The geometric Satake isomorphism for p-adic groups

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1 Geometric Satake

Let me start by recalling a fundamental theorem in the Geometric Langlands Program, which is the Geometric Satake isomorphism.

Fix a reductive group G/k, and set Gr to be the affine Grassmannian for G. We'll recall the definition later on, but for now let's just recall some basic facts:

- Gr is an ind-projective variety over k, i.e.

$$\mathrm{Gr} = \varinjlim X_i$$

  where $X_i \hookrightarrow X_{i+1}$ is a closed embedding of projective varieties.

- Gr admits an action of the *loop group* LG,

- $\mathrm{Gr}(k) = G(k((t)))/G(k[[t]])$.

- We can define a category of *perverse sheaves* on Gr:

$$\mathcal{P}(\mathrm{Gr}) = \varinjlim \mathcal{P}(X_i).$$

- The X_i can even be chosen to be L^+G-invariant, so we can define a category of *equivariant* perverse sheaves

$$\mathcal{P}_{L^+G}(\mathrm{Gr}) = \varinjlim \mathcal{P}_{L^+G}(X_i).$$

- The category is equipped with a monoidal structure via the convolution product

$$*\colon \mathcal{P}_{L^+G}(\mathrm{Gr}) \times \mathcal{P}_{L^+G}(\mathrm{Gr}) \to \mathcal{P}_{L^+G}(\mathrm{Gr}).$$

  Since we will be making heavy use of this, let's recall the definition. Denote here and throughout $F = k((t))$ and $\mathcal{O} = k[[t]]$. Let $G(F) \times^{G(\mathcal{O})} G(F)/G(\mathcal{O})$ be the product modulo the diagonal action. We can consider the multiplication

$$G(F) \times^{G(\mathcal{O})} G(F)/G(\mathcal{O}) \xrightarrow{m} G(F)/G(\mathcal{O})$$

We can consider also the projections

$$\begin{array}{ccc} & G(F)\times^{G(O)} G(F)/G(O) \xrightarrow{\quad m \quad} G(F)/G(O) & \\ \swarrow & & \searrow \\ G(F)/G(O) & & G(O)\times^{G(O)} G(F)/G(O) \end{array}$$

Then for $\mathcal{F}$ and $\mathcal{G} \in \mathcal{P}_{L^+G}(\mathrm{Gr})$, we define

$$\mathcal{F} * \mathcal{G} = m_*(\mathcal{F}\widetilde{\boxtimes}\mathcal{G}).$$

Theorem 1.1 (Geometric Satake)**.** *Taking hypercohomology induces an isomorphism*

$$H^\bullet\colon (\mathcal{P}_{L^+G}(\mathrm{Gr}), *) \xrightarrow{\sim} \mathrm{Rep}(\widehat{G}_{\overline{\mathbb{Q}_\ell}}.$$

Applications.

1. The *formulation* of Geometric Langlands in general requires this theorem.
2. Applications to the representation theory of $\widehat{G}$.
3. Applications to classical Langlands by V. Lafforgue. More precisely, Lafforgue uses correspondences on moduli stacks of Shtukas to define creation/annihilation "excursion operators."

If $k = \mathbb{F}_q$, then this theorem is a categorification of classical Satake, which says

$$C(G(\mathbb{F}_q[[t]])\backslash G(\mathbb{F}_q((t)))/G(\mathbb{F}_q[[t]]) \cong R(\widehat{G}) \otimes \overline{\mathbb{Q}_\ell}$$

where $R(\widehat{G})$ is the representation ring of $\widehat{G}$.

In fact, the classical statement holds replacing $\mathbb{F}_q((t))$ with $\mathbb{Q}_p$. This suggests:

Question. Do we have a geometric Satake correspondence for G (say split) over $\mathbb{Q}_p$?

If you wanted to apply V. Lafforgue's ideas to the number field situation, you would need such a thing.

Theorem 1.2 (Zhu)**.** *The answer is yes.*

Of course, this needs to be made more precise. In this talk we'll just try to indicate the ideas. First let me mention some potential applications.

Application. One can define certain (very special) excursion operators for Shimura varieties.

2 Discussion of Theorem

For simplicity, consider $G = \mathrm{GL}_n / \mathbb{Q}_p$. There are two basic difficulties in creating a p-adic version of geometric Satake.

