Introduction to Affine Grassmannians

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Contents

1 The Affine Grassmannian

1.1 Construction

Let F be a local field (for us, $F = k((t))$, where k is a finite field). Let $V = F^n$. As a set we want the affine Grassmannian Gr parametrize the set of *lattices* in V, i.e. finitely generated $\mathcal{O}$-submodules Λ of V such that $\Lambda \otimes_{\mathcal{O}} F \cong V$.

Shortly, we will want to upgrade this construction into an algebro-geometric object, which will require a more general "functor of points" description.

Example 1.1. Suppose $\gamma \colon V \to V$ is an F-linear endomorphism. Then we can define a subset

$$\mathrm{Sp}_\gamma = \{\Lambda \in \mathrm{Gr} \mid \gamma(\Lambda) \subset \Lambda\}.$$

A priori, this is just a subset of Gr. However, we can endow Sp_γ with an algebro-geometric structure, which makes it an *affine Springer fiber* (see the lectures of Zhiwei Yun).

Remark 1.2. If k is a finite field then it makes sense to "count the number of points" of Sp_γ. Once we have a geometric structure, we can use geometric techniques (e.g. Lefschetz trace formula) to count the number of points.

As Grothendieck taught us, a scheme-theoretic structure can be interpreted as a *functor of points*. Therefore, we describe Gr as a functor

$$\mathrm{Gr} \colon \mathbf{Alg}_k \to \mathbf{Set}.$$

For a k-algebra R, viewed as a scheme over Spec k, we want to develop a notion of "family of lattices in V over Spec R."

Definition 1.3. An *R-family of lattices* in $V = k((t))^n$ is a finitely generated, projective $R[[t]]$-module $\Lambda \subset R((t))^n$ such that $\Lambda \otimes R((t)) \cong R((t))^n$.

Definition 1.4. The functor $\mathrm{Gr} \colon \mathbf{Alg}_k \to \mathbf{Set}$ is defined by $R \mapsto \{R\text{-family of lattices in } V\}$.

Theorem 1.5. *The functor* Gr *is representable by an ind-scheme, which is ind-projective over k.*

This means concretely that

$$\mathrm{Gr} = \bigcup_i X_i$$

where X_i is a subfunctor of Gr represented by a projective scheme, such that $X_i \hookrightarrow X_{i+1}$ is a closed embedding. (We have and will continue to abuse notation throughout by identifying a representable scheme with its functor of points.)

Proof. Given any $\Lambda \in \mathrm{Gr}(R)$, there exists $i > 0$ such that

$$t^i R[[t]]^n \subset \Lambda \subset t^{-i} R[[t]]^n. \tag{1}$$

Set $X_i(R) = \{\Lambda \in \mathrm{Gr}(R) \mid \Lambda \text{ satisfies (1)}\}$.

Definition 1.6. Let $X_i' \colon \mathbf{Alg}_k \to \mathbf{Set}$ be the functor defined by

$$R \mapsto \left\{ \begin{matrix} R[[t]]\text{–module quotients } Q \text{ of } t^{-i}R[[t]]^n/t^iR[[t]]^n \\ \text{that are projective as } R\text{-modules} \end{matrix} \right\}.$$

Lemma 1.7. *The functor X_i' is represented by a projective scheme (also denoted X_i'), and there are natural closed embeddings $X_i' \hookrightarrow X_{i+1}'$.*

Lemma 1.8. *There exists a natural isomorphism $X_i \cong X_i'$.*

The proof of the theorem will obviously be concluded after establishing these two Lemmas. ◻

Proof of Lemma 1.7. We are considering projective quotients of a free R-module of rank $2in$. For quotients of fixed rank over R, this is the closed subscheme of the Grassmannian parametrizing quotients which are $R[[t]]$ modules, i.e. stable under the action of t. Since t acts nilpotently on $t^{-i}R[[t]]^n/t^iR[[t]]^n$, we see that this X_i' is in fact a Springer fiber. ◻

Proof of Lemma 1.8. We have a map $X_i \to X_i'$ sending $\Lambda \mapsto t^{-i}R[[t]]^n/\Lambda =: Q$, viewed as a quotient of $t^{-i}R[[t]]^n/t^iR[[t]]^n$ in the natural way.

To see that this is really well defined, we must verify that Q is projective. We have a short exact sequence

$$0 \to Q = t^{-i}R[[t]]^n/\Lambda \hookrightarrow R((t))^n/\Lambda \to R((t))^n/t^{-i}R[[t]] \to 0.$$

To see that Q is projective, it suffices to see that the second two modules are projective. The last one is clearly projective, as it's isomorphic in an obivous way to $\bigoplus_{j \le -i} R$ by forgetting the action of t. For the middle term, note that after inverting t we have $\Lambda \otimes R((t)) \cong R((t))^n$. Therefore, it can be filtered by powers of t:

$$R((t))^n/\Lambda \cong \bigoplus_{j \le -1} t^j\Lambda/t^{j+1}\Lambda \cong \bigoplus_{j \le -1} \underbrace{\Lambda/t\Lambda}_{\cong R}.$$

The inverse construction, if it exists, must be by associating to Q the kernel of the map $t^{-i}R[[t]]^n \to Q$ and setting that to be Λ. What's not obvious is that this kernel is a projective $R[[t]]$-module.

