1. Exercise I

1. (Affine Grassmannian of $\mathbb{G}_a$.) Show that $\mathrm{Gr}_{\mathbb{G}_a} \simeq \varinjlim \mathbb{A}^n$, where $\mathbb{A}^n \to \mathbb{A}^{n+1}$ is the standard inclusion.

2. (Loop group and affine Grassmanian of $\mathbb{G}_m$.) (i) Let R be a commutative ring whose spectrum is connected. Show that every invertible element $f(t) \in R((t))^\times$ can be uniquely written as

$$f(t) = r \cdot t^n \cdot f_+(t) \cdot f_-(t)$$

for some $r \in R^\times, n \in \mathbb{Z}$,

$$f_+(t) = 1 + \sum_{i \geq 1} r_i t^i \in R[[t]], \tag{1.0.1}$$

and

$$f_-(t) = 1 + \sum_{i \geq 1} r_{-i} t^{-i} \in R[t^{-1}], \quad r_{-i} \text{ nilpotent.} \tag{1.0.2}$$

(ii) Let $\mathbb{W}$ be the presheaf that assigns every R the set of power series as in (1.0.1). Let $\widehat{\mathbb{W}}$ be the presheaf that assigns every R the set of polynomials as in (1.0.2). Show that $\mathbb{W}$ is represented by a group subscheme of $L\mathbb{G}_m$ and $\widehat{\mathbb{W}}$ by a group sub-ind-scheme. Usually, $\mathbb{W}$ is called the ring of big Witt vectors.

(iii) Show that as group ind-schemes,

$$L\mathbb{G}_m \simeq \mathbb{G}_m \times \mathbb{Z} \times \mathbb{W} \times \widehat{\mathbb{W}}.$$

In particular $\mathrm{Gr}_{\mathbb{G}_m} \simeq \mathbb{Z} \times \widehat{\mathbb{W}}$ is not reduced.

(iv) Show that $\mathrm{Gr}_{\mathbb{G}_m}$ is formally smooth. (In fact, $\mathrm{Gr}_{\underline{G}}$ is formally smooth in general).

(v) Show that the morphism $L\mathbb{G}_m \to L\mathbb{A}^1$ is not an open embedding.

Remark 1.1. At the level of k' points, where $k' \supset k$ is a field, there is a canonical isomorphism $L\mathbb{G}_m(k') = L\mathbb{A}^1(k) \setminus \{0\}$. Intuitive, different connected components of $L\mathbb{G}_m$ (labelled by $\mathbb{Z}$) glue together. This is an important phenomenon in geometric representation theory. (More in Sasha Braverman's lecture?)

3. (Relative positions) (i) In the case $G = \mathrm{GL}_n$, identify $\mathbb{X}_\bullet(T)_+$ with $\{(m_1, \ldots, m_n) \in \mathbb{Z}^n \mid m_1 \geq m_2 \geq \cdots \geq m_n\}$ in a way such that $(m_1, \ldots, m_n) \leq (m_1', \ldots, m_n')$ if and only if

$$\begin{aligned} & m_1 \leq m_1', \\ & m_1 + m_2 \leq m_1' + m_2', \\ & \qquad \cdots \\ & m_1 + \cdots + m_{n-1} \leq m_1' + \cdots + m_{n-1}', \\ & m_1 + \cdots + m_n = m_1' + \cdots + m_n'. \end{aligned}$$

(ii)Show that given two rank n projective $R[[t]]$-modules $\mathcal{E}_1$ and $\mathcal{E}_2$, and an isomorphism $\beta : \mathcal{E}_1 \otimes R((t)) \simeq \mathcal{E}_2 \otimes R((t))$. The set

$$\{x \in \mathrm{Spec} R \mid \mathrm{Inv}_x(\beta) \leq (r, r, \ldots, r)\}$$

is a closed subset.

