}^*(\omega) + \pi_{2,\mathcal{X}}^*(\omega)) = x_1^*(\omega) + x_2^*(\omega).$$

□

Construction of ϕ_X

We have now established that the pairing

$$p^r H^0(\mathcal{X}, \Omega_{\mathcal{X}}) \times \mathcal{X}(\mathcal{O}_{\overline{K}}) \to \Omega$$

is $\mathcal{O}_K$-linear in the first variable and $\mathbf{Z}[G_K]$-linear in the second. Identifying $\mathcal{X}(\mathcal{O}_{\overline{K}}) \simeq X(\overline{K})$ via Lemma 5.6, we obtain an $\mathcal{O}_K$-linear homomorphism

$$p^r H^0(\mathcal{X}, \Omega_{\mathcal{X}}) \to \operatorname{Hom}_{\mathbf{Z}[G_K]}(X(\overline{K}), \Omega).$$

Recall that $W_p(X) = \operatorname{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p, X)$ (Definition 3.2) can be interpreted as all systems of points of $X(\overline{K})$ compatible under the multiplication-by-p map. In particular, we have an inclusion of the Tate module $T_p(X) \subset W_p(X)$. The preceding map induces

$$p^r H^0(\mathcal{X}, \Omega_{\mathcal{X}}) \to \operatorname{Hom}_{\mathbf{Z}[G_K]}(W_p(X), W_p(\Omega)),$$

which by extension of scalars to K yields a K-linear map

$$\phi_X^0 \colon H^0(X, \Omega_X) \simeq K \otimes p^r H^0(\mathcal{X}, \Omega_{\mathcal{X}}) \to \operatorname{Hom}_{\mathbf{Z}[G_K]}(W_p(X), W_p(\Omega)) \tag{5.11}$$

Now note that for any $\omega \in H^0(X, \Omega_X)$, the restriction of ω to $T_p(X)$ is a $\mathbf{Z}_p$-linear homomorphism to $W_p(\Omega)$. Using the identification $\xi : W_p(\Omega) \simeq \mathbf{C}_p(1)$ of Corollary 3.14 we obtain a map

$$\phi_X \colon H^0(X, \Omega_X) \to \operatorname{Hom}_{\mathbf{Z}_p[G]}(T_p(X), \mathbf{C}_p(1)). \tag{5.12}$$

Proposition 5.8. *The map ϕ_X is independent of the choice of r and $\mathcal{X}$, and is functorial in X.*

Proof. The independence of the choice of r is clear. Suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are two different extensions of X satisfying the conditions above. First assume that $\mathcal{X}_2$ extends $\mathcal{X}_1$, in the sense that the identity morphism $X \to X$ extends to a morphism $\tilde{\iota} : \mathcal{X}_1 \to \mathcal{X}_2$.

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[r, "="] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} \mathcal{O}_K \arrow[r] & \mathcal{X}_1 \arrow[r, "\tilde{\iota}"] & \mathcal{X}_2.
\end{tikzcd}$$

Then the pullback of differentials $H^0(\mathcal{X}_2, \Omega_{\mathcal{X}_2}) \to H^0(X, \Omega_X)$ factors through the pullback $\tilde{\iota}^* : H^0(\mathcal{X}_2, \Omega_{\mathcal{X}_2}) \to H^0(\mathcal{X}_1, \Omega_{\mathcal{X}_1})$. Since we know that this map becomes an isomorphism after tensoring with K, we see that $\mathcal{X}_1$ and $\mathcal{X}_2$ induce the same ϕ_X. In general, we may let $\mathcal{X}_3$ be a scheme that extends both $\mathcal{X}_1$ and $\mathcal{X}_2$ by taking $\mathcal{X}_3$ to be the scheme-theoretic closure of the morphism

$$X \overset{\Delta}{\to} X \times X \to \mathcal{X}_1 \times \mathcal{X}_2.$$

By the preceding argument, the maps induced from $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ are all equal. Finally, functoriality asserts that for any morphism of abelian varieties $\varphi : X_1 \to X_2$, the following diagram commutes:

$$\begin{tikzcd}
H^0(X_2, \Omega_{X_2}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{Z}[G]}(W_p(X_2), W_p(\Omega)) \arrow[d] \\
H^0(X_1, \Omega_{X_1}) \arrow[r] & \mathrm{Hom}_{\mathbf{Z}[G]}(W_p(X_1), W_p(\Omega))
\end{tikzcd}$$

This follows from the fact that we may choose $\mathcal{X}_1$ and $\mathcal{X}_2$ such that φ extends to a morphism $\mathcal{X}_1 \to \mathcal{X}_2$, for instance by taking $\mathcal{X}_2$ by the closure of the image of X_2 in $\mathcal{X}_1 \times \mathcal{X}_2$. In this situation, the commutativity of this square follows from immediately from the definition. ☐

5.3 Proof of Theorem 5.3

We begin by showing that the passage from ϕ_X^0 to ϕ_X, which was obtained by restricting from $W_p(X)$ to $T_p(X)$, does not suffer any loss of information.

Lemma 5.9. *The homomorphism*

$$Hom_{\mathbf{Z}[G]}(W_p(X), \mathbf{C}_p(1)) \to Hom_{\mathbf{Z}[G]}(T_p(X), \mathbf{C}_p(1))$$

induced by the inclusion $T_p(X) \subset W_p(X)$ is injective.

Proof. Let $D_p(X)$ be the quotient of $X(\overline{K})$ by all of its p-power torsion, so that the following sequence is exact:

$$0 \to X[p^\infty] \to X(\overline{K}) \to D_p(X) \to 0. \tag{5.13}$$

Observe that $D_p(X)$ is a p-divisible group (in the purely group-theoretic sense) with the additional property that any $x \in D_p(X)$ has a *unique* p^{th} root. Since $W_p(D_p(x))$ consists of all p-divisible systems of elements of $D_p(X)$, we see that

$$W_p(D_p(x)) \simeq D_p(X),$$

for instance by sending $\varphi \mapsto \varphi(1)$. Therefore, applying $W_p(-)$ to the sequence (5.13) yields an exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p, X[p^\infty]) \to W_p(X) \to D_p(X) \to 0.$$

We claim that the first term is isomorphic to $V_p(x) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p(X)$. Indeed, recall that $\mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p, -) = \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}_p[p^{-1}], -)$. For any $\varphi \in \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[p^{-1}], X[p^\infty])$, the element $\varphi(1)$ lies in $X[p^r]$ for some r and the data of $x_n := \varphi(p^{-n})$ specifies a system of compatible $p^{n\text{th}}$ roots, corresponding to the element $p^{-r} \otimes \{p^r x_n\}_n \in \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p(X)$.

We have now established that the sequence

$$0 \to V_p(X) \to W_p(X) \to D_p(X) \to 0$$

is exact, so applying the functor $\mathrm{Hom}_{\mathbf{Z}[G_K]}(-, \mathbf{C}_p(1))$ leaves us with the exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{Z}[G_K]}(D_p(X), \mathbf{C}_p(1)) \to \mathrm{Hom}_{\mathbf{Z}[G_K]}(W_p(X), \mathbf{C}_p(1)) \to \mathrm{Hom}_{\mathbf{Z}[G_K]}(T_p(X), \mathbf{C}_p(1)).$$

Therefore, our task is to show that $\mathrm{Hom}_{\mathbf{Z}[G_K]}(D_p(X), \mathbf{C}_p(1)) = 0$. But

$$D_p(X) = \varinjlim_{H \lhd G_K \text{ open}} D_p(X)^H,$$

so any $\varphi \in \mathrm{Hom}_{\mathbf{Z}[G_K]}(T_p(X), \mathbf{C}_p(1))$ satisfies

$$\varphi(D_p(X)) = \varinjlim_{H \lhd G_K \text{ open}} \varphi(D_p(X)^H) \subset \varinjlim_{H \lhd G_K \text{ open}} \mathbf{C}_p(1)^H = 0$$

by Corollary 2.16. ☐

It now suffices to show that ϕ_X^0 is injective. The first step is a characterization of the completion of the local ring of $\mathcal{X}$ at a closed point. In general, we know that this is a free formal power series ring at a *smooth* point, and the following lemma shows that for any smooth point $x \in X$, some model $\mathcal{X}$ may be chosen so that this holds for the closed point in the image.

Lemma 5.10. *Let X/K be a projective variety of dimension d and $x \in X$ a smooth K-point. Then there exists a model $\mathcal{X}$ for X over $\mathcal{O}_K$ such that if $\overline{x}$ is the closed point in the closure of x in $\mathcal{X}$, then the $\mathfrak{m}_{\mathcal{X},\overline{x}}$-adic completion of $\mathscr{O}_{\mathcal{X},\overline{x}}$ is isomorphic to the ring of formal power series in d variables over $\mathcal{O}_K$.*

Proof. Let us first describe some reductions. We may assume that X is embedded in $\mathbb{P}^n_K$. By performing a linear change of coordinates if necessary, we may assume that $x = [1\colon 0\colon \dots : 0]$. Since x is smooth, we may find polynomials $F_1, \dots, F_{n-d}$ in the ideal of X that cut out the tangent space at x. Letting the coordinates of $\mathbb{P}^n_K$ be given by $x_0, \dots, x_n$, this amounts to saying that the matrix $\frac{\partial F_i}{\partial x_j}$ has rank $n - d$ at x. By performing another linear change of coordinates, we may assume that the bottom $(n-d) \times (n-d)$ matrix is invertible at x, and after performing a linear change of coordinates again, we may further assume that it is the identity matrix at x.

Let J be the ideal generated by $x_1, \dots, x_d$. By these reductions, we see that $F_i \equiv x_0^{d_i - 1} x_{i+d} \pmod{J}$, where $d_i = \deg F_i$. We will describe a rescaling of the coordinates that will modify the F_i into polynomials with coefficients in $\mathcal{O}_K$. Let ϖ be a uniformizer of $\mathcal{O}_K$, and for each i, let s_i be an integer such that $\varpi^{s_i} F_i \in \mathcal{O}_K[x_0, \dots, x_n]$. Then pick an integer s greater than all of the s_i. We define:

- $x_0 = x_0'$.
- $x_i = \varpi^{2s} x_i'$ for $i = 1, \dots, d$.
- $x_i = \varpi^{s} x_i'$ for $i = d+1, \dots, n$.

Now let $G_i = \varpi^{-s} F_i$ as polynomials in the $x_0', \dots, x_n'$. We claim that $G_i \in \mathcal{O}_K[x_0', \dots, x_i']$. Recall that we had

$$F_i(x_0, \dots, x_n) = x_0^{d_i - 1} x_{i+d} + F_i'(x_0, \dots, x_n),$$

where $F_i' \in (x_1, \dots, x_d)$. Therefore,

$$F_i(x_0', \dots, x_n') = (x_0')^{d_i - 1} \varpi^s x_i' + \varpi^{2s} F_i''(x_0', \dots, x_n')$$

where $F_i'' \in (x_1', \dots, x_d')$. Now, we assumed that $\varpi^{s_i} F$ had coefficients in $\mathcal{O}_K$, and $s > s_i$, so $\varpi^s F''$ has coefficients in $\mathcal{O}_K$. Therefore, $G_i = \varpi^{-s} F_i(x_0', \dots, x_n')$ has coefficients in $\mathcal{O}_K$. Moreover, by construction we have

$$G_i \equiv (X_0')^{d_i - 1} X_{i+d} \pmod{\varpi \mathcal{O}_K[X_1', \dots, X_n'].}$$

The choice of coordinates $x_0', \dots, x_n'$ furnishes a map

$$\text{Proj } K[x_0', \dots, x_n'] \to \text{Proj } \mathcal{O}_K[x_0', \dots, x_n'].$$

We then let $\mathcal{X}$ be the closure of X in the embedding $\mathbb{P}^n_K \to \mathbb{P}^n_{\mathcal{O}_K}$ thus obtained.

If we let $x_{i/j} := x_i / x_j$, then it is clear that the images of $x_{1/0}, \dots, x_{d/0}$ generate $\mathfrak{m}_{X,x} / \mathfrak{m}_{X,x}^2$, and $\mathscr{O}_{X,x}$ is a regular local ring of dimension d. By construction, the polynomials G_i generate the ideal of $\mathcal{X}$ locally at $\overline{x}$, and hence cut out the tangent space along with ϖ. Therefore, the images of $x_{1/0}, \dots, x_{d/0}$ along with ϖ generate $\mathfrak{m}_{\mathcal{X},\overline{x}}/\mathfrak{m}^2_{\mathcal{X},\overline{x}}$, and we see that $\mathscr{O}_{\mathcal{X},\overline{x}}$ is a regular local ring of dimension $d+1$.

The map $\mathcal{O}_K[[x_{1/0}, \dots, x_{d/0}]] \to \widehat{\mathscr{O}_{\mathcal{X},\overline{x}}}$ is a surjection of regular local rings that induces an isomorphism on cotangent spaces, hence an isomorphism of rings. □

Write $\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}} \simeq \mathcal{O}_K[[t_1, \dots, t_d]]$. Then the Kahler differentials that represent *continuous* derivations out of $\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}$ form a free $\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}$-module with basis $dt_1, \dots, dt_d$:

$$\widehat{\Omega}_{\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}/\mathcal{O}_K} \simeq \widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}\langle dt_1, \dots, dt_d \rangle.$$

This is isomorphic to the separable completion of the usual Kahler differentials:

$$\widehat{\Omega}_{\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}/\mathcal{O}_K} \simeq \varprojlim \Omega_{\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}/\mathcal{O}_K}/(\mathfrak{m}_{\mathcal{X},\overline{x}})^n \Omega_{\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}/\mathcal{O}_K}.$$

Recall that our goal is to show that the map $p^r H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K}) \to \mathrm{Hom}_{\mathbf{Z}[G_K]}(X(\overline{K}), \Omega)$ is injective for some choice of r. We establish this in two parts:

Lemma 5.11. *The natural map $p^r H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K}) \to \widehat{\Omega}_{\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}/\mathcal{O}_K}$, given by restriction to the stalk, is injective for some choice of r.*

Lemma 5.12. *The natural map $\widehat{\Omega}_{\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}/\mathcal{O}_K} \to Hom_{\mathbf{Z}[G_K]}(X(\overline{K}), \Omega)$, given by*

$$\omega \mapsto (u \mapsto u^* \omega),$$

is injective.

Proof of Lemma 5.11. By Krull's intersection theorem and the fact that Kähler differentials respect injections, the map

$$\Omega_{\mathscr{O}_{\mathcal{X},\overline{x}}/\mathcal{O}_K} \to \widehat{\Omega}_{\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}/\mathcal{O}_K}$$

is injective, so it suffices to show that the restriction map $p^r H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K}) \to \Omega_{\mathscr{O}_{\mathcal{X},\overline{x}}/\mathcal{O}_K}$ is injective when X is an abelian variety. Recall that we chose an integer r such that the natural map is an injection:

$$p^r H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K}) \subset H^0(X, \Omega_{X/K}).$$

The following is diagram of finite modules over their respective bases commutes after tensoring with K, so increasing r if necessary, we may assume that it commutes.

$$\begin{tikzcd}
p^r H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K}) \arrow[r, hook] \arrow[d] & H^0(X, \Omega_{X/K}) \arrow[d] \\
p^r \Omega_{\mathscr{O}_{\mathcal{X},\overline{x}}/\mathcal{O}_K} \arrow[r, hook] & \Omega_{\mathscr{O}_{X,x}/K}
\end{tikzcd}$$

Therefore, an element $\omega \in p^r H^0(\mathcal{X}, \Omega_{\mathcal{X}/\mathcal{O}_K})$ maps to zero in $p^r \Omega_{\mathscr{O}_{\mathcal{X},\overline{x}}/\mathcal{O}_K}$ if and only if it maps to zero in $\Omega_{\mathscr{O}_{X,x}/K}$, which happens if and only if $\omega = 0$ because ω is an invariant form on X. □

Proof of Lemma 5.12. To complete the proof of Theorem 5.3, we must establish that $\widehat{\Omega}_{\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}/\mathcal{O}_K} \to \mathrm{Hom}_{\mathbf{Z}[G_K]}(X(\overline{K}), \Omega)$ is injective. Observe that $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}_K}(\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}, \mathcal{O}_{\overline{K}})$ may be viewed as a subset of the $\mathcal{O}_K$-points of $\mathcal{X}$, which we have identified with $X(\overline{K})$, so it suffices to show that the map

$$\widehat{\Omega}_{\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}}/\mathcal{O}_K} \to \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}_K}(\widehat{\mathscr{O}}_{X,x}, \mathcal{O}_{\overline{K}}), \Omega)$$

is injective. Since a continuous map $\widehat{\mathscr{O}}_{\mathcal{X},\overline{x}} \to \mathcal{O}_{\overline{K}}$ corresponds to a choice of d elements of the maximal ideal $\mathfrak{m}_{\overline{K}}$, this map sends $\omega = \sum \alpha_i(t_1, \ldots, t_d)dt_i$ to the homomophism defined by

$$\varphi \mapsto \sum \alpha_i(\varphi(t_1), \ldots, \varphi(t_d))d\varphi(t_i).$$

We have now reduced the problem to the following algebraic lemma. □

Lemma 5.13. *Let $\omega = \sum \alpha_i(t_1, \ldots, t_g)dt_i$ be a non-zero differential, with $\alpha_i(t_1, \ldots, t_d) \in \mathcal{O}_K[[t_1, \ldots, t_d]]$. Then there exist $x_1, \ldots, x_d \in \mathfrak{m}_{\overline{K}}$ such that*

$$\sum_{i=1}^{d} \alpha_i(x_1, \ldots, x_d)dx_i \neq 0 \in \Omega.$$

Proof. We first consider the case $g = 1$. We may write

$$\omega = \sum a_i t^i dt.$$

Since the valuation on K is discrete, we may choose i such that $v(a_i) = r$ is minimal, and satisfying the condition that if $j < i$ then $v(a_j) > v(a_i)$. On the other hand, since the value group of $\overline{K}$ is $\mathbf{Q}$, we may choose $x \in \mathfrak{m}_{\overline{K}}$ to be an element of some finite extension L/K such that:

(i) $v(x) < 1/i$, and

(ii) $v(\mathcal{D}_{L/K}) > r + 1$.