The main difficulty is the case where R is not a noetherian ring. To reduce to the noetherian case, note that X_i' is a projective scheme, there exists $R' \subset R$ finitely generated over k such that the sequence

$$0 \to \Lambda \to t^{-i}R[[t]]^n \to Q \to 0$$

is base changed from

$$0 \to \Lambda' \to t^{-i}R'[[t]]^n \to Q' \to 0.$$

(In other words, the diagram descends to the finite type case.) In fact, we can regard Q' as a quotient of the polynomial ring rather than the power series ring, because $t^{-i}R[[t]]^n/t^iR[[t]]^n \cong t^{-i}R[t]^n/t^iR[t]^n$, so we may further descend descend to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda' & \longrightarrow & t^{-i}R'[[t]]^n & \longrightarrow & Q' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \Lambda'' & \longrightarrow & t^{-i}R'[t]^n & \longrightarrow & Q' & \longrightarrow & 0 \end{array}$$

Since $R'[t] \to R'[[t]]$ is flat (thanks to R' being *noetherian*), we have $\Lambda' \cong \Lambda'' \otimes_{R[t]} R'[[t]]$. So it is enough to prove that Λ'' is a projective $R'[t]$-module.

Now, (using noetherianness) it is enough to show that $\Lambda'' \otimes_{R'[t]} K[t]$ is $K[t]$-projective for K a field, by the local criterion for flatness ♠♠♠ TONY: [ehhh?] (since finitely generated plus flat implies projective). But that is clear, as $K[t]$ is a Dedekind domain and $Lambda'' \otimes_{R'[t]} K[t]$ is torsion-free.

□

1.2 Beauville-Laszlo interpretation

The Beauville-Laszlo interpretation realizes the affine Grassmannian in terms of any global curve.

Definition 1.9. Let X/k be a curve and $x \in X(k)$ a smooth point. Let $X^* = X - \{x\}$. Then we define the functor $\mathrm{Gr}_x \colon \mathbf{Alg}_k \to \mathbf{Set}$ by

$$\mathrm{Gr}_x(R) = \left\{ (E,\beta) \mid \begin{smallmatrix} E \text{ a rank } n \text{ vector bundle on } X_R \\ \beta\colon E|_{X_R^*} \xrightarrow{\sim} \mathcal{O}^n_{X_R^*} \end{smallmatrix} \right\}.$$

There is a map $\mathrm{Gr}_x \to \mathrm{Gr}$ obtained by restricting the vector bundle to the completion of the stalk at x, which is (non-canonically) isomorphic to $R[[t_x]]$: $(E,\beta) \mapsto (E|_{R[[t_x]]}, \beta|_{R[[t_x]]})$. We can think of this intuitively as restricting to an open disc about the point x.

Theorem 1.10. *The map* $\mathrm{Gr}_x \to \mathrm{Gr}$ *is an isomorphism.*

Proof Sketch. We need to construct the inverse. The idea is that $(E|_{R[[t_x]]}, \beta|_{R[[t_x]]})$ describes a vector bundle in a disk around x, which we want to extend to a vector bundle on all of X. We can take the trivial bundle away from x, and glue it to the given one via β.

The general difficulty is that for general R, Spec $R[[t_x]] \cup X_R^* \to X_R$ is not faithfully flat, so we can't use Grothendieck's theory of faithfully flat descent. However, we saw in the proof of the theorem that any R-point comes from an R'-point for *noetherian* R', and in that case this map *is* faithfully flat, so we can descend. □

1.3 Loop space interpretation

Definition 1.11. Let X/k be an affine variety. Define the *arc space* functor $L^+X \colon \mathbf{Alg}_k \to \mathbf{Set}$ by $R \mapsto X(R[[t]])$ and the *loop space* $LX \colon \mathbf{Alg}_k \to \mathbf{Set}$ by $R \mapsto X(R((t)))$.

Theorem 1.12. *There is a natural isomorphism of functors* $\mathrm{Gr} = L\,\mathrm{GL}_n / L^+\,\mathrm{GL}_n$*, and in particular an isomorphism of sets*

$$\mathrm{Gr}(k) = \mathrm{GL}_n(k((t)))/\,\mathrm{GL}_n(k[[t]]).$$

Proof. Define

$$\widetilde{\mathrm{Gr}}(R) = \{(\Lambda, \epsilon) \mid \Lambda \subset R((t))^n, \epsilon\colon \Lambda \cong R[[t]]^n\}.$$

This admits a natural map to $\mathrm{Gr}(R)$ by forgetting the trivialization. The fiber is a torsor for $L^+\,\mathrm{GL}_n(R)$.

$$\begin{array}{c} \widetilde{\mathrm{Gr}}(R) \\ \big\downarrow {\scriptstyle L^+\,\mathrm{GL}_n(R)} \\ \mathrm{Gr}(R) \end{array}$$

Therefore, it's enough to show that $\widetilde{\mathrm{Gr}} \cong L\,\mathrm{GL}_n$. That is easy: the isomorphism takes a pair (Λ, ϵ) to the linear automorphism $R((t))^n \xrightarrow{\epsilon^{-1}} \Lambda[t^{-1}] \cong R((t))^n$. The inverse associates to $g \in \mathrm{GL}_n(R((t))$ the lattice $g \cdot R[[t]]^n \subset R((t))^n$ (i.e. the image of the standard lattice under g). □

We can generalize the construction of the affine Grassmannian to an arbitrary reductive group.

Definition 1.13. For a reductive group G/k, define the functor $\mathrm{Gr}_G\colon \mathbf{Alg}_k \to \mathbf{Set}$ by

$$\mathrm{Gr}_G(R) = \left\{(E, \beta) \mid {E \text{ principal } G\text{-bundle on } X_R \atop \beta\colon E|_{X_R^*} \cong E^0|_{X_R^*}}\right\}$$

where E^0 is the trivial G-bundle.