(iii) Assume that G is simple and simply-connected over k. Show that $\mathrm{Gr}_{\leq \mu}(k)$ can be interpreted as the subset $(\mathcal{E}, \beta) \in \mathrm{Gr}_G(k)$ such that for every fundamental representation V_{ω_i} of G, the induced map of $k((t))$-modules

$$\beta_i : \mathcal{E}_{V_{\omega_i}} \otimes k((t)) \simeq V_{\omega_i} \otimes k((t))$$

extends to a morphism of $k[[t]]$-modules

$$\beta_i : \mathcal{E}_{V_{\omega_i}} \hookrightarrow V_{\omega_i} \otimes t^{-(\omega_i, \mu)} k[[t]].$$

Here $\mathcal{E}_{V_{\omega_i}}$ denotes the vector bundle $\mathcal{E} \times^G V_{\omega_i}$.

Remark 1.2. (Assume that char$k = 0$ for simplicity.) It is still an open question whether in (iii), one can replace k by any k-algebra R to get a moduli interpretation of $\mathrm{Gr}_{\leq\mu}$.

(iv) Conclude that $\mathrm{Gr}_{\leq\mu}$ is closed in Gr_G.

(iv) Generalize (iii) to a not necessarily simply-connected group. (You may need the relation between affine Grassmanians under central isogeny.)

(v) For $G = \mathrm{GL}_n$. Show that there is a canonical bijection of sets

$$M_n(k[[t]]) \cap \mathrm{GL}_n(k((t))) \simeq \bigsqcup_{n\geq 0} (L\mathrm{GL}_n)_{\leq n\omega_1}(k).$$

Note that this bijection is closely related to Remark 1.1.

4. A dominant cocharacter μ of G is called minuscule if $\mu \neq 0$ and for any positive root α, $(\mu, \alpha) \leq 1$. The Schubert variety $\mathrm{Gr}_{\leq\mu}$ corresponding a minuscule cocharacter is called a minuscule Schubert variety.

(i) Show that a minuscule Schubert variety is smooth.

(ii) Describe minuscule Schubert varieties for $\mathrm{PGL}_n, \mathrm{SO}_{2n+1}, \mathrm{PSp}_{2n}, \mathrm{PSO}_{2n}$ more explicitly.

A dominant cocharacter μ of G is called quasi-minuscule if $\mu \neq 0$ and for any positive root α, $(\mu, \alpha) \leq 2$. The Schubert variety $\mathrm{Gr}_{\leq\mu}$ corresponding a quasi-minuscule cocharacter is called a quasi-minuscule Schubert variety.

(iii) Show that the quasi-minuscule Schubert variety for SL_2 is isomorphic to the projective cone of a quadratic curve in $\mathbb{P}^2$.

(iv) In general, show that the *short dominant coroot* θ is a quasi-minuscule cocharacter of a simple simply-connected group. Show that $\mathrm{Gr}_{\leq\theta}$ is a projective cone over a partial flag variety of G. What is Gr_θ in this case?

(v) Let θ be as above. Let e_θ denote a root vector corresponding to θ, and $\overline{\mathcal{O}}_{e_\theta}$ the closure of the corresponding nilpotent orbit. Construct an open embedding $\overline{\mathcal{O}}_{e_\theta} \to \mathrm{Gr}_{\leq\theta}$. (You may need the big open cell on the affine Grassmannian.)

5. We consider $G = \mathrm{GL}_n$, and $\mu = m\omega_1$. Let

$$\widetilde{\mathrm{Gr}}_{\leq\mu} = \left\{ \mathcal{E}_m \subset \mathcal{E}_{m-1} \subset \cdots \subset \mathcal{E}^0 = R[[t]]^n \;\middle|\; \begin{array}{l} \mathcal{E}_i \text{ is an } R\text{-family of lattices} \\ \mathcal{E}_i/\mathcal{E}_{i+1} \text{ is a line bundle on Spec}R \end{array} \right\}.$$

(i) Show that there is a natural map

$$\pi : \widetilde{\mathrm{Gr}}_{\leq\mu} \to \mathrm{Gr}_{\leq\mu}$$

which is a resolution of singularities.

(ii) Identify $\pi^{-1}(t_{\omega_m})$ as a Springer fiber in the flag variety of GL_m.

(iii) Can you generalize (ii) to other points $t_\lambda \in \mathrm{Gr}_{\leq\mu}$?