For instance, we may take x to be the uniformizer in a sufficiently large cyclotomic extension. Then $iv(x) < 1$, so that $r < v(\sum a_i x^i) < r+1$. By Lemma 3.6 the annihilator of dx is $\mathcal{O}_{\overline{K}}\mathcal{D}_{L/K}$, so $a_j x^j dx = 0$ if $j \neq i$, and $a_i x^i dx \neq 0$. If $g > 1$, then the following lemma shows that we may specialize any such power series to a power series in a single variable, thus reducing to the case already considered.

Lemma 5.14. *Let $\alpha_1(x_1, \dots, x_d), \dots, \alpha_d(x_1, \dots, x_d)$ be d non-zero power series in the variables $x_1, \dots, x_d$ with coefficients in an infinite, integral commutative ring R. Then there exist formal series $\varphi_1, \dots, \varphi_d$ in a single variable t, with coefficients in R and with no constant terms, such that*

$$\sum_{i=1}^{d} \alpha_i(\varphi_1(t), \dots, \varphi_d(t))\varphi_i(t)'$$

is a non-zero power series in $R[[t]]$.

Proof. We will choose φ_i to be of the form $\varphi_i = a_i t + b_i t^2$. We consider the lowest order term in the expansion of

$$f(t) := \sum \alpha_i(a_1 t + b_1 t^2, \dots, a_d t + b_d t^2)(a_i + 2b_i t).$$

Write

$$\alpha_i = \sum_{m=0}^{\infty} \alpha_{i,m}$$

where each $\alpha_{i,m}$ is homogeneous of degree m. Let r be the least integer such that $\alpha_{i,r} \neq 0$ for some i. Writing $f(t) = \sum f_m t^m$, we compute that

$$\begin{aligned} f_r &= \sum_i a_i \alpha_{i,r}(a_1, \dots, a_d) t^r \\ f_{r+1} &= \sum_i a_i \alpha_{i,r+1}(a_1, \dots, a_d) + 2\sum_i b_i \alpha_{i,r}(a_1, \dots, a_d) \\ &\quad + \sum_{i,j} a_i b_j \frac{\partial \alpha_{i,r}}{\partial x_j}(a_1, \dots, a_d). \end{aligned}$$

Now, if the polynomial $\sum_i x_i \alpha_{i,r}(x_1, \dots, x_d)$ corresponding to f_r is non-zero, then we may find $a_1, \dots, a_d \in R$ such that $f_r(a_1, \dots, a_d) \neq 0$. This guarantees that $f(t) \neq 0$.

Otherwise, if the polynomial $\sum x_i \alpha_{i,r+1}(x_1, \dots, x_d)$ corresponding to the first summand in the expression for f_{r+1} above is non-zero, then we may similarly find $a_1, \dots, a_d$ such that $\sum a_i \alpha_{i,r+1}(a_1, \dots, a_d) \neq 0$, and by setting $b_i = 0$ for all i we again ensure that $f(t) \neq 0$.

Finally, suppose that both $\sum_i x_i \alpha_{i,r}(x_1, \ldots, x_d)$ and $\sum t\alpha_{i,r+1}(x_1, \ldots, x_d)$ are the zero polynomial. Differentiating the first expression with respect to x_j, we see that for all j

$$\alpha_{j,r}(x_1, \ldots, x_d) + \sum_i x_i \frac{\partial \alpha_{i,r}}{\partial x_j}(x_1, \ldots, x_d) = 0.$$

Substituting this equation into the expression for f_{r+1} above, we find that

$$f_{r+1} = \sum_j b_j \left(\sum_i a_i \frac{\partial \alpha_{i,r}}{\partial x_j}(a_1, \ldots, a_d) \right) = -\sum_j b_j \alpha_{j,r}(a_1, \ldots, a_d).$$

To ensure that this is non-zero, we start by choosing some j such that $\alpha_{j,r} \neq 0$, which is possible by the infinitude of the field, and set $b_j = 1$, $b_i = 0$ if $i \neq j$. The only remaining non-zero term is $\alpha_{j,r}$ and we may then choose a_i such that $\alpha_{j,r}(a_1, \ldots, a_d) \neq 0$. □

□

Chapter 6

A Comparison Theorem for Abelian Schemes

In this section, we give an alternative proof of the comparison theorem for an abelian variety with good reduction. While the existence of the Hodge-Tate decomposition in this case is already known from the result of the previous chapter, the approach taken here gives a more explicit comparison.

In Chapter 5, we saw that Tate-Raynaud Theorem is equivalent to the existence of an isomorphism of Galois modules

$$\mathbf{C}_p \otimes_K H^0(X, \Omega_X) \simeq \mathrm{Hom}_{\mathbf{Z}_p}(T_p(X), \mathbf{C}_p(1)).$$

In §5.2 we constructed a map

$$\phi_X \colon H^0(X, \Omega_X) \to \mathrm{Hom}_{\mathbf{Z}_p[G_K]}(T_p(X), \mathbf{C}_p(1)) \tag{6.1}$$

and then proceeded to show that it was an injection. From the properties of the Galois cohomology of $\mathbf{C}_p$, we then deduced that it must be an isomorphism and that the other map involved in the Hodge-Tate decomposition (5.1) could then be obtained by duality.

Observe that the Weil pairing

$$\langle \cdot, \cdot \rangle_W : X \times \widehat{X} \to \mathbf{Z}_p(1)$$

induces an isomorphism $\mathrm{Hom}_{\mathbf{Z}_p[G]}(T_p(X), \mathbf{C}_p(1)) \simeq \mathbf{C}_p \otimes T_p(\widehat{X})$. Abusing notation, let us also denote by ϕ_X the map obtained by extending scalars to $\mathbf{C}_p$ in (6.1), and identifying $\mathbf{C}_p \otimes \mathrm{Hom}(T_p(X), \mathbf{C}_p(1)) \simeq \mathbf{C}_p \otimes T_p(\widehat{X})$:

$$\phi_X \colon \mathbf{C}_p \otimes H^0(X, \Omega_X) \to \mathbf{C}_p \otimes T_p(\widehat{X}).$$

We note down, for future use, that the construction implies that for $x \in T_p(X)$ and $\omega \in H^0(X, \Omega_X)$,

$$\langle x, \phi_X(\omega)\rangle_W = \xi(x^*\omega) \in \mathbf{C}_p(1) \tag{6.2}$$

where ξ is the map defined in Definition 3.4.

We know that ϕ_X is an isomorphism, so there should be a canonical inverse map from Tate module of the dual abelian variety to the global differentials of X. In this section, we will describe an explicit inverse

$$\theta_{\widehat{X}} \colon V_{\mathbf{C}_p}(\widehat{X}) \to H^0(X, \Omega_X) \otimes \mathbf{C}_p.$$

in the case where X has good reduction. This gives an alternate, more explicit proof of the Hodge-Tate decomposition for abelian varieties in this case. The argument is due to Coleman [Col84].

By duality, it suffices to define a map

$$\theta_X \colon V_{\mathbf{C}_p}(X) \to H^0(\widehat{X}, \Omega_{\widehat{X}}) \otimes \mathbf{C}_p$$

that is inverse to $\phi_{\widehat{X}}$. The strategy is quite simple: for an element $\{x_n\}_{n=0}^{\infty} \in T_p(X)$, we wish to define a global differential on $\widehat{X}$. By the duality theory of abelian varieties, the sequence $\{x_n\}$ corresponds to a sequence of compatible torsion divisors $\{D_n\}$ on $\widehat{X}$. In particular, there are rational function f_n on $\widehat{X}$ such that $(f_n) = p^n D_n$. We show that the sequence of differentials $\{\frac{df_n}{f_n}\}$ converges in some appropriate topology, and is independent of our choice of f_n, and then set

$$\theta_X(\{x_n\}) = \lim_{n\to\infty} \frac{df_n}{f_n}.$$

That the resulting θ_X is indeed inverse to $\phi_{\widehat{X}}$ follows from a careful analysis of the Weil pairing. In particular, we shall see that the Weil pairing induces a decomposition of $\mathbf{C}_p \otimes T_p(X)^*$ that precisely corresponds to the Hodge-Tate decomposition.

6.1 Abelian schemes and logarithmic differentials

Let A be an abelian scheme over R. For a map $\operatorname{Spec} R' \to \operatorname{Spec} R$, let $A_{R'}$ denote the base change of A and $\pi_A, \pi_{R'}$ the canonical projection maps as in the following diagram.

$$\begin{tikzcd}
 & A_{R'} \arrow[dl, "\pi_{R'}"'] \arrow[dr, "\pi_A"] & \\
\operatorname{Spec} R' \arrow[dr] & & A \arrow[dl] \\
 & \operatorname{Spec} R &
\end{tikzcd}$$

We then have isomorphisms ([Har77] §II.8)

$$\Omega_{A'/R} \simeq \pi_{R'}^* \Omega_{R'/R} \oplus \pi_A^* \Omega_{A/R} \quad \text{and} \quad \pi_A^* \Omega_{A/R} \simeq \Omega_{A'/R'}.$$

The pairing we alluded to in §5.2

$$\begin{gathered} A(R') \times \Omega_{A/R'}(A) \to \Omega_R(R') \\ (x, \omega) \mapsto x^* \omega \end{gathered}$$

is easily seen to be G_K-equivariant and bilinear.

Lemma 6.1. *Let $U \subset A_{R'}$ be an open set and $x \in U(R')$, $\omega \in \Omega_{A_{R'}/R'}(U)$. Suppose that $nx \in A(R)$. Then $n(x^*\omega) = 0$.*

Proof. Since $A_{R'}$ is an abelian scheme, $\Omega_{A_{R'}/R'}$ is generated by global sections. Therefore,

$$\Omega_{A_{R'}/R'}(U) \simeq \mathscr{O}_{A_{R'}}(U) \otimes \Omega_{R'}(A') \simeq \mathscr{O}_{A_{R'}}(U) \otimes \Omega_R(A).$$

By bilinearity, $n(x^*\omega) = (nx)^*\omega$. Since $(nx) \in A(R)$, the pullback $(nx)^*\omega$ lies in $\Omega_R(R) = 0$. □

Let X/K be an abelian variety with good reduction and A/R an abelian scheme whose generic fiber is X. If f is a rational function on X, then the "logarithmic differential" $\frac{df}{f}$ is a holomorphic differential on the open set away from the divisor of (f). The theme of this section is to prove that we can extend $\frac{df}{f}$ to a larger open set on A. A key technical result is the "algebraic Hartog's Lemma," which says that any function on a normal scheme that is regular away from a codimension 2 set is regular everywhere.

Proposition 6.2. *Let A be an integrally closed noetherian domain. Then*

$$A = \bigcap_{ht\ \mathfrak{p}=1} A_{\mathfrak{p}}.$$

where the intersection is taken over all height 1 primes.

Proof. See [Har77], II.6.3A. ☐

In particular, this applies to abelian schemes, which are smooth.

Lemma 6.3. *Let D be the divisor of f on X, and $U = A - \overline{\operatorname{supp}(D)}$. Then there exists $c \in K$ such that $cf \in \mathscr{O}_A(U)^*$.*

The point is that f is invertible on the complement of a hypersurface in the geneirc, and we may extend it to an invertible function on the complement of a hypersurface on the entire abelian scheme by multiplying by a constant.

Proof. By construction, f is invertible on $U \cap A_K$, so by Proposition 6.2, it suffices to show that g is invertible in some open neighborhood of the special fiber (the inverse of g would then be a section on an open set containing all codimension ≤ 1 primes in U, and hence extend to all of U).

To this end, pick some affine neighborhood $V \simeq \operatorname{Spec} B$ of the intersection of U with the special fiber. We have $f \in (B \otimes K)^\times$, i.e. f is a invertible on the generic fiber of V; say $ff' = 1$, where $f' \in B \otimes K$. Since A is an abelian scheme, the map $R \to B$ makes B into a smooth R-algebra whose special fiber is irreducible. Because R is a discrete valuation ring, its (unique) prime ideal is generated by a uniformizer ϖ, and by the irreducibility of the special fiber, ϖB is also a prime element of B.

We may choose c, c' to be sufficiently large powers of ϖ so that cf and cf' lie in B, but are not divisible by ϖ. Then $(cf)(cf')$ is equal to a unit times some power of ϖ. The assumption that ϖB is prime implies that this cannot be a positive power, so $(cf) \in B^*$. ☐

Remark 6.4. The argument used above is easily extended to prove the lemma for abelian schemes over any unique factorization domain.

The lemma shows that $\frac{df}{f} \in \Omega_{A/R}(U)$. We next show that if (f) is divisible by n, then we can "approximate" $\frac{df}{f}$ with a global holomorphic differential modulo n.

Lemma 6.5. *Let f be a rational function on X and suppose that $(f) = nD$ for some $n \in \mathbf{Z}$. Then there exists $\omega \in \Omega_{A/R}(A)$ such that $\omega|_U - \frac{df}{f} \in n\Omega_{A/R}(U)$.*

Proof. Let $m : A \times A \to A$ denote the multiplication map and $\pi_1, \pi_2 : A \times A \to A$ the two projections. Since D is a torsion divisor, [Lan83] §IV.2.2 implies that the divisor

$$D' = m^*D - (\pi_1^*D + \pi_2^*D)$$

is principal on $A_K \times A_K$. Let $V = A \times A - \overline{\operatorname{supp}(D')}$. By the preceding lemma, there exists some $g \in \mathscr{O}_{A\times A}(V)^*$ such that $(g) = D'$. Then

$$\frac{m^* f}{(\pi_1^* f)(\pi_2^* f)} = g^n c$$

for some $c \in K^*$. Therefore

$$m^* \frac{df}{f} - \left(\pi_1^* \frac{df}{f} + \pi_2^* \frac{df}{f} \right) = n \frac{dg}{g} \text{ on V.} \tag{6.3}$$

This equation tells us that $\frac{df}{f}$ is an *invariant* differential "modulo n," hence can be extended through the group law modulo n. To make this idea precise, let $R_n = R/nR$ and for any object S over R, let S_n denote its pullback via $\operatorname{Spec} R_n \to \operatorname{Spec} R$. Defining $\nu = \left(\frac{df}{f}\right)_n$ to be the pullback of $\frac{df}{f}$ induced by the morphism $A_n \to A_n$, (6.3) implies that

$$m_n^* \nu = \pi_{1,n}^* \nu + \pi_{2,n}^* \nu.$$

Since A_n is an abelian scheme over R_n, ν extends to an invariant differential ω_n on A_n. Since the pullback induces an isomorphism $\Omega_{A/R} \simeq \Omega_{A_n/R_n}$, there is a (unique) $\omega \in \Omega_{A/R}$ mapping to ω_n.