Theorem 1.14. *There is a natural isomorphism* $\mathrm{Gr}_G \cong LG/L^+G$*.*

2 Properties of Gr_G

2.1 Stratification by varieties

Proposition 2.1 (Cartan Decomposition)**.** *Let G be a reductive group, and choose $G \supset B \supset T$ as usual. Then we have*

$$G(k((t))) = \coprod_{\mu \text{ dominant coweight}} G(k[[t]])t_\mu G(k[[t]]).$$

Recall that a coweight is a map $\mu\colon \mathbb{G}_m \to T$. In particular, μ induces $F[t, t^{-1}] \to G(F)$, and we denote by t_μ the image of t under this map. We denote the cocharacter group by $X_\bullet(T)$ and the dominant coweights by $X_\bullet(T)_+$.

Suppose that we have two principal G-bundles E_1 and E_2 on $D := \operatorname{Spec} k[[t]]$, and an isomorphism

$$\beta\colon E_1|_{D^*} \otimes k((t)) \xrightarrow{\sim} E_2|_{D^*} \otimes k((t)).$$

Since $E_1|_{D^*} \cong E_1 \otimes_{k[[t]]} k((t))$ and $E_2|_{D^*} \cong E_2 \otimes_{k[[t]]} k((t))$, we may view β as an element of $G(k((t)))$. However, this is not quite well-defined: only its class in $G(k[[t]])\backslash G(k((t)))/G(k[[t]])$ is well-defined, since we can apply automorphisms of E_1 and E_2 on D.

Therefore, to β we can associate an element $\mathrm{inv}(\beta) \in G(k[[t]])\backslash G(k((t)))/G(k[[t]]) = X_\bullet(T)_+$ (by the Cartan decomposition), which measures the "relative position" of E_1 and E_2. There is a partial order on $X_\bullet(T)_+$ determined by $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a non-negative integral combination of simple roots.

Recall that

$$\begin{aligned}
\mathrm{Gr}_G(k) &= \left\{(E, \beta) \mid {}^{E \text{ principal } G\text{-bundle on } k[[t]]}_{\beta\colon E \otimes k((t)) \cong E^0 \otimes k((t))}\right\} \\
&= \left\{(E, \beta) \mid {}^{E \text{ principal } G\text{-bundle on } X}_{\beta\colon E|_{X-x} \cong E^0|_{X-x}}\right\} \\
&= G(F)/G(\mathcal{O})
\end{aligned}$$

the last equality being the loop space interpretation.

Definition 2.2. We define $\mathrm{Gr}_{\leq \mu} = \{(E, \beta) \mid \mathrm{inv}(\beta) \leq \mu\}$.

Theorem 2.3. *We have $(\mathrm{Gr}_G)_{\mathrm{red}} = \varinjlim \mathrm{Gr}_{\leq \mu}$, and each $\mathrm{Gr}_{\leq \mu}$ is an irreducible, projective variety of dimension $(2\rho, \mu)$.*

Remark 2.4. $\mathrm{Gr}_{<\mu}$ is an orbit under L^+G, by the Cartan decomposition.

2.2 Functoriality

Theorem 2.5. *We have the following:*

1. *The natural map $LG \to G$ induces isomorphisms*

$$\begin{array}{ccc}
\pi_0(LG) & \xrightarrow{\cong} & \pi_1(G) \\
{\scriptstyle\cong}\downarrow & & \\
\pi_0(\mathrm{Gr}_G) & &
\end{array}$$

2. *If $G_1 \to G_2$ is a central isogeny (i.e. the kernel is contained in the center) and* ch $k = p \gg 0$*, then we have a commutative diagram*

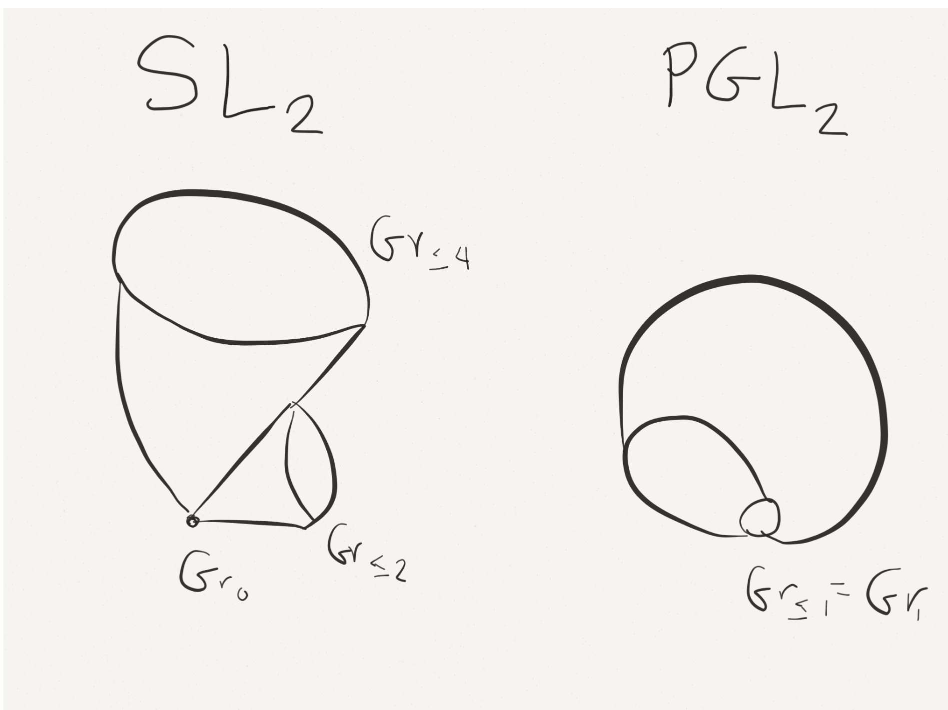
$$\begin{array}{ccc} (\mathrm{Gr}_{G_1})_{\mathrm{red}} & \longrightarrow & (\mathrm{Gr}_{G_2})_{\mathrm{red}} \\ \downarrow & & \downarrow \\ \pi_1(G_1) & \longrightarrow & \pi_1(G_2) \end{array}$$

and $(\mathrm{Gr}_G)^{\circ}_{\mathrm{red}} \cong \mathrm{Gr}_{G_{sc}}$*, where G_{sc} is the simply connected form of G.*

Remark 2.6. (1) The horizontal isomorphism reflects the familiar fact from topology, and the vertical isomorphism follows from the arc space L^+G being connected.

