Local Picard groups

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1 Introduction

1.1 Setup

- $(A, \mathfrak{m}, k)$ is a Noetherian local ring and $f \in \mathfrak{m}$ is a non-zerodivisor.
- Let $X = \operatorname{Spec} A$ and $U = X \setminus \{\mathfrak{m}\}$.
- We define the hypersurface $X_0 = \operatorname{Spec}(A/fA) \hookrightarrow X$ and $U_0 = X_0 \setminus \{\mathfrak{m}\}$.

We will think about the restriction map $\operatorname{Pic}(U) \to \operatorname{Pic}(U_0)$.

1.2 Statement of Results

Theorem 1.1. *Suppose that A is excellent and normal (hence a domain), contains a field, and has* $\dim A \geq 4$.

1. *If* $\operatorname{depth}(A/fA) \geq 2$ *and the field is characteristic* 0*, then the map* $\operatorname{Pic}(U) \to \operatorname{Pic}(U_0)$ *is injective.*
2. *If the field has characteristic $p > 0$, then* $\ker(\operatorname{Pic}(U) \to \operatorname{Pic}(U_0))$ *is p-power torsion.*

Theorem 1.2 (Kollár)**.** *Let A be essentially finite type over k and (S_2). Suppose X is equidimensional of dimension at least* 4 *and* $\operatorname{depth}(A/fA) \geq 2$*. Then* $\operatorname{Pic}(U) \hookrightarrow \operatorname{Pic}(U_0)$ *is injective.*

Proposition 1.3. *If* $\operatorname{depth}(A/fA) \geq 2$*, then* $\ker(\operatorname{Pic}(U) \to \operatorname{Pic}(U_0))$ *is torsion-free.*

Remark 1.4. Grothendieck proved the injectivity under the assumption that the depth is at least 4. The proof of Kollár's theorem uses the first theorem.

1.3 Motivation

Why are these results useful? The motivation is to study families of Cartier divisors. Suppose we have a flat morphism $Y \to C$ (a curve) and Y_0 is the fiber over some closed point $0 \in C$. We're going to have an invertible sheaf which isn't everywhere defined, but is after restricting to the relevant fiber. By zooming in on the generic point of the locus where it isn't defined, we can make it a closed point.

So suppose $\mathcal{N}$ is invertible on $Y \setminus \{y\}$ and $\mathcal{N}|_{Y_0 \setminus \{y\}}$ extends to an invertible sheaf on Y_0, then does $\mathcal{N}$ extend to Y?

Theorem 1.2 says that this is true if some technical conditions are satisfied, i.e. if the fibers are (S_2) an $\dim Y \geq 4$.

Theorem 1.2 is sharp in terms of the dimension constraint, as the following theorem shows.

Example 1.5. Let $A = \mathbb{C}[x, y, z, t]/(x^2 + y^2 + z^2 + t^2)$ and $f = t$. Then the kernel is $\mathbb{Z}$. This is a nice exercise.

2 Proofs

2.1 Sketch of Theorem 1.2 (b) a la Kollár

We first reduce to the complete case. Then let A^+ be the absolute integral closure, i.e. the integral closure of A in an algebraic closure of its fraction field.

Theorem 2.1 (Hochster-Huneke)**.** *A^+ is Cohen-Macaulay.*

Let U^+ be the punctured spectrum of A^+. Then it follows from the depth assumption that $H^2(U^+, \mathcal{O}_{U^+}) = 0$ for $0 < i < \dim A - 1$ (this is another definition of depth, if you like). Then $H^i(U_0^+, \mathcal{O}_{U_0^+}) = 0$ for $0 < i < \dim A - 2$ (by a long exact sequence).

Proposition 2.2. *We have* $\operatorname{Pic}(U^+) \hookrightarrow \operatorname{Pic}(U_0^+)$ *if* $\dim A \geq 4$.