2. Exercise II

1. (Birkoff decomposition) (i) Show that there is canonical bijection between

$$\mathrm{GL}_n(k[t^{-1}])\backslash \mathrm{GL}_n(k((t))/\mathrm{GL}_n(k[[t]])$$

and the set of isomorphism classes of rank n vector bundle on $\mathbb{P}^1_k$.

(ii) Prove the following theorem of Grothendieck: every vector bundle on $\mathbb{P}^1_k$ is a direct sum of line bundles.

(iii) Conclude that there is a canonical bijection between $\mathrm{GL}_n(k[t^{-1}])\backslash \mathrm{GL}_n(k((t))/\mathrm{GL}_n(k[[t]])$ and the set $\{(r_1,\ldots,r_n)\in\mathbb{Z}^n \mid r_1\geq\cdots\geq r_n\}$.

2. (Thick Grassmannian) Fix $x\in \mathrm{Bun}_G$. Let $\mathrm{Bun}_{G,\infty x}$ be the stack that classifies for every k-algebra R, the groupoid of pairs $(\mathcal{E},\epsilon)$, where $\mathcal{E}$ is a G-bundle on X_R, and ϵ is a trivialization of $\mathcal{E}$ on the formal neighbourhood of $\{x\}\times \mathrm{Spec}R$, i.e. the formal completion of X_R along the closed subscheme $\{x\}\times\mathrm{Spec}R$. It is known that $\mathrm{Bun}_{G,\infty x}$ is represented by a scheme (rather than an ind-scheme).

(i) Let $X=\mathbb{P}^1$, and $x=0$. Show that $\mathrm{Bun}_{G,\infty x}(k)=G(k((t))/G[t^{-1}]$. In this case, $\mathrm{Bun}_{G,\infty x}$ is also called the thick affine Grassmannian. The fundamental difference is that Gr_G is a local object but $\mathrm{Bun}_{G,\infty x}$ is global.

(ii) Let y be another point on X, different from x. Show that there is a morphism from the affine Grassmannian $\mathrm{Gr}_{G,y}$ (i.e. the moduli of G-bundles on X with a trivialization on $X-\{y\}$) to $\mathrm{Bun}_{G,\infty x}$. In particular, if $X=\mathbb{P}^1, x=0, y=\infty$, at the level of k-points, we get a map $G(k((t^{-1}))/G(k[[t^{-1}]])\to G(k((t)))/G(k[t^{-1}])$. Could you describe this map more explicitly?

(iii) Assume that X is a general curve. Show that the natural action of L^+G on $\mathrm{Bun}_{G,\infty x}$ given by changing the trivialization ϵ extends to a natural action of LG. Show that if G is semi-simple, the resulting action of LG on $\mathrm{Bun}_{G,\infty x}$ is transitive.

3. (Determinant line bundle) Let $G=\mathrm{GL}_n$ where $V=k((t))^n$, and write Gr_G by Gr for simplicity. We introduce the determinant line bundle on $\mathrm{Gr}\times\mathrm{Gr}$. For $\Lambda_1,\Lambda_2\in\mathrm{Gr}(R)$, let

$$\det(\Lambda_1|\Lambda_2)=(\wedge^{\mathrm{top}}L\hat{\otimes}R/\Lambda_1)\otimes(\wedge^{\mathrm{top}}L\hat{\otimes}R/\Lambda_2)^{-1},$$

where L is some lattice in V such that $(L\hat{\otimes}R)/\Lambda_1$ and $(L\hat{\otimes}R)/\Lambda_2$ are projective R-modules. This is independent of the choice of L up to a canonical isomorphism. This is called the relative determinant line for (Λ_1,Λ_2). This way, we define a line bundle $\mathcal{L}_{\det}$ on $\mathrm{Gr}\times\mathrm{Gr}$.

