CHOW GROUPS OF QUOTIENT STACKS

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Everything here is following Totaro [Tot99].

1. Motivation

1.1. **Principal G-bundles.** Topology is concerned with topological spaces and continuous maps between them. But the data is a topological space of so complicated and infinite in nature that it can be very difficult even to tell when two topological spaces are "the same." For instance, all n-dimensional manifolds look locally the same. A central theme in algebraic topology is to attach *algebraic invariants* to topological spaces, such as homotopy groups, homology groups, cohomoloy groups, etc.

Today we will be discussing a very specific kind of topological object, which is nonetheless ubiquitous: the *principal* G-*bundle*.

Definition 1.1. Let G be a topological group. A *principal* G-*bundle over* X is a topological space P equipped with a continuous, free action of G and a map

$$\pi \colon P \to X$$

such that

(1) π identifies the quotient space $G\backslash P$ with X, and
(2) π is locally trivial, i.e. for all $x \in X$ there is an open neighborhood $x \in U \subset X$ such that

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\cong} & U \times G \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

Remark 1.2. We say that G acts *freely* on Y if the map $G \times Y \to Y \times Y$ sending $(g, y) \mapsto (y, gy)$ is a homeomorphism onto its image. The bijectivity is equivalent to all stabilizers being trivial, which is the familiar notion of free action for discrete groups.

Example 1.3. The trivial G-bundle on X is the product space $G \times X$ with the obvious projection map.

Any G-bundle $\pi \colon P \to X$ admitting a global section $s \colon X \to P$ is trivial, as we can view s as giving a coherent choice of identity element in each fiber. Concretely, we have a map $G \times X \to P$ sending $(g, x) \mapsto gs(x)$, which is necessarily an isomorphism.

Example 1.4. As with vector bundles, one can think of principal G-bundles in terms of gluing. Explicitly, if $\{(U_\alpha, \phi_\alpha)\}$is a trivialization of $\pi \colon P \to X$, then P is determined by the

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transition functions $\tau_{\beta\alpha}\colon U \to G$:

$$G \times U_\alpha \xrightarrow{\phi_\alpha^{-1}} p^{-1}(U_\alpha) \xrightarrow{\phi_\beta} G \times U_\beta, \qquad \tau_{\beta\alpha}\colon G \times U_\alpha \to G \times U_\beta$$

These transition functions must satisfy the *cocycle conditions* to be consistent. Thus G is called the "structure group."

One immediate consequence of this definition is that if $G = GL_n$, then our transition functions also define a vector bundle. So we see that there is an equivalence between principal GL_n-bundles and vector bundles! In fact whenever G is the automorphism group of a certain structure, a principal G-bundle will have an alternate interpretation in terms of that structure.

Even if you are not familiar with principal G-bundles, you have probably encountered plenty of vector bundles and appreciate their importance. Again, the datum of a vector bundle is complicated, "infinite," and global in nature, and it can be difficult even to tell when two vector bundles are isomorphic. Luckily, we can again attempt to attach algebraic invariants to vector bundles. Namely, to any vector bundle $V \to X$ one can associate elements of $H^*(X)$ *in a functorial way* (meaning compatible with pullbacks).

Example 1.5. If $V \to X$ is a complex vector bundle of (complex) rank n, then there are *Chern classes* $c_1(V), c_2(V), \dots, c_n(V) \in H^*(X; \mathbb{Z})$.

Example 1.6. If $V \to X$ is a real vector bundle of rank n, then there are *Stiefel-Whitney classes* $w_1(V), w_2(V), \dots, w_n(V) \in H^*(X; \mathbb{Z}/2)$.

One might ask *why* these characteristic classes exist, and why there aren't any more out there waiting to be discovered. Classifying spaces answer these questions in a very elegant way.

Definition 1.7. The space BG (well-defined up to homotopy) is a space representing the functor **Top** $\to$ **Set** sending

$$X \mapsto \{\text{principal } G\text{-bundles on } X\}/\text{isom}.$$

In other words, there is a natural bijection

$$\text{Hom}(X, BG)/\text{homotopy} \leftrightarrow \{\text{principal } G\text{-bundles on } X\}/\text{isom}$$

It is a theorem that such a space always exists. In fact, here is a "concrete" construction. Take a contractible space EG on which G acts freely. [Why does such a thing always exist?] Then $BG = EG/G$.