(2) Although the affine Grassmannian may not be connected, its components are all isomorphic because it is a homogeneous space for the loop group, hence $(\mathrm{Gr}_G)^{\circ}_{\mathrm{red}} \cong \mathrm{Gr}_{G_{sc}}$ (which is connected by the first part).

Example 2.7. Consider $G = \mathrm{SL}_2 \to \mathrm{PGL}_2$. Then $X^\bullet(T) \cong \mathbb{Z}$, under which identification we have $2\rho = \alpha = 2$. For the coweights, we have $X_\bullet(T)_+ = \{0, 1, 2, \ldots\}$.



These are isomorphic as schemes, but their L^+ actions are different.

2.3 Some sub ind-schemes of Gr_G

Let $\varphi\colon \mathrm{Gr}_G \to \mathrm{Gr}_G$ be a morphism. For $\mu \in X_\bullet(T)_+$ and $\gamma \in G(\bar{k}((t)))$, we can define a closed subscheme of Gr_G by

$$X(\mu,\gamma\varphi)(k) = \left\{ (E,\beta) \in \mathrm{Gr}_G(k) \mid \begin{smallmatrix} \beta\colon E|_{D^*} \to E^0|_{D^*} \\ \gamma\varphi(E) \xrightarrow{\gamma\varphi} E^0|_{D^*} \\ \mathrm{inv}(\psi := (\beta^{-1} \circ \gamma\varphi) \le \mu) \end{smallmatrix} \right\}.$$

To elaborate on the somewhat confusing conditions, note that $\varphi(E,\beta)$ consists of a the data of another principal G-bundle $\varphi(E)$ and a trivialization $\varphi(E)|_{D^*} \cong E^0|_D^*$. We can

$$\begin{array}{ccc} E|_{D^*} & \xrightarrow{\beta} & E^0|_{D^*} \\ {\scriptstyle\psi}\downarrow & & \| \\ \gamma\varphi(E)|_{D^*} & \xrightarrow{\gamma\varphi} & E^0|_{D^*} \end{array}$$

Here's another way to think about it, which highlights a similarity with affine Springer fibers:

$$X(\mu,\gamma\varphi)(k) = \{g \in G(F) \mid G(O)g^{-1}\gamma\varphi(gG(O)) \in \overline{G(O)t_\mu G(O)}\}.$$

Since this is a closed subscheme of an ind-scheme, it is itself an ind-scheme.

Example 2.8. If $\varphi = \mathrm{Id}$, then $X(\mu,\gamma)$ is a group version of affine Springer fibers.

Example 2.9. If $k = \mathbb{F}_p$ and $\varphi = \mathrm{Frob}\colon \mathrm{Gr}_G \to \mathrm{Gr}_G$, then $X(\mu,\gamma\varphi)$ is an affine "Deligne-Lusztig variety."

Example 2.10. If $k = \mathbb{F}_p$ and $\varphi\colon \mathrm{Gr}_G \to \mathrm{Gr}_G$ is induced by the absolute ♠♠♠ TONY: [isn't this what we would call relative?] Frobenius of $k[[t]] \to k[[t]]$ sending $\sum a_i t^i \mapsto \sum a_i t^{ip}$, then $X(\mu,\gamma\varphi)$ is called a "Kisin variety". This has significance in the context of Galois representations.

Example 2.11. If $k = \mathbb{C}$ then φ is induced by $t \mapsto qt$ for $q \in \mathbb{C}^\times$, then $X(\mu,\gamma\varphi)$ is closely related to moduli of elliptic curves, via the uniformization $\mathbb{C}^\times/q^{\mathbb{Z}}$.

Suppose that $\varphi\colon \mathrm{Gr}_G \to \mathrm{Gr}_G$ is induced by $\varphi\colon LG \to LG$ preserving L^+G. We can define a twisted conjugation of LG by LG via

$$g_1 \cdot^\varphi g_2 = g_1^{-1} g_2 \varphi(g_1).$$

Then we have a fiber diagram

$$\begin{array}{ccc} LG/^\varphi L^+G \supset \{\overline{G(O)t_\mu G(O)}/^\varphi L^+G\} & \hookleftarrow & X(\mu,\gamma\varphi) \\ \downarrow & & \downarrow \\ LG/^\varphi LG & \hookleftarrow & \gamma \end{array}$$

The point is that $LG/^\varphi L^+G \to LG/^\varphi LG$ is a moduli space, but very badly behaved. It is easier to study the fibers.

3 Uniformization

3.1 Uniformization of $\mathbb{P}^1$

There is an action of L^+G on Gr_G, whose orbit closures are Schubert varieties. This gives a nice stratification.

Definition 3.1. We define a functor $L^-G \colon \mathbf{Alg}_k \to \mathbf{Set}$ sending $R \mapsto G(R[t^{-1}])$. This is represented by a group ind-scheme.