1. We need an algebro-geometric analogue of the affine Grassmannian, i.e. $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ should be realized as the $\mathbb{F}_p$-points of some algebro-geometric object (namely an ind-scheme).

2. Then you need to prove the analogue of the geometric Satake correspondence.

In the usual proof, although the statement is purely local the proof uses a global curve (in fact, a product of two global curves). As far as I know, there isn't an analogue of this.

2.1 Witt vector affine Grassmannian

Let's first think about the first question. What could be the algebro-geometric structure on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$? You can think of this as

$$\{\text{free rank } n \ \mathbb{Z}_p\text{-modules in } \mathbb{Q}_p^n\}.$$

To give this algebro-geometric structure, we have to make a family version of this. The classical answer in the equal-characteristic situation is that we are considering the set o ffree rank n $k[[t]]$-modules in $k((t))^n$. In families, if R is any k-algebra then we can consider the set of rank n $R[[t]]$-projective modules in $R((t))^n$. If you define the affine Grassmannian as a functor which assigns to R this set, then it turns out to be an ind-projective scheme.

Now over $\mathbb{Q}_p$, we should define $\mathrm{Gr}_{G_{\mathbb{Q}_p}}(R)$, for R an algebra over $\mathbb{F}_p$, to be the set of rank n projective $W(R)$-modules inside $W(R)[1/p]^n$, where $W(R)$ is the ring of Witt vectors for R. Secretly think of $W(R) = R\widehat{\otimes}\mathbb{Z}_p$, in analogy to $R[[t]] = R\widehat{\otimes}k[[t]]$.

This is a nice attempt, but it may not (probably doesn't) work. The reason is that the ring of Witt vectors for general $R/\mathbb{F}_p$ is very badly behaved. For example, an element of $W(R)$ can be represented as $(a_0, a_1, \ldots$ with $p(a_0, a_1, \ldots) = (0, a_0^p, a_1^p)$ so for instance p could be a zero-divisor if R is non-reduced, and $W(R)/p$ may not necessarily be isomorphic to R.

However, there is a situation in which the Witt vectors are very nice:

1. $W(R)$ is well-behaved if R is a perfect ring, i.e. the map $r \mapsto r^p$ induces an isomorphism $R \cong R$. In particular, in this situation $W(R)/p \cong R$.

2. If $X/\mathbb{F}_p$ is a variety, and you consider $\{X(R) \colon R \text{ perfect}\}$ then this collection determines the *perfection* of X,
$$X^{\mathrm{perf}} = \varprojlim_{\mathrm{Frob}} X.$$

3. The étale topos of X depends only on X^{perf}.

This means that we can just consider $\mathrm{Gr}\colon \mathbf{Aff}^{\mathrm{perf}}_{\mathbb{F}_p} \to \mathbf{Set}$. Then we can ask if this is representable by a (perfect) ind-scheme on this subcategory.

Remark 2.1. We can consider presheaves on $\mathbf{Aff}^{\mathrm{perf}}_{\mathbb{F}_p}$. Inside this category we can consider the subcategory of schemes, algebraic spaces... In the usual world of schemes, there is a notion of "perfect scheme" or "perfect algebraic space" (these are the objects for which Frobenius is an isomorphism). So the notion of representability certainly makes sense.

Theorem 2.2 (Bhatt-Scholze)**.** Gr *is represented by an ind-projective scheme.*

Previously I had proved a weaker result that Gr is represented by an algebraic space.

2.2 Proof of Geometric Satake

Now it makes sense to talk about the monoidal category $\mathbf{Sat}_G := (\mathcal{P}_{L^+G}(\mathrm{Gr}), *)$, by replacing $F, \mathcal{O}$ by $\mathbb{Q}_p, \mathbb{Z}_p$.