The map $EG \to BG$ is the "universal principal G-bundle," and it corresponds to the identity map $BG \to BG$. Given a map $f\colon X \to BG$, the corresponding principal G-bundle is the pullback

$$\begin{array}{ccc} f^*EG & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

1.2. **Characteristic classes.** Therefore, $H^*(BG)$ parametrizes all functorial assignments of cohomology classes to principal G-bundles. Given a vector bundle $P \to X$, we get a map $X \to BG$ pulling back the universal bundle to P.

$$\begin{array}{ccc} P & \dashrightarrow & EG \\ \downarrow & & \downarrow \\ X & \dashrightarrow & BG \end{array}$$

We can then assign to P the pullback of a cohomology class of BG along this map. Conversely, suppose we have such an assignment. Then it is completely determined by its value on the universal bundle, as any other bundle is a pullback of this one.

We claim that if we have a group homomorphism $H \to G$, then we get a map $BH \to BG$. Indeed, by definition giving a map $BH \to BG$ is the same as giving a functorial recipe for turning a principal H bundle into a principal G-bundle. One perspective on a principle H-bundle is in terms of transition functions with values in H satisfying the cocycle conditions. But if we compose that with the homomorphism to G, then we get transition functions valued in G satisfying the cocycle conditions, hence a principal G-bundle.

If $H \subset G$ is a subgroup, we can choose the map $BH \to BG$ to be a fibration with fiber G/H. To see this, note that H acts freely on EG a fortiori, and the fibers of $EG/H \to EG/G$ are evidently G/H.

The result we state now is probably not the optimal one, but it suffices for our purposes.

Proposition 1.8. *If* $H \hookrightarrow G$ *is a weak homotopy equivalence, then* $BH \to BG$ *is a weak homotopy equivalence.*

Proof. Recall that $H \hookrightarrow G$ is a *weak homotopy equivalence* if it induces isomorphisms on all homotopy groups, which implies (Hurewicz's Theorem) that it induces isomorphisms on all (co)homology groups.

By the long exact sequence of homotopy groups for the fibration $H \to G \to G/H$, we see that $\pi_i(G/H) = 0$ for $i > 0$. Next applying the long exact sequence of homotopy groups for the fibration $G/H \to BH \to BG$ shows that BH and BG are weakly homotopy equivalent. ☐

Example 1.9. By the Proposition, $B\,GL(1, \mathbb{R}) \cong B\mathbb{Z}/2$. What is $B\mathbb{Z}/2$? Well, $\mathbb{Z}/2$ acts freely on S^∞, which is contractible. So $B\mathbb{Z}/2 \cong \mathbb{RP}^\infty$. This has a cell structure, with one cell of each dimension and in $\mathbb{Z}/2$-(co)homology, the boundary maps are 0 (that's what makes it easy to calculate!). In fact, $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1]$ where $|w_1| = 1$.

Given any real line bundle $L \to X$, we get a map $f \colon X \to \mathbb{RP}^\infty$ such that the pullback of the tautological bundle is L. The *first Stiefel-Whitney class* $w_1(L)$ is precisely $f^*[w_1]$.

Example 1.10. By the fact, $B\,GL(1, \mathbb{C}) \cong BS^1$. Again, S^1 acts on $S^\infty \subset \mathbb{C}^\infty$ by multiplication, and the quotient is $\mathbb{CP}^\infty$. The cohomology ring is $H^*(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[c_1]$.

Given any *complex* line bundle $L \to X$, we get a map $f \colon X \to \mathbb{CP}^\infty$ such that the pullback of the tautological bundle is L. The *first Chern class* $c_1(L)$ is precisely $f^*[c_1]$.

2. CHOW GROUPS

2.1. **The definition.** One can think of Chow groups as being something like a "homology theory" for algebraic varieties. We can think of elements of $CH_*(X)$ as representing subvarieties of X, just as a map of closed, oriented, connected manifolds $Y \to X$ induces an element of $H_*(X; \mathbb{Z})$, namely the image of the fundamental class of Y.

Definition 2.1. Formally, we define the group of *cycles* $Z_k(X)$ to be the free abelian group on k-dimensional subvarieties (which we take by definition to be closed, irreducible, reduced). Then $Z_*(X) = \bigoplus_k Z_k(X)$.

We say that two cycles in $Z_k(X)$ are *rationally equivalent* if there exists a cycle on $\mathbb{P}^1 \times X$ whose restrictions to the fibers $\{t_0\} \times X$ and $\{t_1\} \times X$ are A_0 and A_1.

The subgroup $B_k(X)$ is generated by differences of rationally equivalent varieties. (Warning: this is non-standard notation!) We set $B_*(X) := \bigoplus_k B_k(X)$.

The (graded) *Chow group* $CH_*(X)$ is the quotient $Z_*(X)/B_*(X)$, and we have the natural quotient grading

$$CH_*(X) = \bigoplus_k CH_k(X) := Z_k(X)/B_k(X).$$

One can think of rational equivalence as stating that there is a "family" parametrized by a segment in $\mathbb{P}^1$ whose boundary is $A_0 - A_1$. This is reminiscent of cobordism.