For any flat R-algebra R', the sequence

$$0 \to \Omega_{R'/R} \xrightarrow{n} \Omega_{R'/R} \to \iota_* \Omega_{R'_n/R} \to 0$$

is short exact: indeed, $\Omega_{R'/R}$ is locally free and hence flat because R' is, so $\Omega_{R'/R}$ is torsion-free. The exact sequence for Kähler differentials (3.5) shows that $\Omega_{R'_n/R} \simeq \Omega_{R'_n/R_n}$. Therefore, there is a short exact sequence of sheaves

$$0 \to \Omega_{A/R} \xrightarrow{n} \Omega_{A/R} \to \iota_* \Omega_{A_n/R_n} \to 0$$

Since $\omega|_U - \frac{df}{f}$ maps to zero in $\iota_* \Omega_{A_n/R_n}$, it is in $n\Omega_{A/R}(U)$. □

6.2 Construction of θ_X

For the rest of the chapter, we let X be an abelian variety with good reduction over a p-adic field K and A an abelian scheme over $R = \mathcal{O}_K$ such that $X = A_K$ is its generic fiber.

Differentials of the third kind. Here we recall the notion of "differentials of the third kind," which should be intuitively thought of as describing those differentials with only simple poles. More precisely, let

$$\begin{aligned} \mathrm{dlog} : \mathscr{O}_X^\times &\to \Omega_{X/K} \\ f &\mapsto \frac{df}{f} \end{aligned}$$

be the logarithmic differentiation map. Set $\mathscr{L}$ to be the quotient sheaf of differentials by the image of dlog. Then a *differential of the third kind* is a section of $\Omega_{X/K}(U)$ whose image in $\mathscr{L}(U)$ is the restriction of a global section of $\mathscr{L}$. Thus, a differential of the third kind is locally the sum of a holomorphic differential and some global logarithmic differential.

Let L denote the K-vector space of differentials of the third kind of X and $M \subset L$ the R-submodule consisting of $\eta \in L$ for which there is a hypersurface Y on X such that η lies in the image of $\Omega_{A/R}(A - \overline{Y})$. We topologize L by declaring $\{p^n M\}_{n=0}^\infty$ to be a basis of neighborhoods of the origin. Observe that $\frac{df}{f} \in M$ for all $f \in K(A)^*$.

Lemma 6.6. $M \cap \Omega_{X/K}(X) = \Omega_{A/R}(A)$.

Proof. The inclusion $\supset$ is obvious. For the inclusion $\subset$, observe that if $\omega \in \Omega_K(X)$ and ω is defined on the complement of some hypersurface *and* on the generic fiber, then it is defined on all primes of codimension 1, hence extends to all of A by Proposition 6.2.

□

Therefore, the subspace topology on $\Omega_{X/K}(X) \subset L$ agrees with its natural p-adic topology as a finite dimensional vector space over K.

Since $T_p(A) \simeq T_p(X)$ and $\Omega_{K/X}(X) \simeq K \otimes \Omega_{A/R}(A)$, we may and will define a map $\theta_{\widehat{A}} : T_p(\widehat{A}) \to \Omega_{A/R}(A)$, and let $\theta_{\widehat{X}}$ be the map obtained from extending scalars to K.

We may now carry out the strategy outlined earlier for defining θ_X. Suppose that the $x = \{x_n\}_{n=0}^\infty$ represents an element of $T_p(\widehat{X})$ with each point defined over K. Then there is a corresponding sequence of divisors $D = \{D_n\}_{n=0}^\infty$ on X such that:

(i) $p^n D_n = (f_n)$ is principal, and

(ii) For $m \geq n$, $p^{m-n} D_m - D_n = (g_{m,n})$ is principal.

Proposition 6.7. *The limit*

$$\theta_{\widehat{A}}(x) := \lim_{n\to\infty} \frac{df_n}{f_n}$$

exists, is an element of $\Omega_{A/R}(A)$*, and is well-defined.*

Proof. By Lemma 6.5, for each n there exists $\omega_n \in \Omega_{A/R}(A)$ such that $\omega_n - \frac{df_n}{f_n} \in p^n M$. Therefore,

$$\frac{df_m}{f_m} - \frac{df_n}{f_n} = p^n \frac{dg_{m,n}}{g_{m,n}}$$

so that $\omega_m - \omega_n \in p^n \Omega_{A/R}(A)$. Since this space is p-adically complete and separated, being a finite R-module, the sequence $\{\omega_n\}_{n=0}^{\infty}$ converges to a unique differential $\omega \in \Omega_{A/R}(A)$. Therefore,

$$\theta_{\widehat{A}}(x) = \omega.$$

To see that it is well-defined, suppose that $\{D'_n\}_{n=0}^{\infty}$ is another sequence of divisors representing x and $\{f'_n\}_{n=0}^{\infty}$ is a choice of functions such that

(i) $(f'_n) = p^n D'_n$ and

(ii) $(h_n) = D_n - D'_n$.

Then

$$\frac{df_n}{f_n} - \frac{df'_n}{f'_n} = p^n \frac{dh_n}{h_n}.$$

Therefore,

$$\lim_{n\to\infty} \frac{df_n}{f_n} = \lim_{n\to\infty} \frac{df'_n}{f'_n}.$$

□

We note down an immediate consequence of this proof that describes the image of $\theta_{\widehat{A}}$ modulo powers of p.

Corollary 6.8. *With the same notation as in the proof of Theorem 6.7,*

$$\theta_{\widehat{A}}(x) \equiv \frac{df_n}{f_n} \pmod{p^n \Omega_{A/R}(A - \overline{\operatorname{supp}(D_n)})}.$$

Proof. From the construction used in the proof of Theorem 6.7, we have

$$\theta_{\widehat{A}}(x) \equiv \omega_n \quad (\text{mod } p^n \Omega_{A/R}(A).)$$

On the other hand, Lemma 6.5 shows that

$$\omega_n \equiv \frac{df_n}{f_n} \mod p^n \Omega_{A/R}(A - \overline{\operatorname{supp}(D_n)}).$$

□

We now establish a functoriality property of this construction with respect to isogenies.

Corollary 6.9. *Suppose $\alpha\colon A \to A'$ is an isogeny of abelian schemes over R and $x' \in T_p(\widehat{A'})$. then*

$$\theta_{\widehat{A}}(\widehat{\alpha}_*(x')) = \alpha^*\theta_{\widehat{A'}}(x').$$

Proof. Indeed, if the sequence of divisors $\{D'_n\}$ represent x' and $p^n D'_n = (f'_n)$, then $f_n := \alpha^* f'_n$ has divisor $p^n \widehat{\alpha}_*(D'_n)$, where the sequence of divisors D'_n represent $\widehat{\alpha}_*(x')$. Therefore, we may use the functions $\frac{df_n}{f_n} = \alpha^* \frac{df'_n}{f'_n}$ in the construction of $\theta_{\widehat{A'}}$, yielding $\alpha^*\theta_{\widehat{A'}}(x)$. $\square$

6.3 Interaction with the Weil pairing

Our next step is to understand the interaction between the maps defined and the Weil pairing, from which the Hodge-Tate decomposition falls out as an orthogonal decomposition. This is the most intricate part of the proof.

For the rest of this chapter, we let $R^a = \mathcal{O}_{\overline{K}}$ and $A_{R^a} = A_{R^a}$.

Zero-cycles and Lang Reciprocity

Definition 6.10. A **zero-cycle** $\mathfrak{c}$ on $A(R)$ is a formal finite linear combination over $\mathbf{Z}$ of points in $A(R)$

$$\mathfrak{c} = \sum_{x \in A(R)} n_x x.$$

where all but finitely many n_x are zero.

We also fix the following notations for a zero-cycle $\mathfrak{c} = \sum_{x \in A(R)} n_x x$.

- $\mathrm{supp}(\mathfrak{c}) = \{x \in A(R) \colon n_x \neq 0\}$,
- $\deg \mathfrak{c} = \sum_x n_x \in \mathbf{Z}$, and
- $S(\mathfrak{c}) = \sum_x n_x x \in A(R)$ is the sum of the points of $\mathfrak{c}$, taken with multiplicity, in the group law of A.
- If $f \in K(A)^*$ satisfies $\overline{\mathrm{supp}(f)} \cap \mathrm{supp}\, \mathfrak{c} = \emptyset$, then

$$f(\mathfrak{c}) = \prod_{x \in A(R)} f(x)^{n_x}.$$

Since all but finitely many factors of the product are 1, this is well-defined.

Now let A and $\widehat{A}$ be dual abelian schemes and D a Poincaré divisor on the generic fiber $X \times \widehat{X}$.

Definition 6.11. If $\mathfrak{c}$ is a zero cycle on $\widehat{A}_{R^a}$, then we let $D(\mathfrak{c})$ be the divisor on A_{R^a} given by

$$D(\mathfrak{c}) := \pi_1(D \cdot \pi_2^{-1}(\mathfrak{c}))$$

if the right hand side is defined, where π_1 and π_2 are the obvious projection maps on $A_{R^a} \times \widehat{A}_{R^a}$ and $\cdot$ is the intersection product.

Observe that if $D(\mathfrak{c})$ is defined, then $S(\mathfrak{c}) = 0$ if and only if $D(\mathfrak{c})$ is a principal divisor. Indeed, since $\widehat{A}$ parametrizes isomorphism classes of line bundles on A, the property that $S(\mathfrak{c}) = 0$ is equivalent to the tensor product of the corresponding line bundles being trivial.

Theorem 6.12 (Lang reciprocity)**.** *Let A, B be abelian varieties, D a divisor on $A \times B$, and $\mathfrak{a}$ and $\mathfrak{b}$ zero-cycles on A and B, respectively, satisfying $S(\mathfrak{a}) = S(\mathfrak{b}) = 0$. If $(\mathfrak{a} \times \mathfrak{b}) \cap D = \emptyset$, then $D(\mathfrak{a})$ and $D(\mathfrak{b})$ are defined, and for any $f \in K(B)^*$ satisfying $(f) = D(\mathfrak{a})$ and g in $K(A)^*$ satisfying $(g) = D(\mathfrak{b})$, we have*

$$f(\mathfrak{b}) = g(\mathfrak{a}).$$

Proof. See [Lan83], VI.4.9. ☐

Given a subset $S \subset A(\mathbf{C}_p)$ and an integer $n \in \mathbf{Z}$, we denote

$$nS = \{nx \colon x \in S\} \subset A(\mathbf{C}_p)$$

(here nx is multiplication by n in the group law on $A(\mathbf{C}_p)$). Recall that for a scheme A/R, and $\operatorname{Spec} R' \to \operatorname{Spec} R$ a map of affine schemes, we have

$$\Omega_{A_{R'}/R} \simeq \pi_{R'}^* \Omega_{R'/R} \oplus \pi_A^* A/R.$$

Abusing notation, let the projection map $\Omega_{A_{R'}/R'} \to \pi_{R'}^* \Omega_{R'/R}$ also be denoted $\pi_{R'}$ and the projection map $\Omega_{A_{R'}/R'} \to \pi_A^* \Omega_{A/R}$ also be denoted π_A.

For the next proposition, we adopt the following conventions to simplify notation.

- For $R = \mathcal{O}_K$, we let $R^a = \mathcal{O}_{\overline{K}}$.
- We let $d_{A^a} f \in \Omega_R(A_{R^a})$ denote the differential of f with respect to A_{R^a} and $d_A(f)$ denote the differential of f with respect to A. For a function f on A_{R^a} that is defined over R, we may also consider f as a function on A, in which case the two differentials are related by

  $$d_{A^a} f = \pi_A^* d_A(f),$$

  since $\pi_{R^a} d_{A^a} f = 0$ because f is defined over R.

Proposition 6.13. *Suppose* $\mathfrak{a}$ *is a zero cycle on* A_{R^a} *such that* $\deg \mathfrak{a} = 0$*,* $p^n S(\mathfrak{a}) = 0$*,* $p^n \operatorname{supp}(\mathfrak{a}) \subset A(K^{\mathrm{unr}})$*, and* $D(\mathfrak{a})$ *is defined. Let* $U = A_{R^a} - \overline{\operatorname{supp} D(\mathfrak{a})}$*. Then for any* $f \in \mathscr{O}_{A_{R^a}}(U)^\times$ *such that* $(f) = p^n D(\mathfrak{a})$*, we have*

$$\pi_{R^a} \frac{d_{A^a} f}{f} \in \Omega_{R^a/R}(R^a)$$

where $\frac{d_{A^a} f}{f}$ *is considered as an element of* $H^0(U, \Omega_{A_{R^a}/R})$*.*

The point of the proposition is that a priori,

$$\pi_{R^a} \frac{d_{A^a} f}{f} \in \pi_{R^a}^* \Omega_{R^a/R}(U) \simeq \mathscr{O}_{A_{R^a}}(U) \otimes \Omega_{R^a/R}$$

but it in fact lies in the image of $\Omega_{R^a/R}$ in the tensor product.

Proof. The proof is done in two steps.

Lemma 6.14. *For any zero-cycle* $\mathfrak{b}$ *on* A *satisfying*

$$\deg \mathfrak{b} = 0, \qquad S(\mathfrak{b}) = 0, \qquad \textit{and} \qquad \operatorname{supp} \mathfrak{b} \subset U(R) \tag{6.4}$$

we have

$$\mathfrak{b}^* \pi_{R^a} \frac{d_{A^a} f}{f} = 0. \tag{6.5}$$

Proof. Let $\mathfrak{b}$ be a zero-cycle on A satisfying (6.4). Then $D(\mathfrak{b})$ is principal since $S(\mathfrak{b}) = 0$, and if we set $U_{\mathfrak{b}} = \widehat{A} - \overline{\operatorname{supp} D(\mathfrak{b})}$, then $\operatorname{supp} \mathfrak{a} \subset U_{\mathfrak{b}}(R^a)$. We choose $g \in \mathscr{O}_{\widehat{A}}(U_{\mathfrak{b}})^\times$ defined over R such that $(g) = D(\mathfrak{b})$, which is possible since $\mathfrak{b}$ is defined over R. By the Lang Reciprocity (Theorem 6.12),

$$f(\mathfrak{b}) = g(p^n \mathfrak{a}) = g(\mathfrak{a})^{p^n}.$$

Therefore,

$$\frac{d_{A^a} f(\mathfrak{b})}{f(\mathfrak{b})} = p^n \frac{d_{A^a} g(\mathfrak{a})}{g(\mathfrak{a})}.$$

The preceding equation may be re-written as

$$\mathfrak{b}^* \frac{d_{A^a} f}{f} = p^n \mathfrak{a}^* \frac{d_{A^a} g}{g} \tag{6.6}$$

Because g is defined over R,

$$\frac{d_{A^a} g}{g} = \pi_A^* \frac{d_A g}{g} \in \Omega_{A_{R^a}/R^a}(U_{\mathfrak{b}}).$$

Since $p^n \operatorname{supp}(\mathfrak{a}) \subset A(K)$, Lemma 6.1 implies that $p^n(\mathfrak{a}^* \frac{d_A g}{g}) = 0$. Therefore, (6.6) implies that

$$\mathfrak{b}^* \frac{d_{A^a} f}{f} = 0. \tag{6.7}$$

Since $\operatorname{supp} \mathfrak{b} \subset U(R)$ by assumption, Lemma 6.1 again implies that

$$\mathfrak{b}^* \pi_A \frac{d_{A^a} f}{f} = 0.$$

Subtracting this from (6.7) yields

$$\mathfrak{b}^* \pi_{R^a} \frac{d_{A^a} f}{f} = 0, \tag{6.8}$$

as desired. □

Recall that $\Omega_R(R^{\mathrm{unr}}) = 0$. The tower of inclusions $R \subset R^{\mathrm{unr}} \subset R^a$ induces an exact sequence

$$R^a \otimes \Omega_{R^{\mathrm{unr}}/R} \to \Omega_{R^a/R} \to \Omega_{R^a/R^{\mathrm{unr}}} \to 0,$$

which exhibits the isomorphism $\Omega_{R^a/R} \simeq \Omega_{R^a/R^{\mathrm{unr}}}$. Therefore, we may base-change to assume that $K = K^{\mathrm{unr}}$, so that the residue field k is algebraically closed. The proof is then concluded by the next Lemma.