We have an inclusion $L^-G \hookrightarrow LG$, equivariant for the action of Gr_G. This gives an action of L^-G on Gr_G, but in contrast to the case of L^+G, where the orbit closures were finite-dimensional closed subvarieties, the orbit closures of L^-G are infinite-dimensional.

Definition 3.2. Let $\mathrm{Bun}_G(\mathbb{P}^1)$ be the moduli space of G-bundles on $\mathbb{P}^1$.

By the description of the affine Grassmannian on a global curve, we have a map

$$\mathrm{Gr}_G \to \mathrm{Bun}_G(\mathbb{P}^1)$$

forgetting the trivialization: $(E, \beta) \mapsto E$. The fibers of this map are different trivializations of E away from 0. But given any trivialization, any other trivialization is related by something in L^-G, since $\mathrm{Aut}(E^0|_{\mathbb{P}^1-0}) \cong L^-G$. So we see that L^-G acts transitively on the fibers.

Theorem 3.3. *We have an isomorphism of stacks:*

$$[L^-G \backslash \mathrm{Gr}_G] \cong \mathrm{Bun}_G(\mathbb{P}^1).$$

There are several different levels at which one can understand this theorem. The first is at the level of k-points, where it says that every G-bundle on $\mathbb{P}^1$ arises uniquely from $L^-G(k)\backslash \mathrm{Gr}_G(k)$. Since we described the affine Grassmannian in terms of gluing the trivial G-bundle on the complement of x to a G-bundle locally near x, this is equivalent to:

Proposition 3.4. *Every G-bundle on $\mathbb{P}^1$ is trivial on $\mathbb{P}^1 - \{0\}$.*

The reason is that a G-bundle can be reduced to a B-bundle, which is then reduced to $\mathbb{G}_m$ and $\mathbb{G}_a$ bundles on $\mathbb{A}^1$, and all such are trivial.

At another level, the fact that Bun_G is an algebraic stack amounts to the concrete assertion that for every G-bundle on X_R ($X = \mathbb{P}^1$), there exists a faithfully flat map $R \to R'$ such that $E|_{X_R}$ is trivial.

Applications. The theorem implies that the $L^-G(k)$-orbits of Gr_G are in bijection with $\mathrm{Bun}_G(\mathbb{P}^1)(k) \cong X_\bullet(T)_+$.

Example 3.5. For $G = \mathrm{SL}_2$, every SL_2-bundle on $\mathbb{P}^1$ is equivalent to $\mathcal{O}(n) \oplus \mathcal{O}(-n)$.

Here's another corollary. There exists an $\mathbb{A}^1$-family of rank two bundles E_t on $\mathbb{P}^1$, such that at $0 \in \mathbb{A}^1$ we have $E_0 \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$ and for $t \neq 0$ we have $E_t = \mathcal{O} \oplus \mathcal{O}$.

Exercise 3.6. Prove it by considering the extension group between $\mathcal{O}(1)$ and $\mathcal{O}(-1)$.

This shows that $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ should be in the closure of $\mathcal{O} \oplus \mathcal{O}$. In fact,

$$\mathrm{Bun}_G(\mathbb{P}^1) = \coprod_{\mu \in X_\bullet(T)_+} Z_\mu = L^- G \cdot t_\mu.$$

where $\overline{Z_\mu} = \coprod_{\lambda \geq \mu} Z_\lambda$ (so the inclusion relation is *opposite* of the Bruhat order).

Then Z_0 is open (corresponding to the trivial G-torsor), corresponding to the open orbit $L^- G \cdot L^+ G / L^+ G \subset \mathrm{Gr}_G$. This is called the "big open cell."

Theorem 3.7. *We have:*

1. $\mathrm{Bun}_G(\mathbb{P}^1) \setminus Z_0 =: \Theta$ *is pure of codimension one, hence* Θ *is an effective divisor (using that* Bun_G *is smooth),*

2. *If G is simple and simply-connected, then* $\mathrm{Pic}(\mathrm{Bun}_G(\mathbb{P}^1)) \cong \mathbb{Z}\Theta \xrightarrow{\sim} \mathrm{Pic}(\mathrm{Gr}_G)$.

3.2 Affine Loop Group

Fix an algebraically closed field $k = \overline{k}$ and $F = k((t)) \supset \mathcal{O} = k[[t]]$. Let G be a simple, simply-connected group. As discussed in the previous section, we have a map

$$u_0 \colon \mathrm{Gr}_G \to \mathrm{Bun}_G(\mathbb{P}^1).$$

The important thing about this map is that it is a torsor under a group ind-scheme $L^- G = \mathrm{Aut}(E^0|_{\mathbb{P}^1 - 0})$.

The Picard group of this moduli space is generated by $\mathcal{O}(1) = u_0^* \mathcal{L}(\Theta)$ where Θ is the divisor $\mathrm{Bun}_G(\mathbb{P}^1) \setminus Z_0$. It is a fact that $\mathcal{O}(1)$ is ample. This seems strange, because it is a pullback, but there is no contradiction to the usual theorems because the map itself is so horribly infinite type that none of the usual theorems apply.

We have a central extension

$$1 \to \mathbb{G}_m \to \widehat{LG} \to LG \to 1$$

where

$$\widehat{LG} = \{(g, \alpha) \mid g \in LG, \alpha \colon g^* \mathcal{O}(1) \cong \mathcal{O}(1)\}$$

so the action of LG on Gr_G lifts to an action of $\widehat{LG}$ on (the total space) $\mathcal{O}(1) \to \mathrm{Gr}_G$. It is interesting to ask whether or not the action of LG lifts to $\mathcal{O}(1)$, i.e. whether or not this central extension splits.