Example 2.2. Any two points in $\mathbb{A}^n$ are rationally equivalent, because we can pick a line between them. In fact, a point is rationally equivalent to the empty set because we can "push it off" to ∞.

Any hypersurface in $\mathbb{A}^n$ is rationally equivalent to the empty set, as the graph of $f\colon \mathbb{A}^n \to \mathbb{P}^1$ is a cycle in $\mathbb{A}^n \times \mathbb{P}^1$ whose fiber over ∞ is empty.

In fact, we claim that

$$CH^*(\mathbb{A}^n) \cong \begin{cases} \mathbb{Z} & * = n, \\ 0 & * \neq 0 \end{cases}$$

It suffices to show that any proper subvvariety $W \subset \mathbb{A}^n$ is rationally equivalent to the empty set. We will try the same pushing off trick. As W is a proper subvariety, we may assume that $O \notin Y$. Consider the subvariety $\widetilde{W} \subset \mathbb{A}^n \times (\mathbb{A}^1 - \{O\})$ defined by

$$\widetilde{W} = \{(z,t) \mid \frac{z}{t} \in W\}.$$

In terms of geometry, this if the family whose fiber at t is the dilation of W by t. In terms of equations, this is cut out $f(z/t)$ such that $f(z)$ vanishes on Y. Thus the closure of $\widetilde{W}$ is a family in $\mathbb{A}^n \times \mathbb{P}^1$, whose fiber over $t = 1$ is precisely Y. The fiber over ∞ should morally be , as we have "pushed away" all the points. Let's see this explicitly.

As $O \notin W$, there exists a polynomial $g(z)$ vanishing on Y and having non-zero constant term: $g(z) = c + \ldots$. Then $g(z/t) = c + t^{-1} \ldots$ has the value c on the fiber $\mathbb{A}^n \times \{\infty\}$.

Example 2.3. Any two points on a genus $g > 0$ projective curve are *not* rationally equivalent - if they were, then the corresponding cycle in $X \times \mathbb{P}^1$ would give a birational map $X \to \mathbb{P}^1$.

Here is another characterization of $B(X)$. For any rational function f on a subvariety $Y \subset X$, we can associate a divisor

$$\mathrm{Div}(f) = \sum_{\substack{W \subset Y \\ \text{codim } 1}} \mathrm{ord}_W(f)[W].$$

Then $B(X)$ is generated by $\mathrm{Div}(f)$ as f and Y vary.

One direction is quite trivial: given a rational function f, we get a rational map $f\colon Y \to \mathbb{P}^1$. The graph of f (the closure of the usual graph on an open subset) is a cycle in $Y \times \mathbb{P}^1 \subset X \times \mathbb{P}^1$, the difference of whose fibers over 0 and ∞ is precisely $\mathrm{Div}(f)$. This shows that $\mathrm{Div}(f) \subset B(X)$.

The other direction is a bit more subtle. Given a cycle $\widetilde{V} \subset X \times \mathbb{P}^1$ with fiber $A_0 - A_\infty$, the projection map determines a rational function $\widetilde{f}$ on V such that $\mathrm{Div}(\widetilde{f}) = [A_0] - [A_\infty]$. The map $V \to X$ is generically finite over its image unless $\widetilde{V} = X \times \mathbb{P}^1$, and then $\mathrm{Nm}(\widetilde{f})$ turns out to do the trick.

Example 2.4. $CH_n(X) = Z_n(X)$ is the free abelian group on the irreducible (connected) components of X. $Z_{n-1}(X)$ is just the group of divisors of X. $B_{n-1}(X)$

If X has pure dimension n, then $CH_{n-1}(X) \cong \mathrm{Cl}(X)$, i.e. divisors modulo principal divisors.

There is a map $CH_*(X) \to H_*(X)$, essentially by inclusion of the fundamental class (as we discussed previously). This gets a little messy because you have to define the fundamental class of a singular variety, but it works out. The map is actually more like a cobordism theory than a homology theory, and Totaro showed that it factors through the complex cobordism ring MU^*.

A very naïve conjecture would be that if $X/\mathbb{C}$ is a smooth projective variety, then the map $CH_*(X) \to H_*(X; \mathbb{Z})$ is *surjective*, i.e. any element of $H_*(X; \mathbb{Z})$ is obtained as the fundamental class of some an algebraic subvariety. This fails for at least two reasons: first, one cannot expect this to be true integrally, but only rationally. Second, there are some constraints from Hodge theory. If one refines the conjecture appropriately to account for these obstructions, then one arrives at the *Hodge conjecture*.