(i) For any $g\in L\mathrm{GL}(R)$, and $\Lambda_1,\Lambda_2\in\mathrm{Gr}(R)$, there is a canonical isomorphism

$$\det(g\Lambda_1|g\Lambda_2)\simeq\det(\Lambda_1|\Lambda_2),$$

such that for g,g', the isomorphism

$$\det(gg'\Lambda_1|gg'\Lambda_2)\simeq\det(g'\Lambda_1|g'\Lambda_2)\simeq\det(\Lambda_1|\Lambda_2)$$

coincides with $\det(gg'\Lambda_1|gg'\Lambda_2)\simeq\det(\Lambda_1|\Lambda_2)$. In other words, the diagonal action of $L\mathrm{GL}_n$ on $\mathrm{Gr}\times\mathrm{Gr}$ lifts to an action on $\mathcal{L}_{\det}$;

(ii) For any $\Lambda_1,\Lambda_2,\Lambda_3$, there is a canonical isomorphism

$$\gamma_{123}:\det(\Lambda_1|\Lambda_2)\otimes\det(\Lambda_2|\Lambda_3)\cong\det(\Lambda_1|\Lambda_3)$$

such that for any $\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4$, $\gamma_{134}\gamma_{123}=\gamma_{124}\gamma_{234}$.

(iii) Let $\Lambda_0=k[[t]]^n\subset k((t))^n$ be the standard lattice. Therefore, we get a line bundle

$$\mathcal{L}_{\det}|_{\{\Lambda_0\}\times\mathrm{Gr}},$$

on Gr still denoted by $\mathcal{L}_{\det}$. We can construct sections of $\mathcal{L}_{\det}$ as follows. Let $L\subset k((t))^n$ be a subspace complementary to Λ_0, i.e. $k((t))^n=L\oplus\Lambda_0$. Let $x:\mathrm{Spec}R\to\mathrm{Gr}$ be a map given by an R-family lattice $\Lambda\subset R((t))^n$. Consider the 2-term complex

$$\Lambda\oplus(L\otimes_k R)\to R((t))^n,$$

of R-modules. Make sense of the determinant of this map and show that the determinant of this map defines a section ϑ_L of $x^*\mathcal{L}_{\det}$. By varying x, we get a section ϑ_L of $\mathcal{L}_{\det}$.

In the sequel, we fix $L = L_0 = (t^{-1}k[t^{-1}])^n$.

(iv) Let $\mathrm{SL}_n \to \mathrm{GL}_n$ denote the natural embedding, which induces a closed embedding

$$i : \mathrm{Gr}_{\mathrm{SL}_n} \to \mathrm{Gr}.$$

Show that the non-vanishing loci of ϑ_{L_0} on $\mathrm{Gr}_{\mathrm{SL}_n}$ is exactly the big open cell

$$L^-\mathrm{SL}_n L^+\mathrm{SL}_n / L^+\mathrm{SL}_n \subset \mathrm{Gr}_{\mathrm{SL}_n}.$$

Show that $i^*\mathcal{L}_{\det} \simeq \mathcal{O}(1)$.

(v) Assume $\mathrm{char} k \neq 2$ Let $G = \mathrm{SO}_n$. Then the natural embedding $\mathrm{SO}_n \to \mathrm{GL}_n$ induces a closed embedding

$$i : \mathrm{Gr}_{\mathrm{SO}_n} \to \mathrm{GL}_n.$$

Show that the non-vanishing loci of ϑ_{L_0} on $\mathrm{Gr}^0_{\mathrm{SO}_n}$ (the neutral connected component of $\mathrm{Gr}_{\mathrm{SO}_n}$) is still the big open cell

$$L^-\mathrm{SO}_n L^+\mathrm{SO}_n / L^+\mathrm{SO}_n \subset \mathrm{Gr}_{\mathrm{SO}_n}.$$

However, show that $i^*\mathcal{L}_{\det} \simeq \mathcal{O}(2)$.

4. We assume that $\mathrm{char} k = 0$.

(i) We assume that $G = \mathrm{SL}_n$. Let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation, which induces a closed embedding

$$\mathrm{Ad} : \mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}(\mathfrak{g})}.$$

Show that $\mathrm{Ad}^*\mathcal{L}_{\det} \simeq \mathcal{O}(2n)$.

(ii) Show that the restriction of $\mathcal{O}(n)$ to each Schubert cell Gr_μ is the anti-canonical bundle of Gr_μ. (Recall that since Gr_μ is a single L^+G-orbit, it is smooth.)