There is another torus $\mathbb{G}_m$ acting on F by rotation on t, so we can form

$$\widetilde{LG} = \widehat{LG} \ltimes \mathbb{G}_m.$$

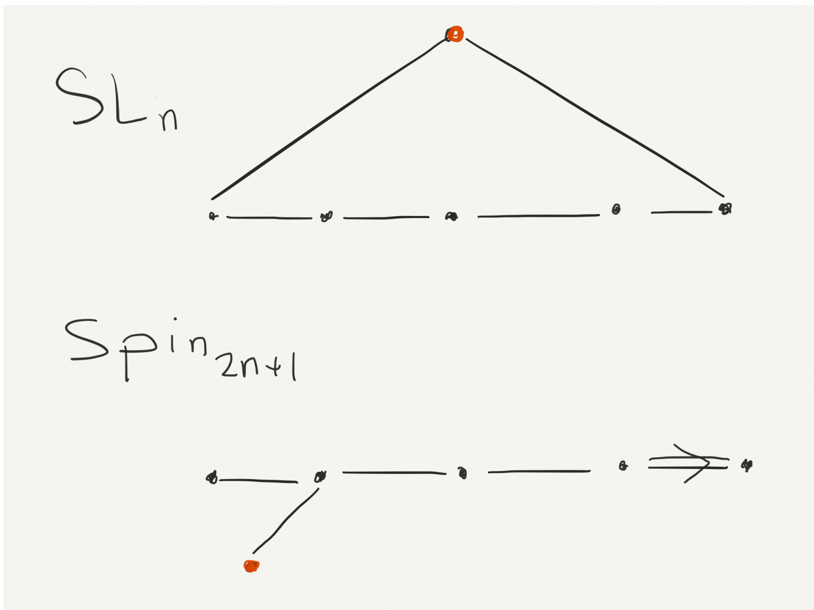
Theorem 3.8. *We have:*

1. $\widetilde{LG}$ *is isomorphic to the Kac-Moody group associated to the affine Dynkin diagram of* G.

2. Gr_G *is isomorphic to the (partial) flag variety of the affine Kac-Moody group, corresponding to the extra node.*

Remark 3.9. In my experience, the second part of the theorem is more useful.

Example 3.10. For $G = \mathrm{SL}_n$ and $G = \mathrm{Spin}_{2n+1}$, the affine Dynkin diagrams are:



We have the following corollary, analogous to Borel-Weil-Bott:

Theorem 3.11. *The space* $\Gamma(\mathrm{Gr}_G, \mathcal{O}(n))^\vee$ *is isomorphic to the level n basic representation of the Kac-Moody algebra.*

3.3 Uniformization of general curves

Now, recall that going back to the affine Grassmannian, we could have taken an arbitrary curve X/k and point $x \in X(k)$. Then we would similarly have

$$u_x\colon \mathrm{Gr}_G \to \mathrm{Bun}_G(X).$$

The fibers of this map are a torsor for $L_{\mathrm{out}}G := \mathrm{Aut}(E^0|_{X-x})$.

Theorem 3.12. *If G is semisimple, then we have an isomorphism of stacks*

$$[L_{out}G \backslash \mathrm{Gr}_G] \cong \mathrm{Bun}_G(X).$$

Again, you can interpret this at two levels. The first is that this is surjective at the level of points. where it has the meaning that any principal G-bundle on $X - x$ is trivial. The second is that at the level of R points, any bundle on X_R can be trivialized after faithfully flat extension on R.

Example 3.13. Let's try to understand what this isomorphism says for $G = \mathrm{SL}_2$. Let $\mathcal{E}$ be an SL_2-bundle on X. We want to show that $\mathcal{E}|_{X-x}$ is trivial.

Proof. We sketch an argument.

1. Any G-bundle has a reduction to the Borel. Concretely, this means that for the associated vector bundle $\mathcal{E}$, we can find a filtration $\mathcal{E}$

   $$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1} \to 0.$$

   (Any $\mathcal{E}$ admits a filtration of line bundles, and the triviality of determinant forces the quotient to be $\mathcal{L}^{-1}$.) Now, any extension on an affine curve is trivial (by the vanishing of cohomology), so $\mathcal{E}|_{X-x} \cong \mathcal{L} \oplus \mathcal{L}^{-1}$. We now want to show that we can choose $\mathcal{L}$ to be trivial.

2. We want to find a global section $\mathcal{O} \xrightarrow{s} \mathcal{L} \oplus \mathcal{L}^{-1}$ such that $s(y) \neq 0$ at any $y \in X - x$. Indeed, you can find a non-zero section $s_1 \colon \mathcal{O} \to \mathcal{L}(nx)$, which vanishes at only finitely many points, and $s_2 \colon \mathcal{O} \to \mathcal{L}^{-1}(nx)$ not vanishing at those points at which s_1 vanishes. Then $s_1 \oplus s_2$ fits the bill.

3. This gives an extension

   $$0 \to \mathcal{O}|_{X-x} \xrightarrow{s} \mathcal{L} \oplus \mathcal{L}^{-1}|_{X-x} \to \mathcal{O}|_{X-x} \to 0.$$

   By the same argument, any such extension splits, so $\mathcal{L} \oplus \mathcal{L}^{-1}|_{X-x} \cong \mathcal{O}_{X-x}^{\oplus 2}$.

□

Example 3.14. Let $G = \mathbb{G}_m$. There exist non-trivial line bundles on some affine curve $X - x$ if X has positive genus. So we see that the semisimplicity assumption cannot be dropped.

3.4 Applications of uniformization

You can use this to study the topology of G-bundles.