Definition 2.5. If X is a compact complex manifold of dimension n, then Poincaré duality "identifies" $H^i(X)$ and $H_{n-i}(X)$. Motivated by this, we define $CH^i(X) := CH_{n-i}(X)$.

2.2. **Functoriality.** As the homology and cohomology are functorial, one might expect functoriality properties for Chow groups. These are a little subtle, but they do exist.

Proper pushforward. If $f\colon Y \to X$ is a proper map, then we can "push forward" subvarieties to subvarieties. However, one has to take care that this map preserve rational equivalences. We define $f_*\colon CH_*Y \to CH_*X$ by

$$f_*([A]) = \begin{cases} 0 & \dim f(A) < \dim A, \\ n[f(A)] & [K(A) : K(f(A))] = n. \end{cases}$$

We will mostly just be thinking of the case where f is a closed embedding, in which case $f_*([A]) = [f(A)]$ on the nose.

Flat pullback. If $f\colon X \to Z$ is flat, then we may define a pullback map $f^*\colon CH^*Y \to CH^*X$ which is determined by

$$f^*([A]) = [f^{-1}(A)]$$

when $f^{-1}(A)$ is reduced.

Push-Pull formula. These two operations satisfy

$$f_*(\alpha \cdot f^*\beta) = f_*\alpha \cdot \beta.$$

2.3. **Properties.** We now discuss some features of the Chow groups that will be useful for computations.

Excision. If $Y \subset X$ is a closed subscheme and $U = Y \setminus X$ is its complement, then the inclusion and restriction maps of cycles give a right exact sequence

$$\mathrm{CH}_*(Y) \xrightarrow{j_*} \mathrm{CH}_*(X) \xrightarrow{\iota^*} \mathrm{CH}_*(U) \to 0. \tag{1}$$

This is analogous to the excision axiom in algebraic topology.

Homotopy invariance. If $\pi\colon V \to X$ is an affine space bundle (i.e. a fiber bundle whose fibers are affine space), then the induced map $\pi^*\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(V)$ is a surjection. If V is actually a *vector* bundle (i.e. there exists a section), then π^* is an isomorphism. This is analogous to the fact that a vector bundle is homotopy equivalent to its base.

2.4. **Ring structure.** In fact, $\mathrm{CH}_*(X)$ has a *ring structure*. This might seem weird at first, if we're thinking of CH_* as some algebraic analogue of homology, but recall that compact complex manifolds *also* have a ring structure coming from the intersection product (dual to the cup product via Poincaré duality), which has the property that the intersection of the homology classes represented by two transversely intersecting, complementary-dimensional submanifolds is precisely the number of intersection points.

Definition 2.6. We say that subvarieties $A, B \subset X$ *intersect generically transversely* if they intersect transversely at a generic point of each component of $A \cap B$.

Theorem 2.7. *There exists a unique product structure on* $\mathrm{CH}^*(X)$ *satisfying the condition that if* A, B *are generically transverse then* $[A] \cdot [B] = [A \cap B]$. *This product structure makes* $\mathrm{CH}^*(X)$ *a commutative graded ring.*

Example 2.8. If X is a quasiprojective surface and D is an ample line bundle on X, then $A + nD$ and $B + nD$ will be very ample for $n \gg 0$. By Bertini's Theorem, we can find representatives in the class of $[A + nD]$ and $[B + nD]$ that intersect generically transversely. Then linearity forces the value of $[A] \cdot [B]$.

2.5. **Examples.**

Example 2.9. We saw earlier that

$$\mathrm{CH}^*(\mathbb{A}^n) \cong \begin{cases} \mathbb{Z} & * = 0, \\ 0 & * > 0. \end{cases}$$

We claim that the same holds for any open subset $U \subset \mathbb{A}^n$. Indeed, $Y := \mathbb{A}^n \setminus U$ is a closed subset of dimension at most $n - 1$, so the excision exact sequence gives a surjection $\mathrm{CH}_*(\mathbb{A}^n) \to \mathrm{CH}_*(U)$.

Example 2.10. Let's compute $\mathrm{CH}^*(\mathbb{P}^n)$. We have an inclusion of $\mathbb{P}^{n-1}$ as a closed subscheme, with the complement being $\mathbb{A}^n$. Therefore, the excision exact sequence is

$$\mathrm{CH}_*(\mathbb{P}^{n-1}) \to \mathrm{CH}_*(\mathbb{P}^n) \to \mathrm{CH}_*(\mathbb{A}^n) \to 0$$

but we know that $\mathrm{CH}^*(\mathbb{A}^n) = \mathbb{Z}$ (generated by the fundamental class) if $* = 0$ and 0 otherwise. Therefore, $\mathrm{CH}^*(\mathbb{P}^n)$ is generated by the fundamental class and $\mathrm{CH}_*(\mathbb{P}^{n-1})$.