(iii) Let G be a general semsimple group, and let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ denote the adjoint representation, which induces $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}(\mathfrak{g})}$. Show that the restriction of $\mathrm{Ad}^*\mathcal{L}_{\det}$ to each Schubert cell Gr_μ is twice of the anti-canonical bundle of Gr_μ.

5. (Convolution Grassmannians) Let $\mathcal{E}_0 = \mathcal{E}^0$ denote the trivial $\underline{G}$-torsor. We define the convolution Grassmannian over X^n as

$$\mathrm{Gr}^{\mathrm{conv}}_{X^n} = \left\{ \{x_i, \mathcal{E}_i, \beta_i\}, i = 1, \ldots, n \;\middle|\; \begin{array}{l} \{x_i\} \in X^n(R), \quad \mathcal{E}_i \text{ are } \underline{G}\text{-torsors on } X_R, \\ \beta_i : \mathcal{E}_i|_{X_R \setminus \Gamma_{x_i}} \simeq \mathcal{E}_{i-1}|_{X_R \setminus \Gamma_{x_i}} \text{ is an isomorphism} \end{array} \right\}$$

(i) Show that $\mathrm{Gr}^{\mathrm{conv}}_{X^n}$ is represented by an ind-scheme over X^n.

(ii) Show that there is a canonical morphism

$$m : \mathrm{Gr}^{\mathrm{Conv}}_{X^n} \to \mathrm{Gr}_{X^n}$$

(where Gr_{X^n} is the Beilinson-Drinfeld Grassmannian as defined in the lecture) which restricts to an isomorphism $\mathrm{Gr}^{\mathrm{Conv}}_{X^n} \times_{X^n} U \simeq \mathrm{Gr}_{X^n} \times_{X^n} U$, where U is the open subset of X^n consisting of points $\{(x_1, \ldots, x_n\}$ that are pairwise distinct.

(iii) Let $n = 2$. Describe the map m over $(x, x) \in X^2$.

3. Exercise III

1. Describe the cohomology group of a smooth quadratic in $\mathbb{P}^n$ using the geometric Satake.

2. Let $\mathrm{Gr}_{\omega_1,\omega_1^*}$ be the variety classifying the chain of lattices $(\Lambda_2 \supset \Lambda_1 \subset \Lambda_0 = k[[t]]^n)$ in $k((t))^n$, with Λ_1 of codimension one in Λ_0 and Λ_2. Let $Z \subset \mathrm{Gr}_{\omega_1,\omega_1^*}$ be the closed sub variety defined by the condition $\Lambda_0 = \Lambda_2$. Show that Z is middle dimensional and calculate the self-intersection number of Z using the geometric Satake.

3. Consider the situation as in Exercise I, Problem 5.

(i) Show that $\widetilde{\mathrm{Gr}}_{\leq m\omega_1} \to \mathrm{Gr}_{\leq m\omega_1}$ as considered in that exercise is the convolution map $m : \mathrm{Gr}_{\omega_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{\omega_1} \to \mathrm{Gr}_{\leq m\omega_1}$.

(ii) Let $\mathrm{Gr}^{\square}_{\leq m\omega_1}$ denote the GL_m-torsor over $\mathrm{Gr}_{\leq m\omega_1}$ classifying $(\mathcal{E} \to \mathcal{E}_0) \in \mathrm{Gr}_{\leq m\omega_1}(R)$ together with an isomorphism $\mathcal{E}_0/\mathcal{E} \simeq R^m$ of R-modules. Show that there is the following diagram

$$\mathcal{N} \xleftarrow{\pi} \mathrm{Gr}^{\square}_{\leq m\omega_1} \xrightarrow{p} \mathrm{Gr}_{\leq m\omega_1},$$

where $\mathcal{N}$ is the nilpotent cone in GL_m. Show that $\mathrm{Gr}^{\square}_{\leq m\omega_1} \to \mathcal{N}$ is smooth of relative dimension mn.