Proof. Assume $\mathcal{L}^+|_{U_0^+} \cong \mathcal{O}_{U_0^+}$ and look at the short exact sequence

$$0 \to \mathcal{L}^+ \xrightarrow{f} \mathcal{L}^+ \to \mathcal{O}_{U_0^+} \to 0.$$

We want to lift a trivializing section, so it suffices to show that the $H^1(U^+, \mathcal{L}^+) = 0$. The assumption implies that $H^1(\mathcal{O}_{U_0^+}) = 0$, so multiplication by f is surjective on $H^1(U^+, \mathcal{L}^+)$, i.e. $H^1(U^+, \mathcal{L}^+)$ is f-divisible. Therefore, it's enough to show that it is killed by a power of f. That follows from the following lemma.

Lemma 2.3. *Let B be a ring, and let $J \subset I \subset B$ be finitely generated ideals. Let $V =$ $\operatorname{Spec} B \setminus V(I)$, and $\mathcal{G}$ be finite locally free on V.*

If $H^i(V, \mathcal{O}_V)$ is annihilated by J^n for some n, then $H^i(V, \mathcal{G})$ is annihilated by J^m for some m.

Proof. The reason is essentially that $\mathcal{G}$ is a direct sum of copies of $\mathcal{O}_V$, plus quasicompactness. □

You apply this with $B = A^+, I = \mathfrak{m}A^+$, and $J = fA^+$.

□

So we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(U^+) & \hookrightarrow & \mathrm{Pic}(U_0^+) \\ \uparrow & & \uparrow \\ \mathrm{Pic}(U) & \longrightarrow & \mathrm{Pic}(U_0) \end{array}$$

Now the kernel of the map $\mathrm{Pic}(U) \to \mathrm{Pic}(U^+)$ is torsion because of the norm trick (a finite morphism of normal schemes is an isomorphism up to multiplication by a constant, because you can take norms).

To finish off (showing that there is only p-power torsion), it suffices to show that $\pi_1(U_0) \twoheadrightarrow \pi_1(U)$, which is a Lefschetz theorem for local π_1. This is because there is a construction from an n-torsion line bundle to an étale covering if $n \neq p$, via $L \rightsquigarrow \mathcal{O} \oplus L \oplus L^2 \ldots \oplus L^{n-1}$, and the triviality of the bundle can be read off from the triviality of this covering.

Lemma 2.4. *Assume that $H^1_{\mathfrak{m}}(A)$ is finite and $H^2_{\mathfrak{m}}(A)$ is killed by f^n for some n, and (A, fA) is henselian. Then*

$$FEt_U \to FEt_{U_0}$$

is fully faithful.

In particular, if U_0 is connected then $\pi_1(U_0, \overline{\xi}) \twoheadrightarrow \pi_1(U, \overline{\xi})$.

2.2 Proof of Proposition 1.3

Consider a triple $(\mathcal{L}, \mathcal{L}_0, \alpha)$ where

- $\mathcal{L}$ is invertible on U,
- $\mathcal{L}_0$ is invertible on X_0 (so $\mathcal{L}_0 \cong \mathcal{O}_{X_0}$),
- $\alpha \colon \mathcal{L}|_{U_0} \cong \mathcal{L}_0|_{U_0}$.

For such a triple we define the following numerical invariant:

$$\chi(\mathcal{L}, \mathcal{L}_0, \alpha) = \mathrm{length}_A(\Gamma(X_0, \mathcal{L}_0)/\alpha(\Gamma(X, \mathcal{L})/f)).$$

It isn't totally obvious that the thing we're quotienting by is even a submodule. The claim is that if we have a global section of $\mathcal{L}$, we can restrict to U_0, and after applying α we actually get a section of $\mathcal{L}_0$ (on X_0).

Lemma 2.5. *The function $n \mapsto \chi(\mathcal{L}^{\otimes n}, \mathcal{L}_0^{\otimes n}, \alpha^{\otimes n})$ is a numerical polynomial $P(n)$.*

The argument is the same as for Kleiman's argument for numerical intersection numbers.

Proof of Proposition 1.3. Suppose $\mathcal{L} \in \ker(\mathrm{Pic}(U) \to \mathrm{Pic}(U_0))$. Then we get a triple as above. If this is e-torsion, then $P(ne) = 0$ for all $n \in \mathbb{Z}$. The reason is that if we replace the triple with the eth tensor power, you get something trivial. Since P was a numerical polynomial, $P(1) = 0$ and that's the lifting result you need. □