We claim that $\mathrm{CH}^*(\mathbb{P}^n) = \mathbb{Z}[h]/h^{n+1}$, where h represents the class of a hyperplane, i.e. the image of the fundamental class of $\mathbb{P}^{n-1}$. Let h' be the hyperplane class of $\mathbb{P}^{n-1}$, which maps to h^2. Then if $a(h')^k = 0$, we would have $ah^{2k} = 0$. Since the intersection product is well-defined, we could intersect an $n-2k$-plane to find that $a = 0$ by Bezout's theorem.

In fact, an easy generalization of this argument shows that whenever X has an affine stratification, i.e. a partition into affine spaces $\{U_i\}$ such that if U_i intersects $\overline{U_j}$, then $U_i \supset U_j$, then CH^*X is generated by the closed strata, i.e. the classes of the $\overline{U_i}$.

3. Classifying spaces in algebraic geometry

Now we are finally ready to begin the fun. We would like to have some notion of "classifying spaces" in algebraic geometry. What could this mean? Ideally, we could find a *scheme* BG such that

$$\mathrm{Hom}_{\mathbf{Sch}}(X, BG) \leftrightarrow \{\text{principal } G\text{-bundles on } X\}.$$

In other words, we want to represent the functor taking a scheme X to *algebraic* principal G-bundles over X. Here I am brushing an important but technical point under the rug: the "local triviality" should not be with respect to the standard Zariski topology (except in lucky cases), but some finer Grothendieck topology.

Anyway, this doesn't really matter because BG doesn't exist (in the category of schemes). It is easy to see why this would be: for instance, as long as G has non-trivial center, a principal G-bundle will have non-trivial automorphisms.

Nevertheless, Totaro gave a definition of the "Chow ring of BG." The motivation goes back to the topological construction of BG as $G\backslash EG$, where EG is a contractible space with a free G-action. We don't have the luxury of an EG in algebraic geometry, but if we can pick an "almost contractible" space with a free G-action then we might expect to get a reasonable definition of Chow groups.

Consider the topological story. If V is contractible and $S \subset V$ has high codimension, say real codimension at least $i+1$, then the inclusion $V - S \hookrightarrow V$ will induce an *isomorphism* on homotopy groups $\pi_0, \ldots, \pi_i$. In particular, $V - S$ will be i-connected. Then the map $V - S \to EG$ will induce an isomorphism on homotopy groups in dimension up to i, hence also an isomorphism on homology groups up to dimension $i+1$ by the Hurewicz theorem.

Definition 3.1. We define $CH^i(BG) = CH^i(V-S)/G$ for any pair (V, S) such that G acts freely on $V - S$ and $\mathrm{codim}_V S > i$.

Definition 3.2. For an algebraic group G acting on a smooth variety X, we define the *G-equivariant Chow ring* by

$$CH^i_G(X) = CH^i(X \times (V-S))/G.$$

In order for this to really be well-defined, we have to check that it is independent of the choices V, S.

Proof. We want to show that if (V, S) are (V', S') are two pairs such that G acts freely on both $V - S$ and $V' - S'$ and $\mathrm{codim}_V S, \mathrm{codim}_{V'} S' > i$ then

$$CH^*(V-S)/G \cong CH^*(V'-S')/G \text{ for } * < i.$$

We use the "double fibration trick" due to Bogomolov in order to reduce to the special case where one pair "dominates" the other, in the sense that $V' = V$ and $S' \supset S$.

We first reduce to the case where the representations are equal by considering a common domination by $V \times V'$. Then $(V-S) \times V'$ is a vector bundle over $V - S$, so $(V-S) \times V'/G$ exists and $S \times V'$ has codimension at least i in $V \times V'$. Similarly, $V \times (V'-S')$ is a vector bundle over $V' - S'$, satisfying the right conditions. By the homotopy axiom, a space as the same Chow groups as any vector bundle over it. This reduces to the case $V = V'$.

Next, replacing S' with $S \cup S'$ allows us to assume that $S' \supset S$. Then we apply the excision axiom (1):

$$CH_*(S' - S)/G \to CH_*(V - S)/G \to CH_*(V - S')/G \to 0$$

is exact. But since S' has codimension greater than i, $CH_*(S' - S)/G$ vanishes up to codimension i, so the map $CH_*(V - S)/G \to CH_*(V - S')/G$ must be an isomorphism in codimension up to i. □

We can informally think of BG as $\varinjlim_i (V_i - S_i)/G$. This may not really make sense, but we have shown that $\varinjlim_i CH^i(V_i - S_i/G)$ really does make sense, so we can just define it to be $CH^i(BG)$.