Theorem 2.3. *This is Tannakian with the fiber functor given by*

$$H^\bullet \colon \mathbf{Sat}_G \xrightarrow{\sim} \mathrm{Rep}(\widehat{G}_{\overline{\mathbb{Q}_\ell}}).$$

Proposition 2.4. $H^\bullet$ *has a canonical monoidal structure*

$$H^\bullet(\mathcal{F} * \mathcal{G}) \xrightarrow{\sim} H^\bullet(\mathcal{F}) \otimes H^\bullet(\mathcal{G}).$$

Proposition 2.5. *There exists a canonical (unique) isomorphism* $\mathcal{F} * \mathcal{G} \cong \mathcal{G} * \mathcal{F}$ *such that*

$$\begin{array}{ccc} H(\mathcal{F} * \mathcal{G}) & \longrightarrow & H(\mathcal{G} * \mathcal{F}) \\ \downarrow \cong & & \downarrow \cong \\ H(\mathcal{F}) \otimes H(\mathcal{G}) & = & H(\mathcal{G}) \otimes H(\mathcal{F}) \end{array}$$

The uniqueness is automatic from the fact that the cohomological functor is fully faithful and conservative, so the real content is existence.

The two propositions easily imply the theorem.

The basic idea is to consider

$$\mathrm{Gr} \widetilde{\times} \mathrm{Gr} := G(F) \times^{G(\mathcal{O})} G(F)/G(\mathcal{O}).$$

Contrast this with

$$\mathrm{Gr} \times \mathrm{Gr} = G(F)/G(\mathcal{O}) \times G(F)/G(\mathcal{O}).$$

We know that for $\mathcal{F} \boxtimes \mathcal{G}$ we easily have

$$H^\bullet(\mathcal{F} \boxtimes \mathcal{G}) \cong H^\bullet(\mathcal{F}) \otimes H^\bullet(\mathcal{G}).$$

The key is to prove that $\mathrm{Gr} \widetilde{\times} \mathrm{Gr}$ is "topologically isomorphic" to $\mathrm{Gr} \times \mathrm{Gr}$. Then one gets the isomorphism by passing "through" this topological isomorphism.

This doesn't make sense in algebraic geometry, but what makes sense it that $G(F) \to \mathrm{Gr}$ is a $\mathrm{Gr}(O)$-torsor. You want to prove that this is topologically trivial, i.e. the chern class is 0. But we have a $G(O)$-action on Gr, so we can consider *equivariant* cohomology, which makes $H_{G(O)}(\mathrm{Gr})$ a *bimodule* for $H^\bullet(BG)$.

Lemma 2.6. *The two $H^\bullet(BG)$-module structures coincide.*

This implies the needed "topological isomorphism."

I don't know of a local proof. The group $G = \mathrm{GL}_n$ has an anti-involution θ (the transpose; in general need a Chevalley involution) which induces an anti-involution

$$\theta\colon G(O)\backslash G(F)/G(O) \to G(O)\backslash G(F)/G(O).$$

By the anti-involution property, one has automatically

$$\theta^*(F * G) \cong \theta^* G * \theta^* F.$$

This ("Gelfand's trick") is the basis for the classical proofs, e.g. of commutativity of the spherical Hecke algebra. What we need is to find an isomorphism $\theta^*\mathcal{F} \cong \mathcal{F}$. Since the category is semisimple, it suffices to do this for *irreducible* $\mathcal{F}$. In fact we know that the irreducible objects are "intersection cohomology sheaves" IC_μ coming from local systems on Schubert varieties.

You might think that you're done, because on the Schubert variety there is a canonical isomorphism

$$\theta^* IC_\mu \cong IC_\mu$$

but in fact you need to introduce a sign $\sqrt{-1}^{(-2\rho,\mu)}$ which is pretty subtle.

Let me give you the main idea of showing the necessary commutativity. We have a canonical isomorphism $\theta^* IC_\mu \cong IC_\mu$ inducing $\theta^*\colon H^\bullet(IC_\mu) \cong H^\bullet(IC_\mu)$.

Proposition 2.7. *θ^* acts as $(-1)^i$ on degree $2i$.*

This plus the aforementioned sign change shows the commutativity of the relevant diagram. This seems easy, but I don't know a direct proof. The only argument I have is through a spectral sequence argument, studying the action of θ^* on the stalks of IC_μ. The stalks of the IC_μ are given by the Kazhdan-Lusztig polynomials, while the other side of the equation turns out be described by LV polynomials. So the equality comes down to something of the form KL = LV, but this is non-trivial, and ends up being a theorem of Lusztig-Yun *using* Geometric Satake in equal characteristic.