Lemma 6.15. *Suppose that the residue field k of K is algebraically closed. Let $U \subset A_{R^a}$ be an open subset and suppose $\omega \in \pi_{R^a}^* \Omega_{R^a/R}$ has the property that for all zero-cycles $\mathfrak{b}$ on A satisfying*

$$\deg \mathfrak{b} = 0, \qquad S(\mathfrak{b}) = 0, \qquad \textit{and} \qquad \operatorname{supp} \mathfrak{b} \subset U(R) \tag{6.9}$$

we have $\mathfrak{b}^*\omega = 0$. Then $\omega \in \Omega_{R^a/R}$.

Proof. Let V be an affine open subscheme of A_{R^a} contained in U. Since $\Omega_{R^a/R}$ is a projective limit of cyclic modules, ω may be written as $g\nu$ for some $\nu \in \Omega_{R^a/R}(R^a)$ and $g \in \mathscr{O}_{A_{R^a}}(V)$. The annihilator of ν in R^a is ηR^a for some $\eta \in R^a$. Letting $V(R) = V(R^a) \cap A(R)$, the previous Lemma implies that for any $\mathfrak{b}$ satisfying (6.4), we have

$$\sum_{b \in \mathfrak{b}} n_b g(b) \nu(b) \equiv 0 \pmod{\eta R^a} \tag{6.10}$$

Since $\deg \mathfrak{b} = 0$, this equation is certainly satisfied if g is constant modulo $\eta\mathscr{O}_{A_{R^a}}(V)$; we claim that this condition is also necessary. From that it follows that $g\nu$ is just an R^a multiple of ν. Since this holds on every affine patch of U and U is connected, we may then conclude that $\omega \in \Omega_{R^a/R}(R^a)$.

Now it suffices to establish the claim. If the special fiber of V is empty, then $\mathscr{O}_{A_{R^a}}(V)$ is a $\overline{K}$-vector space, so $\eta\mathscr{O}_{A_{R^a}}(V) = \mathscr{O}_{A_{R^a}}(V)$. In this case, the claim is trivial. Otherwise, there exists some $c \in V(R)$ since R is Henselian and the residue field is algebraically closed. Let $h = T_c^*(g) - g(c)$ be a function on $V_c := T_c^{-1}(V)$. If a, b are also points in $V_c(R)$ satisfying $a + b \in V_c(R)$, then $a + c, b + c, a + b + c$ are points of $V(R)$ and the zero-cycle

$$\mathfrak{b} = 1(a + b + c) + 1(c) - 1(a + c) - 1(b + c)$$

is a zero-cycle in $V(R)$ satisfying (6.4), implying by (6.10) that

$$h(a + b) \equiv h(a) + h(b) \pmod{\eta R^a}.$$

Now the idea is similar to that in the proof of Lemma 6.5: we may extend h to a global invariant function on the scheme "modulo η." Indeed, let $R_\eta = R^a/\eta R^a$ and for an object S over R, let S_η denote its pullback over R_η. By [Gre66], or the more general Artin approximation theorem [Art69], V_δ is a dense open subscheme of A_δ over R_δ. Let $W = m^{-1}(V) \cap \pi_1^{-1}(V) \cap \pi_2^{-1}(V)$, and open subscheme of A. Then the function

$$m_\delta^* h_\delta - (\pi_{1,\delta}^* h_\delta + \pi_{2,\delta}^* h_\delta)$$

vanishes on the image of $W(R)$ in $W_\delta(R_\delta)$, which is dense (again by [Gre66]), and is therefore zero on all of W_δ. Therefore, the function h_δ may be extended to an invariant function on A_δ via the group law, satisfying

$$m_\delta^* h_\delta = \pi_{1,\delta}^* h_\delta + \pi_{2,\delta}^* h_\delta.$$

However, the only such function is zero. □

□

An alternate characterization of the Weil pairing.

Let

$$\langle \cdot, \cdot \rangle_W : T_p(A) \times T_p(\widehat{A}) \to T_p(\mathbb{G}_m)$$

denote the Weil pairing. Theorem 11 of [Lan83] gives the following construction of this pairing. If $u = (u_n)_{n=0}^{\infty} \in T_p(A)$ and $v = (v_n)_{n=0}^{\infty} \in T_p(\widehat{B})$ are two elements, then let $\{\mathfrak{u}_n\}_{n=0}^{\infty}$ and $\{\mathfrak{v}_n\}_{n=0}^{\infty}$ be two sequences of zero-cyces on $X_{\overline{K}}$ and $\widehat{X}_{\overline{K}}$ such that

$$\deg \mathfrak{u}_n = \deg \mathfrak{v}_n = 0$$

$$S(\mathfrak{u}_n) = u_n \qquad \text{and} \qquad S(\mathfrak{v}_n) = v_n,$$

and

$$\operatorname{supp}(\mathfrak{u}_n \times \mathfrak{v}_n) \cap \operatorname{supp} D = \emptyset.$$

Then $D(\mathfrak{u}_n)$ and $D(\mathfrak{v}_n)$ are defined and

$$\operatorname{supp}(\mathfrak{u}_n) \cap \operatorname{supp} D(\mathfrak{v}_n) = \operatorname{supp} \mathfrak{v}_n \cap \operatorname{supp} D(\mathfrak{u}_n) = \emptyset.$$

If $\{f_{\mathfrak{u}_n}\}_{n=0}^{\infty}$ and $\{f_{\mathfrak{v}_n}\}_{n=0}^{\infty}$ are sequences of rational functions on $\widehat{A}_{\overline{K}}$ and $A_{\overline{K}}$ satisfying

$$(f_{\mathfrak{u}_n}) = p^n D(\mathfrak{u}_n), \qquad (f_{\mathfrak{v}_n}) = p^n D(\mathfrak{v}_n),$$

then

$$(\langle u, v\rangle_W)_n = f_{\mathfrak{v}_n}(\mathfrak{u}_n) f_{\mathfrak{u}_n}(\mathfrak{v}_n)^{-1}. \tag{6.11}$$

Theorem 6.16. *With the notation above,*

$$\langle u, v\rangle_W^* \frac{dT}{T} = u^*\theta_{\widehat{A}}(v) - v^*\theta_A(u).$$

Proof. We will compute $(\langle u, v\rangle_W)_n^* \frac{dT}{T}$ for each n and compare it to the terms in a limiting sequence for $u^*\theta_{\widehat{A}}(v) - v^*\theta_A(u)$. To that end, fix n and let $\mathfrak{u} = \mathfrak{u}_n$ and $\mathfrak{v} = \mathfrak{v}_n$ be a choice of divisors as in the construction of the Weil pairing, satisfying

$$p^n \operatorname{supp} \mathfrak{u} \subset A(K^{\mathrm{unr}})$$
$$p^n \operatorname{supp} \mathfrak{v} \subset A(K^{\mathrm{unr}})$$
$$\operatorname{supp}(\mathfrak{u} \times \mathfrak{v}) \cap \overline{\operatorname{supp}(D)} = \emptyset$$

where D is the Poincaré divisor.

Let $U_{\mathfrak{u}} = \widehat{A}_{R^a} - \overline{\operatorname{supp} D(\mathfrak{u})}$ and $U_{\mathfrak{v}} = A_{R^a} - \overline{\operatorname{supp} D(\mathfrak{v})}$. Then by the third condition, we have $\operatorname{supp} \mathfrak{u} \subset U_{\mathfrak{v}}(R^a)$ and $\operatorname{supp} \mathfrak{v} \subset U_{\mathfrak{u}}(R^a)$. Therefore, we may choose $f_{\mathfrak{u}} \subset \mathscr{O}_{\widehat{A}_{R^a}}(U_{\mathfrak{u}})^\times$ and $f_{\mathfrak{v}} \in \mathscr{O}_{A_{R^a}}(U_{\mathfrak{v}})^\times$, so that $f_{\mathfrak{v}}(\mathfrak{u})$ and $f_{\mathfrak{u}}(\mathfrak{v})$ are both units of R^a. By the characterization of the Weil pairing in (6.11), it follows that

$$(\langle \mathfrak{u}, \mathfrak{v}\rangle)_n^* \frac{dT}{T} = \frac{df_{\mathfrak{v}}(\mathfrak{u})}{f_{\mathfrak{v}}(\mathfrak{u})} - \frac{df_{\mathfrak{u}}(\mathfrak{v})}{f_{\mathfrak{u}}(\mathfrak{v})} = \mathfrak{u}^* \frac{df_{\mathfrak{v}}}{f_{\mathfrak{v}}} - \mathfrak{v}^* \frac{df_{\mathfrak{u}}}{f_{\mathfrak{u}}}.$$

By Proposition 6.13, $\pi_{R^a}\frac{df_\mathfrak{u}}{f_\mathfrak{u}} \in \Omega_{R^a/R}$. Note that

$$\mathfrak{u}^*\frac{df_\mathfrak{v}}{f_\mathfrak{v}} = \mathfrak{u}^*\pi_{R^a}\frac{df_\mathfrak{v}}{f_\mathfrak{v}} + \mathfrak{u}^*\pi_A\frac{df_\mathfrak{v}}{f_\mathfrak{v}}. \tag{6.12}$$

By definition of the pullback map $\pi_{R^a}^* : \Omega_{R^a/R} \to \Omega_{A_{R^a}/R^a}(A_{R^a})$, the composition of $\pi_{R^a}^*$ with pullback via an R^a-point $\operatorname{Spec} R^a \to A$ is the identity on $\Omega_{R^a/R}$ by the commutativity of the diagram:

$$\begin{tikzcd}
\operatorname{Spec} R^a \arrow[r] \arrow[dr, "\mathrm{id}"] & A_{R^a} \arrow[d] \\
 & \operatorname{Spec} R^a \arrow[d] \\
 & \operatorname{Spec} R
\end{tikzcd}$$

In particular, since $\deg \mathfrak{u} = 0$, we have $\mathfrak{u}^*\pi_{R^a}\frac{df_\mathfrak{v}}{f_\mathfrak{v}} = 0$. Therefore, (6.12) implies that

$$\mathfrak{u}^*\frac{df_\mathfrak{v}}{f_\mathfrak{v}} = \mathfrak{u}^*\pi_A\frac{df_\mathfrak{v}}{f_\mathfrak{v}}.$$

Now Corollary 6.8 implies that

$$\pi_A\frac{df_\mathfrak{v}}{f_\mathfrak{v}} - \theta_{\widehat{A}}(v) \in p^n\Omega_{A_{R^a}}(U_\mathfrak{b}).$$

Since $p^n \operatorname{supp}(\mathfrak{u}) \subset A(K^{\mathrm{unr}})$, Lemma 6.1 implies that $\mathfrak{u}^*$ kills $p^n\Omega_{A_{R^a}/R^a}(U_\mathfrak{b})$, so that

$$\mathfrak{u}^*\frac{df_\mathfrak{v}}{f_\mathfrak{v}} = \mathfrak{u}^*\theta_{\widehat{A}}(v).$$

The equation

$$\mathfrak{v}^*\frac{df_\mathfrak{u}}{f_\mathfrak{u}} = \mathfrak{v}^*\theta_A(u)$$

is obtained in an exactly analogous manner, completing the proof. $\square$

6.4 The Decomposition Theorem

With the results at our disposal, the Hodge-Tate decomposition can now be deduced using linear algebra.

Definition 6.17. Let $\rho_A = \phi_{\widehat{A}} \circ \theta_A : V_{\mathbf{C}_p}(A) \to V_{\mathbf{C}_p}(A)$.

Corollary 6.18. *Let $u \in T(A)$ and $v \in T(\widehat{A})$. Then*

$$\langle u, v\rangle_W = \langle \rho_A(u), v\rangle_W \langle v, \rho_{\widehat{A}}(u)\rangle_W.$$

Proof. Theorem 6.16 shows that

$$\langle u,v\rangle_W^* \frac{dT}{T} = u^*\theta_{\widehat{A}}(v) - v^*\theta_A(u).$$

By (6.2),

$$u^*\theta_A(v) = \langle u, \phi_A \circ \theta_{\widehat{A}}(v)\rangle_W \frac{dT}{T} = \langle u, \rho_{\widehat{A}}(v)\rangle_W \frac{dT}{T}$$

and

$$v^*\theta_A(u) = \langle v, \phi_{\widehat{A}} \circ \theta_A(u)\rangle_{\widehat{W}} \frac{dT}{T} = \langle v, \rho_A(u)\rangle_{\widehat{W}} \frac{dT}{T}$$

where $\widehat{W}$ denotes the Weil Pairing on $\widehat{A}\times A$. Therefore, $\langle u,v\rangle_W$ and $\langle u,\rho_{\widehat{A}}(v)\rangle_W(\langle v,\rho_A(u)\rangle_{\widehat{W}})^{-1}$ have the same effect in pulling back $\frac{dT}{T}$, so Corollary 3.13 implies that they are equal:

$$\langle u,v\rangle_W = \langle u,\rho_{\widehat{A}}(v)\rangle_W(\langle v,\rho_A(u)\rangle_{\widehat{W}})^{-1}.$$

The result follows from the fact that $(\langle v,u\rangle_{\widehat{W}})^{-1} = \langle u,v\rangle_W$. ☐

Theorem 6.19. *We have* $V_{\mathbf{C}_p}(A) \simeq Im\,\phi_{\widehat{A}} \oplus \ker\theta_A$*, and moreover*

(i) θ_A *gives a Galois-equivariant isomorphism* $Im\,\phi_{\widehat{A}} \simeq \mathbf{C}_p \otimes H^0(\widehat{X}, \Omega_{\widehat{X}/K})$*, and*

(ii) $\ker\theta_A \simeq \mathbf{C}_p \otimes H^1(\widehat{X}, \mathscr{O}_{\widehat{X}})(1)$ *(as Galois modules).*

This theorem establishes the Hodge-Tate decomposition in a somewhat more explicit manner. We ease into the proof by making a series of fairly simple observations. Let $W_1(A)$ denote the kernel of ρ_A and $W_0(A)$ the image.

Lemma 6.20. *Let* $\dim A = g$*. Then*

(i) $W_1(\widehat{A}) = W_1(A)^\perp$ *under the Weil pairing* $\langle\cdot,\cdot\rangle_W$*, and*

(ii) $\dim W_0(A) = \dim W_1(A) = g$*.*

Proof. If $u \in W_1(A)$ and $v \in W_1(\widehat{A})$, then Corollary 6.18 shows that

$$\langle u,v\rangle = \langle\rho_A(u),v\rangle\langle u,\rho_{\widehat{A}}(v)\rangle = 1.$$

Therefore, $W_1(\widehat{A}) \subset W_1(A)^\perp$. If $\alpha : A \to \widehat{A}$ is any polarization, then Corollary 6.8 implies that

$$\rho_{\widehat{A}} \circ \alpha_* = \alpha_* \circ \rho_A,$$

where α_* is the induced homomorphism $T(A) \to T(\widehat{A})$. Therefore,

$$\dim_{\mathbf{C}_p} W_i(A) = \dim_{\mathbf{C}_p} W_i(\widehat{A}).$$

Since ρ_A factors through θ_A, which maps to the g-dimensional space $\Omega_{\mathbf{C}_p}(\widehat{A}_{\mathbf{C}_p})$, we see that $\dim W_1(A) \geq g$ and $\dim W_1(\widehat{A}) \geq g$. However, our earlier observation that $W_1(\widehat{A}) \subset W_1(A)^\perp$ shows that

$$g \leq \dim_{\mathbf{C}_p} W_1(\widehat{A}) \leq \dim_{\mathbf{C}_p} W_1(\widehat{A})^\perp = 2g - \dim_{\mathbf{C}_p} W_1(A) \leq g.$$

Therefore, we must equality everywhere, which immediately implies (i) and (ii). □

Lemma 6.21. *Under the Weil pairing $\langle \cdot, \cdot \rangle_W$, we have $W_0(\widehat{A}) = W_0(A)^\perp$.*

Proof. Since the dimensions of the two spaces are equal, it suffices to show that $W_0(\widehat{A}) \subset W_0(A)^\perp$. Because ρ_A factors through $\Omega_K(X) \otimes \mathbf{C}_p$, we have $W_0(A) \subset V_{\mathbf{C}_p}(A)^G \otimes \mathbf{C}_p$. Let $u \in V_{\mathbf{C}_p}(A)^G$ and $v \in V_{\mathbf{C}_p}(\widehat{A})^G$. Then

$$\langle u, v \rangle_W = \langle u^\sigma, v^\sigma \rangle_W = \langle u, v \rangle_W^\sigma = \chi(\sigma) \langle u, v \rangle_W$$

by the Galois-equivariance of the Weil pairing. But since K is discrete, it contains only finitely many p-power roots of unity, so the cyclotomic character χ is nontrivial on G_K, implying that $\langle u, v \rangle = 0$. □

Proposition 6.22. *The map ρ_A is a projection onto $W_0(A)$, so we have $V_{\mathbf{C}_p}(A) \simeq W_0(A) \oplus W_1(A)$.*

Proof. It suffices to show that ρ_A fixes $W_0(A)$. Let $u \in W_0(A)$ and $v \in V_{\mathbf{C}_p}(\widehat{A})$. According to Corollary 6.18,

$$\langle u, v \rangle_W = \langle \rho_A(u), v \rangle_W \langle u, \rho_{\widehat{A}}(v) \rangle_W.$$

By Lemma 6.21, $\langle u, \rho_{\widehat{A}}(v) \rangle_W = 1$, so we conclude that

$$\langle u, v \rangle_W = \langle \rho_A(u), v \rangle_W \text{ for all } v \in V_{\mathbf{C}_p}(\widehat{A}).$$

By the non-degeneracy of the Weil pairing, we may conclude that $u = \rho_A(u)$. □

We may finally complete the proof of the Hodge-Tate Decomposition.