We can make all the definitions we want, but why is this a *good* definition?

Theorem 3.3. *Let* G *be a reductive group over a field* k*. Then the above group* $CH^i(BG)$ *is naturally identified with the set of (pullback) functorial assignments for every smooth quasiprojective variety* X*,*

$$\{\textit{principal } G\textit{-bundle over } X\} \to CH^i X.$$

Remark 3.4. This gives a natural ring structure on $CH^i X$, which agrees with what you think it is (namely the inverse limit of the ring structures on the finite approximations).

4. EXAMPLES

A good source of nearly contractible spaces equipped with G-action are *representations*. Therefore, our strategy for computing BG will be to find a representation of G such that the action is free on the complement of a high codimension subset S. We then need to compute the quotient variety $V - S/G$. When G is a finite group we can just take the ring of invariants in $V - S$; when G is a linear algebraic group, the quotient exists as a quasi-projective variety by general theory.

4.1. **Stratifications.**

Example 4.1. Let $G = \mathbb{G}_m$. Then G acts on $\mathbb{A}^{n+1}$ by scalar multiplication, and the action is free on $\mathbb{A}^{n+1} - \{O\}$. The quotient space is one we know and love: $\mathbb{P}^n$. We computed earlier that $CH^*(\mathbb{P}^n) \cong \mathbb{Z}[c_1]$.

Let L_n be the tautological line bundle on $\mathbb{P}^n$. The inclusion via 0-section $\mathbb{P}^n \to L$ sends the class of the hyperplane to the class of a codimension 2 plane. On the one hand, we know that $CH^*(L) \to CH^*(\mathbb{P}^n)$ is an isomorphism by homotopy invariance. On the other hand, we have a pushforward map

$$CH^{n-k}(\mathbb{P}^n) = CH_k(\mathbb{P}^n) \to CH_k(L) \cong CH_{k-1}(\mathbb{P}^n) = CH^{n-k+1}(\mathbb{P}^n).$$

taking the fundamental class of $\mathbb{P}^n$ to the class of the zero section in $CH_k(L)$, to the hyperplane class in $\mathbb{P}^n$. Therefore, this map corresponds to multiplication by $c_1(L_n)$.

As this is the "universal" line bundle, it implies the same in general: if $L \to X$ is any line bundle, then

$$CH_k(X) \to CH_k(L) \cong CH_{k-1}(X)$$

corresponds to multiplication by $c_1(L) \in CH^*(X)$.

Example 4.2. What's BGL_n? Let V be the standard representation of BGL_n. Let $W = \mathrm{Hom}(\mathbb{A}^N, V) \cong V^N$ for $N \gg 0$. Then GL(V) acts freely on the open subset of *surjective* linear maps $\mathrm{Surj}(\mathbb{A}^n, V)$. The quotient space $\mathrm{Surj}(\mathbb{A}^n, V)/\mathrm{GL}(V)$ is isomorphic to $\mathrm{Gr}(N-n, N)$, by associating the kernel.

The codimension of the complement goes to ∞ with N, so we get that

$$\mathrm{CH}^*\mathrm{BGL}(n) = \varinjlim \mathrm{CH}^* \mathrm{Gr}(N-n, N).$$

As the Grassmannian also admits an *algebraic* affine stratification (Schubert cells), its Chow groups are the free abelian group on the set of cells, like the ordinary cohomology ring. Therefore,

$$\mathrm{CH}^*\mathrm{BGL}(n) \cong \mathbb{Z}[c_1, \dots, c_n] \quad |c_i| = i.$$

By the earlier theorem, each c_i furnishes a functorial assignment from rank n vector bundles $V \to X$ to $\mathrm{CH}^i(X)$, which is called the *Chern class*.

4.2. **Finite groups.**

Example 4.3. Let's try to compute $\mathrm{CH}^*B(\mathbb{Z}/2)$.

$i = 0$. $\mathbb{Z}/2$ acts freely on $A^1 - \{O\}$, with quotient again $\mathbb{A}^1 - \{O\}$, so

$$\mathrm{CH}^0(B\mathbb{Z}/2) \cong \mathrm{CH}^0(\mathbb{A}^1 - \{O\}) \cong \mathbb{Z}.$$

$i = 1$. $\mathbb{Z}/2$ acts freely on $\mathbb{A}^2 - \{0\}$ by multiplication by ± 1. The ring of invariants its $k[x^2, xy, y^2] \subset k[x, y]$, which you might recognize as the (affine) quadric cone $Q :=$ Spec $k[u, v, w]/(uw - v^2)$. Removing the origin corresponds to removing the cone point.