Theorem 3.15. *We have isomorphisms*

$$\pi_0(\mathrm{Bun}_G(X)) \xleftarrow{\sim} \pi_0(\mathrm{Gr}_G) \cong \pi_1(G).$$

The isomorphism $\pi_0(\mathrm{Bun}_G(X)) \to \pi_1(G)$ is given by the "Chern class map." To see this, it suffices to show that $L_{\mathrm{out}}(G)$ is connected (but this isn't always true).

In particular, for simply-connected G we see that Bun_G is connected. We then have the analogous result to Theorem 3.7:

Theorem 3.16. *If G is simply-connected, then the bundle $\mathcal{O}(1)$ descends to* $\mathrm{Bun}_G(X)$*, giving a generator of the Picard group.*

And also analogous to Theorem 3.11 is:

Corollary 3.17. *If* $\operatorname{ch} k = 0$*, then we have an isomorphism*

$$\Gamma(\mathrm{Bun}_G, \mathcal{O}(\kappa))^\vee \cong L_\kappa / L_{out}\mathfrak{g}$$

where L_κ is the level κ basic representation of $\widehat{L\mathfrak{g}} = \mathrm{Lie}(\widehat{LG})$, and $L_{out}\mathfrak{g} = \mathrm{Lie}(L_{out}G)$.

Comparing with the situation for $\mathbb{P}^1$, we see two drawbacks. One is that this theory works only for semisimple groups. The second and more serious issue is that we don't really understand $L_{\mathrm{out}}G$.

3.5 The Beilinson-Drinfeld Grassmannian

Recall the classical uniformization theorem of Weil:

Theorem 3.18 (Weil)**.** *We have a canonical isomorphism*

$$\mathrm{Bun}_G(X)(k) \cong G(k(X)) \backslash G(\mathbb{A}) / G(\mathcal{O})$$

where $k(X)$ is the function field of X.

If G is semisimple, then this uniformization theorem is equivalent to a *local* one which recovers what we have been discussing:

$$\mathrm{Bun}_G(X)(k) \cong \Gamma \backslash G(F_x) / G(\mathcal{O}_x).$$

where $\Gamma = L_{\mathrm{out}}(G)(k)$ is a mysterious discrete subgroup that we don't really understand.

Idea. The global uniformization theorem of Weil makes sense in algebraic geometry! This suggests that we consider the following moduli problem.

Define the set

$$\mathrm{Gr}_{\mathrm{rat}}(R) = \left\{ (E, \beta) \mid {}^{E \text{ a } G\text{-bundle on } X,}_{\beta \colon E|_{k(X)} \cong E^0|_{k(X)}} \right\}$$

This admits a map to $\mathrm{Bun}_G(k)$, whose fibers are torsors for $G(k(X))$. We are using the triviality of any G-bundle on $k(X)$, which follows from cohomological dimension one (we are assuming that k is algebraically closed).

How do we think about trivializations? You can think of it as a trivialization on some open subsets, or on the complement of finitely many points. Taking the latter point of view, we have

$$\mathrm{Gr}_{\mathrm{rat}} = \prod_{x \in X}{}' G(F_v)/G(\mathcal{O}_v).$$

This motivates:

Definition 3.19. Let I be a finite non-empty set. The *Beilinson-Drinfeld Grassmannian* associated to a reductive group G/k is defined by

$$\mathrm{Gr}_{X^I}(R) = \left\{ (x_i \in X(R))_{i \in I}, E, \beta \mid {}_{\beta\colon\, E|_{X_R - \cup_{i\in I}\Gamma_{x_i}} \cong E^0|_{X_R - \cup_{i \in I}\Gamma_{x_i}}}^{E \text{ a } G\text{-bundle on } X_R} \right\}$$

where Γ_{x_i} is the graph of $x_i\colon$ Spec $R \to X$.

Then $\{\mathrm{Gr}_{X^I}\}$ is an *algebro-geometric model* of $\mathrm{Gr}_{\mathrm{rat}}$. It admits an obvious map to X^I.

Theorem 3.20. Gr_{X^I} *is represented by an ind-scheme over* X^I*, which is ind-projective if* G *is reductive.*

Example 3.21. We consider $I = \{1\}$ and $I = \{1, 2\}$.

- If $I = \{1\}$, then we get a map $\mathrm{Gr}_X \to X$, whose fiber over some closed point $x \in X$ is just the original affine Grassmannian Gr by the Beauville-Laszlo interpretation.

$$\begin{array}{ccc} \mathrm{Gr}_X & \hookleftarrow & \mathrm{Gr} \\ \downarrow & & \downarrow \\ X & \hookleftarrow & x \end{array}$$

  In general, this is a non-trivial family.

- If $I = \{1, 2\}$ then we have $\mathrm{Gr}^2_X \to X^2$, and we can stratify the base by the diagonal Δ and its (open) complement.

$$\begin{array}{ccccc} & & \mathrm{Gr}_{X^2} & & \\ & & \downarrow & & \\ \Delta & \hookrightarrow & X^2 & \hookleftarrow & U = X^2 \setminus \Delta \end{array}$$

  The fiber over $x \in \Delta$ is Gr again. What about the fiber over $x \in U$? We claim that it is $\mathrm{Gr} \times \mathrm{Gr}$. Indeed, Gr consists of the data of a G-bundle on a formal disk D, together with a gluing to the trivial bundle away from the center. Then the data of $(E, \beta\colon E|_{X-x-y} \cong E^0|_{X-x-y})$ consists of two G-bundles on two disjoint formal disks, together with two sets of gluing functions away away from each center.

$$\begin{array}{ccccc} \mathrm{Gr} & \hookrightarrow & \mathrm{Gr}_{X^2} & \hookleftarrow & \mathrm{Gr} \times \mathrm{Gr} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta & \hookrightarrow & X^2 & \hookleftarrow & U = X^2 \setminus \Delta \end{array}$$

  This seems weird: upon specialization, the fiber gets *smaller*. But again there's no contradiction to the usual theorems, as everything here is horribly infinite type.