(iii) Show that the pullback the Springer resolution $\widetilde{\mathcal{N}} \times_{\mathcal{N}} \mathrm{Gr}^{\square}_{\leq m\omega_1}$ is isomorphic to $\widetilde{\mathrm{Gr}}_{\leq m\omega_1} \times_{\mathrm{Gr}_{\leq m\omega_1}} \mathrm{Gr}^{\square}_{\leq m\omega_1}$. Conclude that the convolution map $m : \mathrm{Gr}_{\omega_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{\omega_1} \to \mathrm{Gr}_{\leq m\omega_1}$ is semismall.

(iv) Let $\mathcal{S}pr$ be the Springer sheaf on $\mathcal{N}$. Show that there is a canonical isomorphism $\pi^*\mathcal{S}pr[mn] \simeq p^*(\mathrm{IC}_{\omega_1} \star \cdots \star \mathrm{IC}_{\omega_1})[m^2]$.

Remark 3.1. There is an action of symmetric group S_m on $\mathcal{S}pr$ coming from the Springer theory and an action of S_m on $\mathrm{IC}_{\omega_1} \star \cdots \star \mathrm{IC}_{\omega_1}$ coming from the symmetric monoidal structure on the Satake category. One can show that the canonical isomorphism above is compatible with the S_m-actions.

4. We consider a Quot Scheme

$$\mathrm{Quot}(\mathcal{O}_X^n, r) = \{q : \mathcal{O}^n_{X_R} \to \mathcal{Q} \mid \mathcal{Q} \text{ is } R\text{-flat torsion sheaf of length } r.\}$$

(i) Show that there is the morphism

$$\pi : \mathrm{Quot}(\mathcal{O}_X^n, r) \to X^{(r)},$$

where $X^{(r)}$ is the rth symmetric power of X.

(ii) Show that there is a natural embedding

$$\mathrm{Quot}(\mathcal{O}_X^n, r) \times_{X^{(r)}} X^r \to \mathrm{Gr}_{\mathrm{GL}_n, X^r}.$$

(iii) Let

$$\tilde{\pi} : \mathrm{Gr}^{\mathrm{conv}}_{X^r,\omega_1,\ldots,\omega_1} \to X^r$$

denote the scheme over X^r, classifying

$$\left\{ \{x_i, \mathcal{E}_i, \beta_i\}, i = 1, \ldots, r \;\middle|\; \begin{array}{l} \{x_i\} \in X^r, \mathcal{E}_i \text{ are locally free sheaves of rank } n \text{ on } X, \beta_i : \mathcal{E}_i \hookrightarrow \mathcal{E}_{i-1} \\ \text{embedding of coherent sheaves such that } \mathcal{E}_{i-1}/\mathcal{E}_i \text{ is a simple skyscraper sheaf at } x_i \end{array} \right\}.$$

Note that $\mathrm{Gr}^{\mathrm{conv}}_{X^r,\omega_1,\ldots,\omega_1}$ is a closed subscheme of the convolution Grassmannian as defined in Example II, Problem 5.

Show that $\mathrm{Gr}^{\mathrm{conv}}_{X^r,\omega_1,\ldots,\omega_1}$ is smooth over X^r and there is a small map

$$p : \mathrm{Gr}^{\mathrm{conv}}_{X^r,\omega_1,\ldots,\omega_1} \to \mathrm{Quot}(\mathcal{O}_X^n, r)$$

such that $\pi \circ p = \mathrm{sym} \circ \tilde{\pi}$, where $\mathrm{sym} : X^r \to X^{(r)}$ is the symmetrisation map. Deduce that, $p_*\mathbb{Q}_\ell$ is a perverse sheaf (up to shift) on $\mathrm{Quot}(\mathcal{O}_X^n, r)$ with an action of the symmetric group

S_r. Show that $(p_*\mathbb{Q}_\ell)^{S_r}$ is the intersection cohomology sheaf of $\mathrm{Quot}(\mathcal{O}_X^n, r)$ (up to shift) and in fact it is the constant sheaf. Conclude that $H^*(\mathrm{Quot}(\mathcal{O}_X^n, r)) \simeq \mathrm{Sym}^r(H^*(\mathbb{P}^{n-1}) \otimes H^*(X))$.