When calculating the first Chow group, we may as well throw the cone point back in since it has codimension 2 (hence doesn't affect CH^1). We claim that $\mathrm{CH}^1 Q \cong \mathbb{Z}/2$, generated by the class of a line through the origin lying on the cone, e.g. $u = 0$.

First let's see why twice the line should be zero. A plane tangent to the line intersects the cone in the double line. As the plane is rationally equivalent to zero on $\mathbb{A}^3$, its intersection is rationally equivalent to zero on the cone.

According to the basic exact sequence, the quotient of CH^1 by the class of this line is just CH^1 of the cone minus the hyperplane section. That corresponds to inverting x, in which case we get Spec $k[x^{\pm}, y]$, which is an open subset of affine space, and hence has trivial CH^1.

In general, $\mathbb{Z}/2$ will act freely on $\mathbb{A}^n - \{O\}$, and you can see that the quotient will be Spec $(k[x_1, \dots, x_n])_{2\bullet}$ minus the origin. That's the affine cone over the Veronese embedding of the smooth quadric in $\mathbb{P}^{n-1}$ minus the cone point. The CH_k of this group will have a class represented by a k-plane contained in the quadric, and twice the k-plane is the intersection of the tangent hyperplane with that k-plane is rationally equivalent to 0.

Why does this generate? Well, the complement of that that plane intersection is the image of the complement of a certain number of quadrics in $\mathbb{A}^n$, i.e. an open subset of $\mathbb{A}^n$.

Moreover, by the usual rules of intersecting k-planes in affine space, we see that the Chow ring will be $\mathbb{Z}[h]/(2h)$.

Example 4.4. Let $P \to X$ be a principal $\mathbb{G}_m$-bundle (with X smooth). Then P is the total space of the corresponding line bundle L minus the 0-section. The excision sequence gives

$$\mathrm{CH}_*(X) \to \mathrm{CH}_*(L) \to \mathrm{CH}_*(L - X = P) \to 0.$$

But the map $\mathrm{CH}_*(X) \to \mathrm{CH}_*(L)$ is multiplication by $c_1(L)$, so we get that $\mathrm{CH}_*(P) = \mathrm{CH}_*(X)/c_1(L)$.

Compare this with the Gysin sequence of a circle bundle:

$$\ldots \to H^{i-2}X \xrightarrow{c_1(L)} H^i X \to H^i P \to H^{i-1}X \to \ldots$$

Example 4.5. What's $CH^*(B\mathbb{Z}/p)$? Let W be a faithful 1-dimensional representation of $\mathbb{Z}/p$ (i.e. via a non-trivial character) and $V = W^{\oplus n}$. As $\mathbb{Z}/p$ acts freely on $V - \{0\}$, an nth level approximation to $B(\mathbb{Z}/p)$ is $(V - \{0\})/(\mathbb{Z}/p)$. Now, this action factors through a representation of $\mathbb{G}_m$, via $\mathbb{Z}/p \hookrightarrow \mathbb{G}_m \hookrightarrow GL(V)$. Therefore, we should have a fiber bundle

$$G_m/(\mathbb{Z}/p) \to B\mathbb{Z}/p \to B\mathbb{G}_m.$$

This means nothing in scheme-land, but concretely $\mathbb{A}^n - \{0\}$ can be used as an approximation to EG for both $\mathbb{G}_m$ and $\mathbb{Z}/p$, so we have a genuine fiber bundle

$$\mathbb{G}_m/(\mathbb{Z}/p) \to (\mathbb{A}^n - \{0\})/(\mathbb{Z}/p) \to (\mathbb{A}^n - \{0\})/\mathbb{G}_m.$$

Of course, we computed the latter objects as $\mathbb{P}^{n-1}$, and $\mathbb{G}_m/(\mathbb{Z}/p) \cong \mathbb{G}_m$. This realizes $(\mathbb{A}^n - \{0\})/(\mathbb{Z}/p)$ as a $\mathbb{G}_m$-bundle over $\mathbb{P}^{n-1}$, which corresponds to the line bundle $\mathcal{O}(-p)$, as it's evidently the pth power of the tautological bundle. Therefore,

$$CH^* B(\mathbb{Z}/p) \cong CH^* \mathbb{P}^\infty / pc_1 \cong \mathbb{Z}[c_1]/pc_1.$$

4.3. **Classical groups.** We now develop the tools to calculate the Chow ring of some classical groups.