Proof of Theorem 6.19. Since $\rho_A = \phi_{\widehat{A}} \circ \theta_A$ has rank g, we see that $\phi_{\widehat{A}}$ is an isomorphism, so that $W_1(A) = \ker \theta_A$ and $W_0(A) = \operatorname{Im} \phi_{\widehat{A}}$. This establishes the direct sum decomposition of Theorem 6.19.

Since ρ_A restricts induces the identity transformation on $W_0(A)$ by Proposition 6.22, θ_A induces an isomorphism and we may conclude (ii). By Lemma

6.21, the Weil pairing induces a G_K-equivariant perfect pairing

$$W_1(A) \times W_0(\widehat{A}) \to \mathbf{C}_p(1),$$

and hence an isomorphism

$$\begin{aligned} W_1(A) &\simeq \mathrm{Hom}_{\mathbf{C}_p[G_K]}(W_0(\widehat{A}), \mathbf{C}_p(1)) \\ &\simeq \mathrm{Hom}_{\mathbf{C}_p[G_K]}(W_0(\widehat{A}), \mathbf{C}_p)(1) \\ &\simeq \mathrm{Hom}_{\mathbf{C}_p[G_K]}(\mathbf{C}_p \otimes H^0(\widehat{X}, \Omega_{\widehat{X}/K}), \mathbf{C}_p)(1) \\ &\simeq H^1(\widehat{X}, \mathscr{O}_{\widehat{X}/K}) \otimes \mathbf{C}_p(1) \end{aligned}$$

□

Part III

A Vista of Period Rings

Chapter 7

The Formalism of Period Rings

In classical Hodge theory, the periods, or integrals of algebraic forms, arise in the comparison theorem between singular and de Rham cohomology. We have just seen the analogous comparison isomorphism for p-adic abelian varieties, and we may analogously ask about the "periods" of this comparison.

This line of inquiry turns out to be incredibly rich. Fontaine introduced a sequence of "period rings" that refine the comparison isomorphism just established, and capture additional structure coming from geometry. For instance, the Hodge-Tate decompostion involves a splitting and is refined by the notion of de Rham representations, which possess a filtration corresponding to the filtration on algebraic de Rham cohomology, and which are in turn refined by the notion of crystalline representations, possessing a Frobenius endomorphism. In this chapter of the story, we will introduce the formalism of Fontaine's theory of *period rings.*

7.1 Regular rings and admissible representations

Let G be a topological group and B a topological commutative ring compatible with a continuous action of G: for all $g \in G$ and $b_1, b_2 \in B$,

$$\begin{aligned} g(b_1 + b_2) &= g(b_1) + g(b_2) \\ g(b_1 b_2) &= g(b_1) g(b_2). \end{aligned}$$

Example 7.1. The basic model is where G acts on B as the Galois group of some field extension. For instance:

- $B = L/K$ is a Galois extension and $G = \mathrm{Gal}(L/K)$ the Galois group, both possessing the discrete topology.

- $B = L/K$ is a Galois extension of local fields, equipped with the p-adic topology, and $G = \mathrm{Gal}(L/K)$ is the Galois group, equipped with the profinite topology.

Let $E = B^G$ and F a closed subfield of E. Note that if B is a domain, then there is a natural extension of the action of G to $\mathrm{Frac}B$ given by

$$g\left(\frac{b_1}{b_2}\right) = \frac{g(b_1)}{g(b_2)}.$$

Definition 7.1. We say that B is an (F, G)-**regular** ring if it satisfies the following conditions

1. B is a domain,
2. $B^G = (\mathrm{Frac}B)^G$, and
3. if $b \in B$ is an element such that the F-vector space spanned by b is stable under G, then $b \in B^\times$.

Remark 7.2. The third condition immediately implies that $E = B^G$ is a field. It is clear that all of the conditions are satisfied if B is a field.

Let $\mathbf{Rep}_F(G)$ denote the category of continuous F-representations of G, in the sense described above.

Definition 7.3. In the setting above, we define the functor D_B from $\mathbf{Rep}_F(G)$ to the category of E-vector spaces by

$$D_B(V) = (B \otimes_F V)^G.$$

There is a natural map

$$\begin{aligned} \alpha_V : B \otimes_E D_B(V) &\to B \otimes_F V \\ \lambda \otimes x &\mapsto \lambda x. \end{aligned}$$

Proposition 7.4. *If B is an (F, G)-regular ring, then the map α_V is injective for all $V \in \boldsymbol{Rep}_F(G)$. In particular,*

$$\dim_E D_B(V) \leq \dim_F V.$$

Proof. The inclusion $B \otimes V \subset \mathrm{Frac}(B) \otimes V$ induces an inclusion $D_B(V) \subset D_{\mathrm{Frac}(B)}(V)$, so we have a commutative diagram

$$\begin{tikzcd}
B \otimes_E D_B(V) \arrow[r, "\alpha_V"] \arrow[d] & B \otimes_F V \arrow[d] \\
\mathrm{Frac}(B) \otimes_E D_{\mathrm{Frac}(B)}(V) \arrow[r, "\beta_V"] & \mathrm{Frac}(B) \otimes_F V
\end{tikzcd}$$

Since V is an F-vector space, the map $B \otimes_F V \to \mathrm{Frac}(B) \otimes_F V$ is an injection. Similarly, noting that $(\mathrm{Frac} B)^G = E$ by definition, the composition

$$B \otimes_E D_B(V) \to B \otimes_E D_{\mathrm{Frac}(B)}(V) \to \mathrm{Frac}(B) \otimes_E D_{\mathrm{Frac}(B)}(V)$$

is an injection. By Proposition 5.5, the map β_V is an injection as well, hence so is α_V. □

Definition 7.5. If B is an (F, G)-regular ring and $V \in \mathbf{Rep}_F(G)$ satisfies

$$\dim_E D_B(V) = \dim_F V$$

then we say that V is a B-**admissible** representation.

Proposition 7.6. *V is B-admissible if and only if α_V is an isomorphism.*

Proof. If α_V is an isomorphism, then $D_B(V)$ has dimension at least the B-rank of $B \otimes_F V$, which is $\dim_F V$.

Conversely, suppose that V is B-admissible. Let $e_1, \ldots, e_n$ be a basis for $D_B(V)$ over E and $v_1, \ldots, v_n$ a basis for V over F. Identifying these elements with their images in $B \otimes_F V$, we have $e_i = \sum b_{ij} v_j$, and $\det(b_{ij}) = b \neq 0$ because α_V is injective. Setting $e = e_1 \wedge \ldots \wedge e_n$ and $v = v_1 \wedge \ldots \wedge v_n$, we have

$$e = bv.$$

For any $g \in G$, we have $g \cdot e = e$ by definition, and $g \cdot v = \chi(g) v$ where $\chi : G \to F^\times$ is the determinantal representation obtained from V. Therefore, applying g to both sides of the equation yields

$$e = (g \cdot b)\chi(g) v \implies g \cdot b = \chi(g)^{-1} b,$$

so the F-line generated by b is stable under G. Since B is regular, $b \in B^\times$.

□

7.2 $\mathbf{Rep}_F^B(G)$ is Tannakian

We now axiomatize general properties of admissible representations under operations of such as dualizing and forming direct sums and tensor products.

Theorem 7.7. *Let $\mathscr{C} := \boldsymbol{Rep}_F^B(G)$ be the subcategory of B-admissible representations. Then $\boldsymbol{Rep}_F^B(G)$ is a Tannakian category, i.e.*

(i) $\mathscr{C}$ contains the trivial representation.

(ii) If $V \in \mathscr{C}$ and $V' \subset V$, then $V' \in \mathscr{C}$ and $V/V' \in \mathscr{C}$.

(iii) If $V_1, V_2 \in \mathscr{C}$, then $V_1 \oplus V_2 \in \mathscr{C}$.

(iv) If $V_1, V_2 \in \mathscr{C}$, then $V_1 \otimes V_2 \in \mathscr{C}$.

(v) If $V \in \mathscr{C}$, then $V^* \in \mathscr{C}$.

Proof. (i) is trivial.

(ii) Since tensoring with vector spaces is exact and applying invariants is left-exact, the sequence

$$0 \to D_B(V') \to D_B(V) \to D_B(V/V')$$

is exact. Therefore,

$$\dim_E D_B(V) \leq \dim_E D_B(V') + \dim_E D_B(V/V') \leq \dim_F V' + \dim_F(V/V') = \dim_F V,$$

so we have equality everywhere.

(iii) The natural map $D_B(V_1) \oplus D_B(V_2) \to D_B(V_1 \oplus V_2)$ factors through the square

$$\begin{tikzcd}
D_B(V_1) \oplus D_B(V_2) \arrow[r] \arrow[d, hook] & D_B(V_1 \oplus V_2) \arrow[d, hook] \\
(B \otimes V_1) \oplus (B \otimes V_2) \arrow[r, "\approx"] & B \otimes (V_1 \oplus V_2)
\end{tikzcd}$$

and is therefore an injection. Since V_1 and V_2 are admissible, we have

$$\dim_E D_B(V_1 \oplus V_2) \geq \dim_E D_B(V_1) + \dim_E D_B(V_2) = \dim_F V_1 + \dim_F V_2 = \dim_F(V_1 \oplus V_2).$$

(iv) The result is immediate from the following Lemma.

Lemma 7.8. *If $V_1, V_2 \in \mathscr{C}$, then there is a natural isomorphism*

$$D_B(V_1) \otimes_E D_B(V_2) \simeq D_B(V_1 \otimes V_2).$$

Proof. The natural transformation $D_B(V_1) \otimes_E D_B(V_2) \to D_B(V_1 \otimes V_2)$ factors through the square

$$\begin{tikzcd}
D_B(V_1) \otimes D_B(V_2) \arrow[r] \arrow[d, hook] & D_B(V_1 \otimes V_2) \arrow[d, hook] \\
(B \otimes V_1) \otimes_B (B \otimes V_2) \arrow[r, "\approx"] & B \otimes (V_1 \oplus V_2)
\end{tikzcd}$$

and is therefore an injection. Since V_1 and V_2 are admissible,

$$\dim_E D_B(V_1 \otimes V_2) \geq \dim_E D_B(V_1) \times \dim_E D_B(V_2) = \dim_F V_1 \times \dim_F V_2 = \dim_F V_1 \otimes V_2.$$

□

Corollary 7.9. *If $V \in \mathscr{C}$, then there are natural isomorphisms*

$$\bigwedge^{r} D_B(V) \simeq D_B(\bigwedge^{r} V) \quad \textit{and} \quad \mathit{Sym}^r D_B(V) \simeq D_B(\mathit{Sym}^r V).$$

Proof. The natural map fits into a square

$$\begin{tikzcd}
D_B(V)^{\otimes r} \arrow[r, "\approx"] \arrow[d, two heads] & D_B(V^{\otimes r}) \arrow[d, two heads] \\
\bigwedge^r D_B(V) \arrow[r] & D_B(\bigwedge^r V)
\end{tikzcd}$$

where the surjectivity follows from fact, proved above, that $D_B(-)$ is exact on admissible representations. This shows that the bottom map is surjective, and counting dimensions completes the proof. □

(v) The result is immediate from the following Lemma.

Lemma 7.10. *If $V \in \mathscr{C}$, then there is a natural isomorphism*

$$D_B(V^*) \simeq D_B(V)^*.$$

Proof. There is a Galois-equivariant non-degenerate pairing

$$\wedge^{n-1} V \times V \to \det V$$

inducing an isomorphism of Galois modules

$$V \simeq \mathrm{Hom}(\wedge^{n-1} V, \det V) \simeq \wedge^{n-1} V^* \otimes \det V.$$

Applying this to the dual space of V, we have a Galois-equivariant isomorphism

$$V^* \simeq \wedge^{n-1} V \otimes (\det V)^*.$$

By the Lemmas 7.9 and 7.8, this reduces to the case where $\dim V = 1$.

In this case, let v be a basis for V over F. We have a square

$$\begin{tikzcd}
D_B(V^*) \arrow[r, dotted] \arrow[d, hook] & D_B(V)^* \arrow[d, hook] \\
B \otimes V^* \arrow[r, "\approx"] & (B \otimes V)^*
\end{tikzcd}$$

where the bottom map is an isomorphism because V is admissible. Therefore, the map $D_B(V^*) \to D_B(V)^*$ is an injection.

If $b \otimes v \in D_B(V)^*$, then $g \cdot v = \chi(g)v$ for some character $\chi : G \to F^\times$, and

$$g \cdot b \otimes g \cdot v = b \otimes v$$

so $g \cdot b = \chi(g)^{-1}$. Now, if v^* represents the dual vector to v, then $g \cdot v^* = \chi(g)^{-1} v^*$, so g fixes $b^{-1} \otimes v^* \in B \otimes V^*$. Therefore, V^* is also admissible and the map is an isomorphism. ☐

☐

7.3 The ring of Hodge-Tate periods

We now consider the important example of *Hodge-Tate* representations. Let K be a local field with residue field of charateristic p and G_K its absolute Galois group. Recall that $\mathbf{C}_p$ is equipped with a continuous G_K-action.

Definition 7.11. The **Hodge-Tate ring** B_{HT} is defined to be

$$B_{\mathrm{HT}} = \bigoplus_{i \in \mathbf{Z}} \mathbf{C}_p(i) \simeq \mathbf{C}_p[t, t^{-1}]$$

where G_K acts on t by the cyclotomic character.

As a consequence of Theorem 2.1, we find:

Theorem 7.12. *$H^0(B_{\mathrm{HT}}, G_K) = K$.*

Proposition 7.13. *The ring B_{HT} is $(\mathbf{Q}_p, G_K)$-regular.*

Proof. There are three conditions to check:

1. B_{HT} is a domain,

2. $(\mathrm{Frac}B_{\mathrm{HT}})^{G_K} = (B_{\mathrm{HT}})^{G_K} = K$, and

3. For every non-zero $b \in B_{\mathrm{HT}}$ such that vector space $\mathbf{Q}_p b$ is G_K-stable, b is invertible.

The first condition is immediate from the definition.

For the second, observe that $\mathrm{Frac}B_{\mathrm{HT}} \subset \widehat{B_{\mathrm{HT}}} \simeq \mathbf{C}_p((t))$, so it suffices to show that $(\widehat{B_{\mathrm{HT}}})^{G_K} = K$. Any element of $\widehat{B_{\mathrm{HT}}}$ may be written uniquely as

$$\begin{aligned} b &= \sum_{i \in \mathbf{Z}} b_i t^i \qquad\quad b_i \in \mathbf{C}_p \\ g(b) &= \sum_{i \in \mathbf{Z}} g(b_i)\chi(g)^i t^i, \end{aligned}$$

so $b \in (\widehat{B_{\mathrm{HT}}})^{G_K}$ if and only if $b_i t^i$ is fixed by G_K. But we know that $C(j)^{G_K} = K$ if $i \neq 0$ and 0 otherwise.

For the third, suppose that

$$b = \sum_{i \in \mathbf{Z}} b_i t^i \in \widehat{B_{\mathrm{HT}}}$$

has the property that the $\mathbf{Q}_p$-vector space $\mathbf{Q}_p b$ is stable by G_K. Then for any $g \in G_K$,

$$\begin{aligned} g(b) = \alpha b &\implies \sum_{i=-\infty}^{\infty} g(b_i)\chi(g)^i t^i = \sum_{i=-\infty}^{\infty} \alpha b_i t^i \\ &\implies g(b_i)\chi(g)^i = \alpha b_i. \end{aligned}$$

If any b_i is non-zero for i non-zero, then by comparing the ratios of these terms we see that for any j, g acts on $\frac{b_j}{b_i}$ through χ^{j-i}, where χ is the cyclotomic character, and hence

$$\frac{b_j}{b_i} \in H^0(G_K, \mathbf{C}_p(j-i)).$$

Since this space is zero, $b_j = 0$ for all $j \neq i$. Therefore, b must reside in $\mathbf{C}_p(i)$, so it is invertible.

□

Definition 7.14. Let V be a $\mathbf{Q}_p$-representation of G_K. We say that V is **Hodge-Tate** if it is B_{HT}-admissible.

Example 7.2. Recall that for a field K of characteristic zero, $T_p(\overline{K}) \simeq \mathbf{Z}_p(1)$ is a free $\mathbf{Z}_p$ module. Choosing a generator t, we may write

$$g(t) = \chi(g)t$$

where $\chi : G_K \to \mathbf{Z}_p^\times$ is the cyclotomic character. Earlier, we defined the *Tate twist* $\mathbf{Z}_p(i)$, which is isomorphic as a G_K-module to $\mathbf{Z}_p t^i$: the free, rank one $\mathbf{Z}_p$-module generated by t.

More generally, if M is any $\mathbf{Z}_p$-module with continuous G_K-action, then $M(i) = M \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(i)$, with G_K-action given by

$$g(m \otimes c) = \chi^i(g)g(m) \otimes c.$$

Then $\bigoplus_{i\in\mathbf{Z}} M(i)^{n_i}$ is Hodge-Tate.

In this language, Theorem 5.1 immediately implies:

Theorem 7.15. *Let X be an abelian variety over a p-adic field K. Then $H^i_{\acute{e}t}(X_{\overline{K}}, \mathbf{Q}_p)$ is Hodge-Tate.*

Chapter 8

The de Rham Period Ring

In this section, we seek to construct a period ring B_{dR} that "refines B_{HT}." Returning to our analogy with complex Hodge theory, there is a refinement of the Hodge decomposition given by algebraic *de Rham cohomology*, which possesses a natural filtration. Similarly, our ring B_{dR} will be such that the functor $D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{Rep}_K(G_K)$ returns representations with a natural filtration, with the property that we may recover D_{HT} by taking the associated graded. The ring B_{dR}, then, should be itself filtered, with B_{HT} as its associated graded.

We might look to the analogy with the situation of a discrete valuation ring R with maximal ideal $\mathfrak{m}$: the associated graded of the fraction field is $\bigoplus_{i \in \mathbf{Z}} \mathfrak{m}^i / \mathfrak{m}^{i-1} \simeq k[t, t^{-1}]$, where k is the residue field $R/\mathfrak{m}$. There is a canonical way to construct a complete discrete valuation ring with residue field k, if k is perfect, given by the theory of Witt vectors.

In our case, the desired residue field is $\mathbf{C}_p$, but this is not amenable to the Witt vector construction. We need a characteristic p ring, so a more promising attempt is to work with its ring of integers $\mathcal{O}_{\mathbf{C}_p}$ modulo the ideal (p). However, this ring is not perfect, so we shall first construct a functor $\mathscr{R}$ that returns a canonically associated perfect ring. Then we may apply the Witt vector construction. The resulting ring has a natural discrete valuation, but it is not complete with respect to this valuation, so we pass to its completion, and essentially arrive at the desired ring.

The presentation draws from [FO] and [CB]. The interested, and very determined, reader may consult either source for further discussion of B_{dR} and other period rings; [CB] is especially comprehensive concerning the technical details. The article [Ber] provides an overview of these subjects, skimming over many of the details.

8.1 Review of Witt vectors

Recall that a ring A *perfect* if A has characteristic p and the map $x \mapsto x^p$ is an automorphism of A. The theory of Witt vectors associates to any such A a ring $W(A)$ of characteristic zero, such that $W(A)$ is separated and complete with respect to the topology defined by the ideals $p^n W(A)$. Moreover, this association is functorial.

Example 8.1. For the field $\mathbf{F}_p$, the Witt vectors are $W(\mathbf{F}_p) \simeq \mathbf{Z}_p$. Any finite field k of characteristic p is perfect, and the Witt vectors $W(k)$ is the ring of integers in the unramified extension of $\mathbf{Q}_p$ with residue field k. In this way, we obtain an equivalence of categories between unramified extensions of $\mathbf{Q}_p$ and extensions of the residue field.

Construction of Witt vectors

We now describe a construction of the Witt vectors. We will omit some proofs; the interested reader may consult [FO] §0.2 for details.

Example 8.2. The Witt vectors are built up from rings $W_n(A)$ which are set-theoretically isomorphic to A^n. The tuple $(x_0, \dots, x_n) \in A^{n+1}$ corresponds to

$$(x_0, \dots, x_n) \leftrightarrow \sum_{i=0}^{n} p^i x_i^{p^{n-i}}.$$

This relationship allows us to transfer the addition and multiplication structure on power series to A^{n+1}. For instance, if $(x_0, \dots, x_n)$ and $(y_0, \dots, y_n)$ are elements of A^{n+1}, then

$$\begin{aligned}(x_0, \dots, x_n) + (y_0, \dots, y_n) &\leftrightarrow \sum_{i=0}^{n} p^i x_i^{p^{n-i}} + \sum_{i=0}^{n} p^i y_i^{p^{n-i}} \\ &\leftrightarrow (S_0(x_0, \dots x_n; y_0, \dots, y_n), S_1(x_0, \dots x_n; y_0, \dots, y_n), \dots)\end{aligned}$$

The polynomials S_i are uniquely determined by forcing this equality to hold for all n, so that the ring structure on each A^n is compatible with the maps $A^{n+1} \to A^n$ forgetting the last coordinate. By this constraint, we may compute S_0 by taking $n = 0$, and we see that $S_0(x_0; y_0) = x_0 + y_0$. If we take $n = 1$, then

$$x_0^p + px_1 + y_0^p + py_1 = (x_0 + y_0)^p + pS_1(x_0, x_1; y_0, y_1)$$

from which we see

$$S_1(x_0, x_1; y_0, y_1) = x_1 + y_1 + \frac{1}{p}\sum_{i=1}^{p-1}\binom{p}{i}x_0^i x_1^{p-i}. \tag{8.1}$$

Similarly, for multiplication

$$\begin{aligned}(x_0, \dots, x_n) \cdot (y_0, \dots, y_n) &\left(\leftrightarrow \sum_{i=0}^{n} p^i x_i^{p^{n-i}}\right)\left(\sum_{i=0}^{n} p^i y_i^{p^{n-i}}\right)\\ &\leftrightarrow (P_0(x_0, \dots x_n; y_0, \dots, y_n), P_1(x_0, \dots x_n; y_0, \dots, y_n), \dots)\end{aligned}$$

Taking $n = 0$, we see that $P_0(x_0, y_0) = x_0 y_0$. Taking $n = 1$,

$$(x_0^p + px_1)(y_0^p + py_1) = x_0^p y_0^p + pP_1(x_0, x_1; y_0, y_1),$$

so

$$P_1(x_0, x_1; y_0, y_1) = x_1 y_0^p + x_0^p y_1 + px_1 y_1. \tag{8.2}$$

It is also clear that multiplication by p induces

$$(x_0, x_1, x_2, \dots) \mapsto (0, x_0^p, x_1^p, \dots).$$

The polynomials S_i and P_i appear to be universal polynomials with integer coefficients, so we can define these operations over any ring A. We now formalize these observations.

For indeterminates $X_0, X_1, \dots$, let

$$w_n(X_0, X_1, \dots) = \sum_{i=0}^{n} p^i x_i^{p^{n-i}}$$

For a polynomial $F \in \mathbf{Z}[X_0, X_1, \dots; Y_0, Y_1, \dots]$, we abbreviate

$$F(\underline{X}, \underline{Y}) = F(X_0, X_1, \dots; Y_0, Y_1, \dots).$$

Lemma 8.1. *Let $\Phi \in \mathbf{Z}[X, Y]$. Then there exists a unique sequence of polynomials $\{\Phi_n\}_{n \in \mathrm{N}}$, with each $\Phi_n \in \mathbf{Z}[X_0, \dots, X_n; Y_0, \dots, Y_n]$, such that*

$$\Phi(w_n(\underline{X}), w_n(\underline{Y})) = w_n(\Phi_0(\underline{X}, \underline{Y}), \Phi_1(\underline{X}, \underline{Y}), \dots, \Phi_n(\underline{X}, \underline{Y})).$$

Proof. See [FO], Lemma 0.22. □

We now let $W_n(A) = A^n$ as a set, and define addition and multiplication as follows: Let $S_0, S_1, \dots$ be the polynomials Φ_i obtained from applying Lemma

8.1 to $\Phi(X, Y) = X + Y$ and $P_0, P_1, \ldots$ be the polynomials Φ_i obtained from applying Lemma 8.1 to $\Phi(X, Y) = XY$. Then for $\underline{a} = (a_0, \ldots, a_{n-1}) \in A^n$ and $\underline{b} = (b_0, \ldots, b_{n-1}) \in A^n$, we set

$$\begin{aligned} \underline{a} + \underline{b} &= (S_0(\underline{a}, \underline{b}), \ldots, S_{n-1}(\underline{a}, \underline{b})) \\ \underline{a} \cdot \underline{b} &= (P_0(\underline{a}, \underline{b}), \ldots, P_{n-1}(\underline{a}, \underline{b})). \end{aligned}$$

Now, the collection of maps

$$\begin{aligned} W_{n+1}(A) &\to W_n(A) \\ (a_0, \ldots, a_n) &\mapsto (a_0, \ldots, a_{n-1}) \end{aligned}$$

form a compatible system of ring homomorphisms, and we may therefore define

$$W(A) := \varprojlim W_n(A).$$

Definition 8.2. If A is a perfect ring, then the ring $W(A)$ constructed above is called the **Witt vectors** of A.

Properties of the Witt vectors

This construction is *functorial* in the following sense.

Proposition 8.3. *If A and A' are perfect rings, then there is a natural bijection*

$$\mathit{Hom}(A, A') \leftrightarrow \mathit{Hom}(W(A), W(A')).$$

Proof. A homomorphism $W(A) \to W(A')$ induces, by reduction, a homomorphism $A \to A'$ as $A \simeq W(A)/(p)$. On the other hand, if $\varphi : A \to A'$ is a homomorphism, then we may define $\tilde{\varphi} : W(A) \to W(A')$ by

$$\tilde{\varphi}((a_0, a_1, \ldots)) = (\varphi(a_0), \varphi(a_1), \ldots).$$

□

The Teichmüller lift. There is a distinguished section $[\cdot] : A \to W(A)$ called the *Teichmüller lift*, sending

$$a \mapsto [a] := (a, 0, 0, \ldots).$$

It is evidently multiplicative (and is the only multiplicative section; see below).

p-rings

It will be useful to phrase some of these properties in terms of p-rings.

Definition 8.4. A **p-ring** is a ring R which is separated and complete for the topology defined by a decreasing filtration of ideals $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \ldots$ such that $\mathfrak{a}_n\mathfrak{a}_m \subset \mathfrak{a}_{n+m}$ for all $n, m \geq 1$ and $R/\mathfrak{a}_1$ is a perfect $\mathbf{F}_p$-algebra.

The separability condition is equivalent to $\bigcap \mathfrak{a}_i = 0$. Note that this definition forces $p \in \mathfrak{a}_1$.

Example 8.3. For instance, the ring of integers in any finite extension of $\mathbf{Q}_p$ constitute a p-ring for the filtration $(p) \supset (p^2) \supset \ldots$. Indeed, if A is *any* perfect ring then $W(A)$ is a strict p-ring.

Definition 8.5. If R is a p-ring and $\mathfrak{a}_i = (p^i)$, and p is a non-zero divisor in R, then we say that R is a **strict p-ring**.

Proposition 8.6. *If R is a p-ring and $A = R/\mathfrak{a}_1$, then there is a unique multiplicative section $A \to R$.*

Proof. See [FO], Proposition 0.31. ☐

This section is defined as follows: for any $x \in A$, choose some $\tilde{x}_n \in R$ lifting $x^{p^{-n}}$. Observe that

$$\alpha \equiv \beta \pmod{p^m A} \implies \alpha^p \equiv \beta^p \pmod{p^{m+1} A}$$

because $\alpha^p - \beta^p = (\alpha - \beta)(\alpha^{p-1} + \ldots + \beta^{p-1})$. Therefore, the sequence $\tilde{x}_n^{p^n}$ converges in R to an element $[x]$ depending only on x, so we set

$$[x] = \lim_{n\to\infty} \tilde{x}_n^{p^n}. \tag{8.3}$$

Observe that when $R = W(A)$, then this is exactly the inverse limit process in our construction, so this section is the Teichmüller lift. Abusing notation, we will also denote this section by $[\cdot]$. So every element $r \in R$ may be expressed uniquely as a series of the form

$$r = \sum_{i=0}^{\infty} [a_i] p^i.$$

The elements $\{\beta_i := a_i^{p^i}\}_{i=0}^{\infty}$ are called the *coordinates* of r.

In fact, strict p-rings are exactly the Witt vectors of perfect fields.

Theorem 8.7. *For every perfect ring A of characteristic p, there is a unique strict p-ring with residue ring A, which is the Witt vectors $W(A)$.*

Proof. See [FO], Theorem 0.37. ☐

Proposition 8.3 generalizes as follows:

Proposition 8.8. *If R is a strict p-ring with residue ring A and R' is any p-ring with residue ring A', then there is a natural bijection*

$$Hom(A, A') \leftrightarrow Hom(R, R').$$

Proof. If $\varphi : R \to R'$ is a ring homomorphism, then reducing modulo p gives a ring homomorphism $R/(p) \to R'/\mathfrak{a}'_1$. To define the inverse map, we apply a similar argument as in the proof of Proposition 8.3 using the canonical series form mentioned above, and the fact that the ring structure is again given by universal polynomials by a similar result to Lemma 8.1. ☐

8.2 The functor $\mathscr{R}$

We wish to apply the Witt construction to the ring $\mathcal{O}_{\mathbf{C}_p}/(p)$. Unfortunately, this is not a perfect ring, but there is a functorial way to associate a perfect ring to it. More generally, for a ring A of characteristic p, we define $\mathscr{R}(A)$ to be the inverse limit

$$\mathscr{R}(A) = \lim_{x \mapsto x^p} \{(x_0, x_1, x_2, \ldots) \mid x_i \in A,\ x_{i+1}^p = x_i\}.$$

Proposition 8.9. *The ring $\mathscr{R}(A)$ is perfect.*

Proof. The ring obviously has characteristic p. The Frobenius morphism $x \mapsto x^p$ is injective since if $(x_i)^p = 0$, then $x_n = x_{n+1}^p = 0$ for each n. It is surjective because $(x_0, x_1, \ldots)$ is the image of $(x_1, x_2, \ldots)$ by definition. ☐

Observe that the map $\mathscr{R}(A) \to A$ given by projection to the first coordinate is final among all maps from perfect rings of characteristic p to A. The functoriality of this construction is evident.

Example 8.4. If A is already perfect, then $\mathscr{R}(A) = A$. This follows immediately from the universal property.

Example 8.5. If F is a field of characteristic p, then $\mathscr{R}(F)$ is the set of elements admitting a p^{th} root, which is the largest perfect subfield.

Now let A be a ring that is separated and complete for the p-adic topology.