Theorem 4.6. *Let G be an affine group scheme over k and V a faithful representation of G. Under the induced map*

$$CH^* BGL(V) \cong \mathbb{Z}[c_1, \ldots, c_n] \to CH^* BG$$

Let $c_i \mapsto c_i V$. Then

$$CH^*(GL(V)/G) \cong CH^* BG/(c_1 V, \ldots, c_n V).$$

Proof. We (morally) have a fibration

$$GL(n)/G \to BG \to B\,GL(n).$$

By "looping" this, we also get

$$GL(n) \to GL(n)/G \to BG.$$

[In actuality, if $V - S$ approximates $E\,GL(n)$ then we get

$$GL(n)/G \to (V - S)/G \to (V - S)/\,GL(n)$$

and

$$GL(n) \to [H \times (V - S)]/G \to (V - S)/G.$$

where the middle has the diagonal G-action.]

This shows that $GL(n)/G$ is a principal $GL(n)$-bundle over BG. That's the same as a vector bundle, so it suffices to show that if $P \to X$ is principal $GL(n)$-bundle, then

$$CH^*(P) = CH^*(X)/(c_1(P), \ldots, c_n(P)).$$

We already saw this in the special case $n = 1$. That implies the result for a direct sum of line bundles, i.e. a $(\mathbb{G}_m)^n$-bundle. Then we get the result for a Borel, by considering a flag (the "splitting principle in algebraic geometry"). □

We can use this to gain information about $CH^* BG$ if we know $CH^*(GL(V)/G)$ (or vice versa). For instance, if $CH^*(GL(V)/G)$ is trivial, then

Example 4.7. What is $CH^*BO(n)_{\mathbb{C}}$? Let V be the standard representation of $O(n)$, inducing an embedding $O(n) \hookrightarrow GL(n)$. What is $GL(n)/O(n)$?

Well, $GL(n)$ acts on symmetric forms on V, i.e. $\mathrm{Sym}^2 V^*$, which is isomorphic to $\mathbb{A}^{n(n+1)/2}$. All *non-degenerate* symmetric bilinear forms are $GL(n)$-equivalent, and the stabilizer of a non-degenerate form is $O(n)$. Therefore, $GL(n)/O(n)$ can be realized as an open subset of $\mathbb{A}^{n(n+1)/2}$, so

$$CH^* GL(n)/O(n) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * > 0 \end{cases}.$$

By the theorem, we may conclude that CH^*BG is generated by $CH^*\mathbb{Z}[c_1, \ldots, c_n]$. What's the kernel?

As the representation V of $O(n)$ is self-dual, we get $c_1 = -c_1$, and in general $c_j = (-1)^j c_j(V)$. Therefore, $2c_j = 0$ for all *odd* j. In fact, this is injective. One way to see this is that the map

$$\mathbb{Z}[c_1, \ldots, c_n]/(2c_{2k+1} = 0) \hookrightarrow H^*(BO(n), \mathbb{Z})$$

is injective, but this factors through Chow.

Example 4.8. What is $CH^*B\,Sp(2n)_{\mathbb{C}}$? Again, let V be the standard representation of $Sp(2n)$. $GL(2n)$ acts transitively on the space of symplectic forms on V, which the non-degenerate ones being isomorphic to an open subset of affine space. Again, CH^*BG will be generated by $CH^*\mathbb{Z}[c_1, \ldots, c_{2n}]$. But what are the relations?

Again, the natural symplectic form makes V self-dual, so by the same reasoning we get $2c_i = 0$ for i odd. In fact, we claim that $c_i = 0$ for i odd. It suffices to show that $CH^*B\,Sp(2n) \hookrightarrow CH^*BT = CH^*B\mathbb{G}_m$ (the maximal torus), as we checked that the latter is torsion-free. As BT is an iterated affine space bundle over BB (the classifying space of the Borel), it suffices to show that $CH^*B\,Sp(2n) \hookrightarrow CH^*BB$. This fits into a fiber bundle

$$Sp(2n)/B \to BB \to B\,Sp(2n).$$

As $Sp(2n)$ is "special," this bundle is *Zariski locally trivial*, so we can take a section over an open subset and then take its closure. This gives an element α mapping to $1 \in CH^0B\,Sp(2n)$. This gives a section for Chow groups, as $f_*(\alpha \cdot f^*x) = x$ for all $x \in CH^*B\,Sp(2n)$.

It is known that $H^*(BSp(2n), \mathbb{Z}) \cong \mathbb{Z}[c_2, c_4, \ldots, c_{2n}]$, so we can conclude that $CH^*B\,Sp(2n) \cong \mathbb{Z}[c_2, c_4, \ldots, c_{2n}]$.

References

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