Proposition 8.10. *There is a natural bijection between $\mathscr{R}(A/pA)$ and the set*

$$S = \{(x^{(n)})_{n=0}^{\infty} \mid x^{(n)} \in A, (x^{(n+1)})^p = x^{(n)}\}.$$

Proof. Given an element $x = \{(x^{(n)})\}_{n=0}^{\infty}$ of S, we obtain an element of $\mathscr{R}(A/pA)$ by reducing each $x^{(n)}$ modulo p.

Conversely, given an element $x = (x_n)_{n=0}^{\infty}$ of $\mathscr{R}(A/pA)$, we let $\tilde{x}_k$ be any lift of x_k and set

$$x^{(n)} = \lim_{m\to\infty} \tilde{x}_{n+m}^{p^m}.$$

This limit converges for the same reason as in the Teichmüller lift (8.3); indeed, it is the same construction without the hypothesis of perfect residue ring.

Since $\tilde{x}_{n+1}^p \equiv \tilde{x} \pmod{pA}$, we see that $\tilde{x}_{(n+1)+m}^p \equiv \tilde{x}_{n+m} \pmod{p^m A}$, hence $(x^{(n+1)})^p = x^{(n)}$. The two maps defined are easily check to be inverses.

□

Notice that the S in the Proposition is the set-theoretic limit of $A, A, \ldots$ with respect to $x \mapsto x^p$. Proposition 8.10 allows us to define a ring structure on this set via the bijection. The multiplication is term-by-term, but the addition is more complicated: the proof above shows that if $(x^{(n)})_{n=0}^{\infty}$ and $(y^{(n)})_{n=0}^{\infty}$ are two elements of S, then

$$(x+y)^{(n)} = \lim_{m\to\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}. \tag{8.4}$$

Observe also that if p is odd, then $-(x^{(n)})_{n=0}^{\infty} = (-x^{(n)})_{n=0}^{\infty}$ since the latter forms a compatible sequence. If $p = 2$, then this does *not* hold, but in this case $-(x^{(n)})_{n=0}^{\infty} = (x^{(n)})_{n=0}^{\infty}$.

Applications to rings of integers

We are mostly interested in applying this construction to $\mathcal{O}_{\mathbf{C}_p}$, and the above proposition allows us to transfer the valuation on $\mathcal{O}_{\mathbf{C}_p}$ to R.

Definition 8.11. We define $R := \mathscr{R}(\mathcal{O}_{\mathbf{C}_p}/(p))$.

Let v_p be the valuation on $\mathbf{C}_p$ normalized by $v_p(p) = 1$. We define a valuation v_R on R by $v_R(x) = v_p(x^{(0)})$, using the identification of R with sequences of p-power compatible elements of $\mathcal{O}_{\mathbf{C}_p}$ furnished by Proposition 8.10. We will keep the notation in the proof of the Proposition, i.e.

$$\begin{aligned} x &= (x_n)_{n=0}^{\infty} & x_n &\in \mathcal{O}_{\mathbf{C}_p}/(p) \\ &= (x^{(n)})_{n=0}^{\infty} & x^{(n)} &\in \mathcal{O}_{C_p}. \end{aligned}$$

Proposition 8.12. *The valuation v_R makes R a complete valuation ring with residue field $\overline{k}$.*

Proof. The value group of R is evidently $\mathbf{Q} \cup \{\infty\}$. It is clear that

$$v_R(x) = \infty \iff x^{(0)} = 0 \iff x = 0$$

and that

$$v_R(xy) = v_R(x) + v_R(y).$$

We must check the valuation form of the ultrametric inequality: $v_R(x+y) \geq \min\{v_R(x), v_R(y)\}$. We can assume that x and y are non-zero. Since

$$v_R(x) = v_p(x^{(0)}) = p^n v_p(x^{(n)}), \tag{8.5}$$

there exists n such that $v_p(x^{(n)}) < 1$ and $v_p(y^{(n)}) < 1$. Since $(x+y)^{(n)} \equiv x^{(n)} + y^{(n)} \pmod{p}$, we have

$$v_p((x+y)^{(n)}) \geq \min\{v_p(x^{(n)}), v(y^{(n)}), 1\} = \min\{v_p(x^{(n)}), v(y^{(n)})\}.$$

By (8.5), we see that $v(x+y) \geq \min\{v(x), v(y)\}$.

Next observe that $v_R(x) \geq p^n \iff v_p(x^{(n)}) \geq 1 \iff x_n = 0$. Let $\theta_n : R \to \mathcal{O}_{\mathbf{C}_p}/(p)$ denote the projection to the n^{th} component. Then

$$\ker \theta_n = \{x \in R \mid v_R(x) \geq p^n\}$$

so the topology defined by v_R is the same as the subspace topology of R inside $\prod_{n=0}^{\infty} \mathcal{O}_{\mathbf{C}_p}/(p)$. In particular, since R is a *closed* subspace, it is complete.

The map $\theta_0 : R \to \mathcal{O}_{\mathbf{C}_p}/(p)$ is evidently surjective, and is injective on residue fields because it is local. Since $\overline{k} \subset \mathcal{O}_{\mathbf{C}_p}/(p)$ and $\mathscr{R}(\overline{k}) = \overline{k}$ by Example 8.5, there is an inclusion $\overline{k} \subset R$ by functoriality. The composition $\overline{k} \to R \to \overline{k}$ induces an isomorphism on residue fields, concluding the proof. □

Since R is a domain, we can construct the field Frac(R) and extend v_R to a valuation on Frac(R). We see immediately:

Corollary 8.13. *Frac(R) is a complete, nonarchimedean, perfect field of characteristic p.*

Example 8.6. Let ϵ be a choice of generator for $T_p(\mathbb{G}_m(\overline{K}))$, i.e. a sequence

$$\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots)$$

of primitive p^r roots of units compatible with respect to $x \mapsto x^p$. This can be thought of as a choice of "orientation" for $\overline{K}$. By Proposition 8.10, ϵ may be viewed as an element of R. Let us compute the valuation of $\epsilon - 1$. By (8.4),

$$(\epsilon - 1)^{(0)} = \lim_{m\to\infty} (\zeta_{p^m} - 1)^{p^m}.$$

If p is odd, then

$$v_R(\epsilon - 1) = \lim_{m\to\infty} p^m v_p(\zeta_{p^m} - 1) = \lim_{m\to\infty} \frac{p^m}{p^{m-1}(p-1)} = \frac{p}{p-1}.$$

If $p = 2$, then

$$\begin{aligned} v_R(\epsilon - 1) &= \lim_{m\to\infty} 2^m v_2(\zeta_{2^m} + 1) = \lim_{m\to\infty} 2^m v_2(\zeta_{2^m} - 1 + 2) \\ &= \lim_{m\to\infty} 2^m v_2(\zeta_{2^m} - 1) = 2. \end{aligned}$$

So we find that in either case,

$$v_R(\epsilon - 1) = \frac{p}{p-1}. \tag{8.6}$$

8.3 The ring B_{dR}

Since R is a perfect ring, we may form its Witt vectors $W(R)$.

Definition 8.14. Define the map $\theta : W(R) \to \mathbf{C}_p$ by

$$\sum [c_n] p^n \mapsto \sum c_n^{(0)} p^n.$$

This should be viewed as related to the homomorphism $\theta_0 : R \to \mathcal{O}_{\mathbf{C}_p}/(p)$ in analogy to Proposition 8.8, though its existence does not follow from the Proposition because $\mathcal{O}_{\mathbf{C}_p}/(p)$ is not perfect. In terms of the Witt coordinates $x = (x_0, x_1, \ldots) \in W(R)$, the map is

$$\theta(x) = \sum_{n=0}^{\infty} p^n x_n^{(n)}.$$

It is not entirely straightforward to show that this is actually a homomorphism. Perhaps the most natural way is to establish the alternate characterization

$$W(R) = \varprojlim W_n(\mathcal{O}_{\mathbf{C}_p}/(p))$$

where the transition maps are $x \mapsto x^p$. The interested reader may consult [FO], §5.2.1.

Now let $\tilde{p} \in R$ be an element satisfying $\tilde{p}^{(0)} = p$, so

$$\tilde{p} = (p, p^{-p}, p^{-p^2}, \ldots)$$

(but the subsequent terms after p are defined only up to roots of unity). Since $\tilde{p}$ reduces to p in $\mathcal{O}_{\mathbf{C}_p}/(p)$, the element $\xi := \tilde{p} - p$ lies in the kernel of the map $R \to \mathcal{O}_{\mathbf{C}_p}/(p)$. By the construction of the map θ, we see that ξ lies in its kernel. The next result asserts that, in fact, ξ generates the kernel.

Proposition 8.15. *The kernel of θ is the principal ideal generated by ξ.*

Proof. First, we reduce to showing that $\ker\theta \subset (\xi, p)$. Suppose that this claim is granted. Then any $x \in \ker\theta$ may be written as $x = \xi y_0 + p x_1$, and we see that

$$0 = \theta(x) = p\theta(x_1) \in \mathcal{O}_{\mathbf{C}_p} \implies \theta(x_1) = 0$$

since $\mathcal{O}_{\mathbf{C}_p}$ has no p-torsion. Then we may apply the same argument to the element x_1, and continuing in this way, we find that for each k, there exist elements $y_0, \ldots, y_k \in W(R)$ such that

$$x = \xi(y_0 + py_1 + \ldots + p^k y_k) + p^{k+1} x_{k+1}.$$

Since $W(R)$ is p-adically separated and complete, the series $y_0 + py_1 + \ldots + p^k y_k$ converges to some $y \in W(R)$, and we have $x = \xi y$.

Now suppose that $x = (x_0, x_1, \ldots) \in \ker\theta$. Then

$$\theta(x) = x_0^{(0)} + \sum_{n=1}^{\infty} p^n x_n^{(n)},$$

so $x_0^{(0)} \in p\mathcal{O}_{\mathbf{C}_p}$. This implies that $v_R(x_0) \geq v_R(\tilde{p})$. Since R is a valuation ring by Proposition 8.12, we have $x_0 = \tilde{p}y$ for some $y \in R$. Then

$$x - [\tilde{p}y] = x - [\tilde{p}][y]$$

reduces to zero modulo p, and hence lies in $pW(R)$. □

The homomorphism $\theta : W(R) \to \mathcal{O}_{\mathbf{C}_p}$ then extends to a homomorphism $W(R)[p^{-1}] \to \mathbf{C}_p$, which we also denote by θ.

Definition 8.16. The ring B_{dR}^+ is defined to be the completion of $W(R)[p^{-1}]$ with respect to $\ker\theta = (\xi)$:

$$B_{\mathrm{dR}}^+ := \varprojlim W(R)[p^{-1}]/(\ker\theta)^n = \varprojlim W(R)[p^{-1}]/(\xi)^n.$$

We define the **de Rham period ring** $B_{\mathrm{dR}} := \mathrm{Frac}(B_{\mathrm{dR}}^{+}) = B_{\mathrm{dR}}^{+}[\xi^{-1}]$.

Lemma 8.17. *In $W(R)$, $\bigcap_{n=1}^{\infty}(\xi^n) = 0$.*

Proof. Suppose $x = (x_0, x_1, \ldots) \in \bigcap_{n=1}^{\infty}(\xi^n) = 0$. Reducing modulo (p), we find that x_0 has arbitrarily high valuation in R. Since R is complete with respect to its valuation, $x_0 = 0$ and $x = px'$. Repeating this argument to x', we find that $x \in \bigcap_{n=1}^{\infty}(p^n)$, but since $W(R)$ is a strict p-ring, this forces $x = 0$. □

The Lemma implies that $W(R)[p^{-1}]$ injects to B_{dR}^{+}. The map $B_{\mathrm{dR}} \to \mathbf{C}_p$ by projection to the first factor of the inverse limit extends $\theta : W(R)[p^{-1}] \to \mathbf{C}_p$, so we again denote this map by θ.

Proposition 8.18. *B_{dR}^{+} is a separated, complete discrete valuation ring with residue field $\mathbf{C}_p$ and fraction field B_{dR}.*

Proof. The result follows from the sequence of observations:

- B_{dR}^{+} has the same image under θ as $W(R)[p^{-1}]$, which is $\mathbf{C}_p$.
- Since $W(R)[p^{-1}]/(\xi) \simeq \mathbf{C}_p$, each $W(R)[p^{-1}]/(\xi^n)$ is an Artinian local ring with maximal ideal generated by ξ. Therefore, an element of B_{dR}^{+} is a unit if and only if lies outside $\ker\theta$. This shows that B_{dR}^{+} is a local ring with maximal ideal $\ker\theta$.
- Any non-unit x maps to some $b_n\xi$ in $W(R)[p^{-1}]/(\xi^n)$, with b_n determined modulo (ξ^{n-1}). Therefore, the sequence $\{b_n\}_{n=1}^{\infty}$ defines a *unique* element $b \in B_{\mathrm{dR}}^{+}$ such that $x = \xi b$. Therefore, ξ is a non-zero divisor, and ξ generates the maximal ideal.
- By Krull's intersection theorem, B_{dR} is separated.

□

Remark 8.19. There are two potential candidates for topologies on B_{dR}:

1. The topology given by the discrete valuation, and
2. The subspace topology of the product topology on $\prod W(R)[p^{-1}]/(\xi^n)$, where each factor inherits the quotient topology from $W(R)[p^{-1}]$.

Note that the under the first topology, the residue field $\mathbf{C}_p$ has the *discrete* topology. We adopt the second candidate, sometimes referred to as the *natural* topology, which induces the usual topology on $\mathbf{C}_p$.

8.4 Some properties of B_{dR}

Let ϵ be the element constucted in Example 8.6, and recall that we computed

$$v_R(\epsilon - 1) = \frac{p}{p-1}.$$

As usual, we let $[\epsilon] \in W(R)$ denote the Teichmüller lift of ϵ. Note that

$$\theta([\epsilon] - 1) = \epsilon^{(0)} - 1 = 0$$

so that $[\epsilon] - 1 \in \ker\theta$. By Proposition 8.18,

$$t := \log([\epsilon]) = \log(1 + [\epsilon] - 1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\epsilon]-1)^n}{n} \tag{8.7}$$

converges in B_{dR}^+. If ϵ' is any other choice of basis for the Tate module, then $\epsilon' = \epsilon^a$ for some $a \in \mathbf{Z}_p^\times$, so

$$t' = \log([\epsilon]') = \log([\epsilon]^a) = at.$$

Therefore, the $\mathbf{Z}_p$-line generated by t is independent of the choice of basis.

This element t is a period for the cyclotomic character. Indeed, $g \in G_K$ acts on $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots)$ by exponentiating to the cyclotomic character: $g \cdot \epsilon = \epsilon^{\chi(g)}$. By the multiplicativity of the Teichmüller section,

$$g \cdot t = \log([\epsilon]^{\chi(g)}) = \chi(g)\log([\epsilon]). \tag{8.8}$$

The line $\mathbf{Z}_p t$ can be thought of as analogous to $2\pi i\mathbf{Z}$ in complex analysis, and the element t as analogous to a choice of $2\pi i$.

Proposition 8.20. *The element t is a uniformizer for B_{dR}^+.*

Proof. We have already observed that $[\epsilon] - 1 \in \ker\theta$, so $([\epsilon]-1)^n \in (\ker\theta)^2$ for each $n \geq 2$. Therefore, it suffices to show that $[\epsilon] - 1$ is not in $(\ker\theta)^2$.

Suppose otherwise for the sake of contradiction. Since $[\epsilon] - 1 \in W(R)$, it lies in $(\xi^2) = (\ker\theta)^2$. Projecting to R, and recalling that $\xi = [\tilde{p}] - p$ has image p under this projection, we find that $\epsilon - 1$ is divisible by p^2. If $p > 2$, then we obtain a contradiction from the computation that $v_R(\epsilon - 1) = \frac{p}{p-1}$.

If $p = 2$, then we must consider the "next order" term. Note that

$$\xi^2 = [\tilde{p}]^2 - 4[\tilde{p}] + 4 = (\tilde{p}^2, 0, \ldots).$$

Suppose that $[\epsilon]-1=\xi^2 x$, where $x=(x_0,x_1,\dots)\in R$. By the formula (8.2) we see

$$\xi^2 x=(x_0\tilde{p}^2,x_1\tilde{p}^4,\dots).$$

On the other hand (8.1) shows that

$$[\epsilon]-1=(\epsilon-1,\epsilon-1,\dots)$$

so, projecting by θ_1 we find that $p^4\mid\epsilon-1$, which again contradicts the fact that $v_R(\epsilon-1)=2$. ☐

Corollary 8.21. *We have* $\mathrm{gr}^\bullet(B_{\mathrm{dR}})=B_{\mathrm{HT}}$.

Proof. Indeed, the residue field of B_{dR} is $\mathbf{C}_p$ by Proposition 8.18, and t is a uniformizer, so

$$\mathrm{gr}^\bullet(B_{\mathrm{dR}})\simeq\bigoplus_{i\in\mathbf{Z}}\mathbf{C}_p t^i.$$

By (8.8), $\mathbf{C}_p t^i\simeq\mathbf{C}_p(i)$. ☐

Theorem 8.22. $H^0(B_{\mathrm{dR}},G_K)=K$.

Proof. Consider $V=H^0(B_{\mathrm{dR}},G_K)$ as a K-vector space. Since G_K respects the grading of B_{dR}, the space $\mathrm{gr}^\bullet(V)$ injects into $\mathrm{gr}^\bullet(B_{\mathrm{dR}})^{G_K}=(B_{\mathrm{HT}})^{G_K}=K$ (Theorem 7.12). But $\dim_K\mathrm{gr}^\bullet(V)=\dim_K V$, so $V=K$. ☐

8.5 de Rham representations

Since B_{dR} is a field, it is automatically $(\mathbf{Q}_p,G_K)$-regular.

Definition 8.23. A representation $V\in\mathbf{Rep}_{\mathbf{Q}_p}(G_K)$ is **de Rham** if it is B_{dR}-admissible.

The filtration on B_{dR} yields a filtration on $D_{\mathrm{dR}}(V):=D_{B_{\mathrm{dR}}}(V)=(B_{\mathrm{dR}}\otimes V)^{G_K}$. Thus, we see that de Rham representations are associated with a filtration structure.

Proposition 8.24. *If V is de Rham, then it is Hodge-Tate.*

Proof. By hypothesis,

$$\dim_K(B_{\mathrm{dR}}\otimes V)^{G_K}=\dim_{\mathbf{Q}_p}V.$$

Since taking G_K-invariants is left exact, the sequence

$$0\to\mathrm{Fil}^{i-1}(B_{\mathrm{dR}}\otimes V)\to\mathrm{Fil}^{i}(B_{\mathrm{dR}}\otimes V)\to\mathrm{gr}^{i}(B_{\mathrm{dR}}\otimes V)\to 0$$

induces, after taking invariants, an inclusion $\mathrm{gr}^i(B_{\mathrm{dR}} \otimes V)^{G_K} \subset \mathrm{gr}^i(B_{\mathrm{dR}} \otimes V)^{G_K}$. Fitting these together, we see that

$$\mathrm{gr}^\bullet(B_{\mathrm{dR}} \otimes V)^{G_K} \subset (\mathrm{gr}^\bullet(B_{\mathrm{dR}} \otimes V))^{G_K} = (B_{\mathrm{HT}} \otimes V)^{G_K}$$

so that $\dim_K(B_{\mathrm{HT}} \otimes V)^{G_K} \geq \dim_{\mathbf{Q}_p} V$, forcing equality. □

As we know that the étale cohomology of a smooth projective variety is Hodge-Tate, we might ask next if it is de Rham. This is in fact the case, and it turns out that more is true: the representations coming from geometry are admissible for even finer period rings.

Theorem 8.25. *If X is a smooth projective variety over a p-adic field K, then $H^i_{\acute{e}t}(X_{\overline{K}}, \mathbf{Q}_p)$ is a de Rham representation of G_K.*

8.6 Example: the periods of a Tate curve

In our proof of the Hodge-Tate decomposition, a crucial step was the observation that $H^1_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes \mathbf{C}_p$ furnished an extension of $\mathbf{C}_p(1)^g$ by $\mathbf{C}_p^g$, which we know is trivial by the general Galois theory of $\mathbf{C}_p$. This splitting can be described explicitly by a Galois-equivariant map

$$T_p(X) \otimes \mathbf{C}_p \to \mathbf{C}_p(1)^g.$$

The existence of such a splitting was established by Galois cohomology, as a consequence of Theorem 2.1. A natural question is whether it can be described more explicitly. We will consider this problem for the case of Tate's elliptic curve, and we will see that B_{dR} furnishes a natural setting in which to analyze it.

The Tate Curve

We summarize the basic theory of Tate curves over a local field K; the interested reader should consult [Sil94], §V.3 for a more complete discussion.

Complex elliptic curves may be described analytically $\mathbf{C}/\Lambda$, where Λ is some lattice in $\mathbf{C}$. Homothetic lattices correspond to isomorphic elliptic curves, so we may, without loss of generality, assume that Λ is generated by the vectors 1 and τ. There is an elliptic function $\wp_\Lambda$ associated to Λ called the *Weierstrass $\wp$-function*, which obeys the equation

$$\left(\frac{d\wp_\Lambda(z)}{dz}\right)^2 = 4\wp_\Lambda(z)^3 - g_2(\tau)\wp_\Lambda(z) - g_3(\tau).$$

Therefore, $z \mapsto (\wp_\Lambda(z), \wp'_\Lambda(z))$ furnishes a map

$$\mathbf{C}/\Lambda \to E_\tau(\mathbf{C}),$$

where E_τ is the elliptic curve defined by $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$.

One might hope to repeat this theory for $\overline{K}$, where K is some local field with residue characteristic p, but a direct analogy cannot succeed because $\overline{K}$ has no discrete subgroups. The solution is to rephrase the theory in multiplicative terms.

Returning to the complex case, we observe that all of the relevant functions are periodic in τ with period 1, since τ and $\tau + 1$ generate the same lattice together with 1. Therefore, we may set $q := e^{2\pi i\tau}$ and re-write the theory in terms of q. Since $e^{2\pi i\mathbf{Z}} = 1$, the analytic uniformization then takes the form $\mathbf{C}^\times/q^{\mathbf{Z}} \to E_q(\mathbf{C})$. Now this is promising in the p-adic case, since $\overline{K}^\times$ does have discrete subgroups. Indeed, the formulas defining the relevant maps and coefficients in the complex-analytic case are given by power series in q with rational coefficients, and these may be transported directly to the p-adic setting, provided that one checks issues of convergence.

Example 8.7. The discriminant of E_q is the modular discriminant:

$$\Delta(q) = q\prod_{n=1}^{\infty}(1 - q^n)^{24}.$$

The j-invariant of E_q is the modular j-function:

$$j(E_q) = \frac{1}{q} + 744 + 196884q + \ldots$$

We see that these converge if $|q| < 1$, so the j-invariant must be *non-integral*, which says that E_q has potentially multiplicative reduction.

Theorem 8.26. *Let K be a p-adic field and let $q \in K^\times$ satisfy $|q| < 1$.*

(i) There are series $a_4(q)$ and $a_6(q)$ converging in K such that the Tate curve

$$E_q : y^2 + xy = x^3 + a_4(q)x + a_6(q)$$

is an elliptic curve with discriminant $\Delta(q)$ and j-invariant $j(q)$.

(ii) There are series $X_q(z)$ and $Y_q(z)$ converging for all $z \in \overline{K}^\times/q^{\mathbf{Z}}$ such that

$$z \mapsto (X_q(z), Y_q(z))$$

defines a Galois-equivariant isomorphism

$$\phi : \overline{K}^{\times}/q^{\mathbf{Z}} \to E_q(\overline{K})$$

(with the convention that $q^{\mathbf{Z}}$ maps to the identity).

Proof. See [Sil94], §V.3. □

The power of this uniformization is that it gives us an explicit handle on the Galois-module structure of the torsion points. Indeed, the m-torsion of $\overline{K}^{\times}/q^{\mathbf{Z}}$ is generated by the m^{th}-roots of unity together with $q^{1/m}$ (this element is defined only up to an m^{th} root of unity. By definition, Galois acts on the roots of unity through the cyclotomic character, and it sends $q^{1/m}$ to $\zeta_m^i q^{1/m}$. Fitting together these observations for $m = p^n$, we see that the p-adic Tate module of E_q is an extension of the Tate module of $\overline{K}^{\times}$, which is isomorphic to $\mathbf{Z}_p(1)$, by $\varprojlim q^{1/p^n} \simeq \mathbf{Z}_p$, i.e. there is an exact sequence

$$0 \to \mathbf{Z}_p(1) \to T_p(E_q) \to \mathbf{Z}_p \to 0. \tag{8.9}$$

Periods of a Tate curve

Note that the Hodge-Tate decomposition for a Tate curve is obvious from tensoring (8.9) with $\mathbf{C}_p$, and using the fact that any such extension must split (Corollary 2.19). We seek to describe this splitting more explicitly. More precisely, a splitting is a G_K-equivariant map $T_p(E_q) \otimes \mathbf{C}_p \to \mathbf{C}_p(1)$ inducing the identity on $\mathbf{C}_p(1)$. If we choose a lift of a generator of the quotient $\mathbf{Z}_p$ in (8.9), we may ask: to which element of $\mathbf{C}_p(1)$ does it go?

By the analytic uniformization of the Tate curve, we may identify $T_p(E_q) \simeq T_p(\overline{K}^{\times}/q^{\mathbf{Z}})$, the latter being generated by the two elements

$$\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \qquad \text{and} \qquad \tilde{q} = (q, q^{1/p}, q^{1/p^2}, \ldots).$$

Note that the Galois action on q^{1/p^n} is through Kummer theory: we have $\frac{g \cdot q^{1/p^n}}{q^{1/p^n}} \in \mu_{p^n}$, so we let

$$\begin{aligned} c_n \colon G_K &\to (\mathbf{Z}/p^n\mathbf{Z})^{\times} \\ g &\mapsto \frac{g \cdot q^{1/p^n}}{q^{1/p^n}} \end{aligned}$$

be the corresponding cocycle. Taking the inverse limit, we obtain a cocycle

$$\begin{aligned} c\colon G_K &\to \mathbf{Z}_p^\times \\ g &\mapsto \varprojlim \frac{g \cdot q^{1/p^n}}{q^{1/p^n}} \end{aligned}$$

Then G_K acts on $\tilde{q}$ through this co-cycle:

$$g \cdot \tilde{q} = \epsilon^{c(g)} \tilde{q}.$$

The Hodge-Tate theory tells us that there is a canonical G_K-equivariant splitting after tensoring with $\mathbf{C}_p$. As discussed above, a splitting is equivalent to a map $T_p(E_q) \otimes \mathbf{C}_p \to \mathbf{C}_p(1)$ inducing the identity on $\mathbf{C}_p(1) \subset T_p(E_1) \otimes \mathbf{C}_p$. Since $T_p(E_q)$ is in fact de Rham, we should be able to witness this splitting in B_{dR}.

Let us first argue informally. We may also consider ϵ and $\tilde{q}$ as elements of R, and we "morally" identify $T_p(E_q) \otimes \mathbf{C}_p$ with the Galois submodule $\mathbf{C}_p\langle t, \log[\tilde{q}]\rangle$, where $t = \log[\epsilon]$ (with ϵ now considered as an element of R) and $[\tilde{q}]$ the Teichmüller lift of $\tilde{q}$ (with $\tilde{q}$ now considered as an element of R). While we have yet justified the existence of the element $\log[\tilde{q}]$, it should be the case that for $g \in G_K$,

$$g \cdot \log[\tilde{q}] = \log([\epsilon]^{c(g)}[\tilde{q}]) = \log \tilde{q} + c(g)t,$$

The identification sends $\epsilon \mapsto t = \log[\epsilon]$ and $\tilde{q} \mapsto \log[\tilde{q}]$, and the preceding equation shows that this identification is Galois equivariant.

To define a splitting $\mathbf{C}_p\langle t, \log[\tilde{q}]\rangle \to \mathbf{C}_p(1)$, we need only specify the image of $\log[\tilde{q}]$. Here we can get another clue from the connection with the Hodge-Tate theory: since $T_p(E_q) \otimes \mathbf{C}_p$ is Hodge-Tate with weights 0 and 1, our space lives inside $\mathrm{Fil}^1(B_{\mathrm{dR}}^+)$. The map to the residue field is given by θ, which sends

$$\log[\tilde{q}] \mapsto \log \tilde{q}^{(0)} = \log q.$$

Therefore, the projection to $\mathrm{Fil}^1(B_{\mathrm{dR}}^+)/\,\mathrm{Fil}^0(B_{\mathrm{dR}}^+)$ sends $\log[\tilde{q}] \mapsto \log[\tilde{q}] - \log q = \log\left(\frac{[\tilde{q}]}{q}\right)$.

The discussion above was not legitimate because we have not shown how to make sense of the expression $\log[\tilde{q}]$. However, the expression $\log\left(\frac{[\tilde{q}]}{q}\right)$ does make sense. Formally, note that since $\theta([\tilde{q}] - q) = 0$ by construction, the series

$$\log\left(1 + \frac{[\tilde{q}] - q}{q}\right) = \sum_{j=1}^{\infty} (-1)^{j+1} \left(\frac{[\tilde{q}] - q}{q}\right)^j \tag{8.10}$$

converges to a well-defined element in B_{dR}, which we may call $\log\left(\frac{[\tilde{q}]}{q}\right)$. The map $T_p(\overline{K}^{\times}/q^{\mathbf{Z}}) \to \mathbf{C}_p(1)$ sending ϵ to 1 and $\tilde{q}$ to $\log\left(\frac{[\tilde{q}]}{q}\right)$ is then a G_K-equivariant splitting, since we may now rigorously say that for $g \in G_K$,

$$g \cdot \log\left(\frac{[\tilde{q}]}{q}\right) = \log\left(\frac{[\epsilon]^{c(g)}[\tilde{q}]}{q}\right) = \log\left(\frac{[\tilde{q}]}{q}\right) + c(g)\log t.$$

The analysis here tells us that $\log\left(\frac{[\tilde{q}]}{q}\right) \in \mathrm{Fil}^1(B_{\mathrm{dR}}^+)$ and maps to 0 in $\mathrm{Fil}^0(B_{\mathrm{dR}}^+)$, hence can be written as $a_q t$ for some $a_q \in \mathbf{C}_p$. This a_q describes the period that we seek.

Let us make some brief remarks on how to "compute" a_q, though it is a transcendental element of $\mathbf{C}_p$ and cannot really be described explicitly. The procedure outlined below shows how to compute some "base p digits" of a_q.

1. Since $[\tilde{q}] - q \in \ker\theta$, the only summand in (8.10) that does not lie in the square of the maximal ideal is $\frac{[\tilde{q}]-q}{q}$. Therefore, a_q is $\theta\left(\frac{[\tilde{q}]-q}{qt}\right)$.

2. Since $[\epsilon] - 1 \in \ker\theta$, the only summand in (8.7) that does not lie in the square of the maximal ideal is $[\epsilon] - 1$. Therefore, a_q is $\frac{1}{q}\theta\left(\frac{[\tilde{q}]-q}{[\epsilon]-1}\right)$.

3. First consider $q = p$. Then $\xi := [\tilde{p}] - p$ and $[\epsilon] - 1$ both lie in $W(R)$, and we further know that ξ divides $[\epsilon] - 1$ by Proposition 8.15. We may compute $\frac{[\epsilon]-1}{\xi}$ using the proof of Proposition 8.15, which lets us then compute a_p.

4. If $q \neq p$ lies in K^{unr}, the maximal subextension of K unramified over $\mathbf{Q}_p$, then $[\tilde{q}] - q \in W(R)$, and is divisible by ξ, so we may compute $\frac{[\tilde{q}]-q}{\xi}$ using the proof of Proposition 8.15, and then combine this with Step 3 to compute a_q.

5. If q is the uniformizer in some ramified extension, then we may write $uq^e = p$ for some $e \in \mathrm{N}$ and u a unit. In B_{dR},

   $$\theta\left(\frac{[\tilde{p}] - p}{[\tilde{q}] - q}\right) = \theta\left(u\frac{[\tilde{q}]^e - q^e}{[\tilde{q}] - q}\right) = ueq^{e-1}.$$

   Combined with Step 3 again, this allows us to compute $\theta\left(\frac{[\epsilon]-1}{[\tilde{q}]-q}\right)$, and hence a_q.

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