  You can restore your intuition by going back to the world of finite type varieties. In particular, you can look at the finite type subvarieties whose fibers over the open stratum are $\mathrm{Gr}_{\le\lambda, X} \times \mathrm{Gr}_{\le\mu, X}$, and if you specialize to the diagonal then you get $\mathrm{Gr}_{\le\lambda+\mu, X}$, which is "larger."

Functoriality. Suppose we have a surjective map of sets $\phi\colon J \twoheadrightarrow I$. Then we get a diagonal embedding

$$\Delta(\phi)\colon X^I \hookrightarrow X^J$$

embedding the ith component of X^I diagonally in the products corresponding to $\phi^{-1}(i)$. This induces

$$\begin{array}{ccc} \mathrm{Gr}_{X^I} & \hookrightarrow & \mathrm{Gr}_X^J \\ \downarrow & & \downarrow \\ X^I & \hookrightarrow & X^J \end{array}$$

Then we can take a limit $\varinjlim \mathrm{Gr}_{X^I}$ over the category of finite sets with surjective maps as morphisms. This is an algebro-geometric model of $G(\mathbb{A})/G(\mathcal{O})$, in the sense that $\varinjlim \mathrm{Gr}_{X^I}(k) = G(\mathbb{A})/G(\mathcal{O})$. However, we should warn that the limit is *not* represented by an ind-scheme, because the limit is not filtered (since the objects have automorphisms). [The problem is basically that the limit should be taken in a category of presheaves, and the limit turns out not to be a sheaf.]

We have an obvious map $\mathrm{Gr}_{X^J} \to \mathrm{Bun}_G(X)$ by forgetting the sections and the trivializations. These are compatible, so induce a map

$$u\colon \varinjlim \mathrm{Gr}_{X^J} \to \mathrm{Bun}_G(X).$$

Taking k-points gives a rational version of Weil's uniformization.

Compared to the local version $Gr_G \to \mathrm{Bun}_G(X) = \Gamma\backslash G(F_x)/G(\mathcal{O}_x)$, in which the group Γ is complicated (depending on the geometry of X), the map $G(\mathbb{A})/G(\mathcal{O}) \to G(k(X))\backslash G(\mathbb{A})/G(\mathcal{O})$ is a torsor for the relatively nice group $G(k(X))$.

Theorem 3.22 (Beilinson-Drinfeld)**.** *The category of line bundles on* $\mathrm{Bun}_G(X)$ *(trivialized at the trivial G-bundle, considered as a point on* Bun_G*) is equivalent to the category of line bundles on* $\varinjlim \mathrm{Gr}_{X^I}$ *trivialized along the unit section.*

What does this actually mean? A line bundle on $\varinjlim Gr_X^I$ is a collection of line bundles on each Gr_{X^I}, compatible under the maps induced by pullbacks. The unit section is where E is a trivial bundle.

Remark 3.23. We mean rigidified line bundles above, so all automorphisms are killed.

Theorem 3.24 (Gaitsgory-Lurie)**.** *The map u induces an isomorphism*

$$H_\bullet(\varinjlim \mathrm{Gr}_{X^I}, \mathbb{Q}_\ell) \cong H_\bullet(\mathrm{Bun}_G, \mathbb{Q}_\ell).$$

Remark 3.25. Lurie uses this in an essential way to calculate $H_*(\mathrm{Bun}_G, \mathbb{Q}_\ell)$.

Let's just try to convey a heuristic understanding of why this should be true. For simplicity, take $G = \mathbb{G}_a$. The fiber of u is $\mathbb{G}_a(k(X)) = k(X)$. This is an infinite-dimensional k-vector space, so its fiber should have no cohomology. Similarly for $G = \mathbb{G}_m$ the fiber is $k(X)^\times$, which is something like an "infinite sphere" hence contractible.

Let's see an application, using a result that we stated previously: Theorem 3.7.

Corollary 3.26. *If G is simple and simply-connected, we have a local uniformization*

$$u_x \colon \mathrm{Gr}_G \to \mathrm{Bun}_G$$

and the pullback induces an isomorphism of Picard groups

$$u_x^* \colon \mathrm{Pic}(\mathrm{Bun}_G) \cong \mathrm{Pic}(\mathrm{Gr}_G).$$

Proof. We know that

$$\mathrm{Pic}(\mathrm{Bun}_G) = \varprojlim \mathrm{Pic}(\mathrm{Gr}_{X^I} / X^I).$$

(We take the *relative* Picard group because we considered line bundles which were trivialized along a section.) But in fact, the relative Picard groups form a *sheaf*:

$$\mathrm{Pic}(\mathrm{Gr}_{X^I} / X^I) = \Gamma(X^I, \mathit{Pic}(\mathrm{Gr}_{X^I} / X^I))$$

We claim that $\mathit{Pic}(\mathrm{Gr}_X / X) \cong \mathbb{Z}_X$. The reason is that Pic is an étale sheaf, so we may reason étale-locally, and thus assume that $X \cong \mathbb{A}^1$. But for $\mathbb{A}^1$, an observation due to Mirkovic-Vilonen is that $\mathrm{Gr}_{\mathbb{A}^1} \cong \mathrm{Gr} \times \mathbb{A}^1$ because of the translation action on $\mathbb{A}^1$. Then it becomes clear that the relative Picard group is $\mathbb{Z}$, since Theorem 3.7 implies that $\mathrm{Pic}(\mathrm{Gr}) \cong \mathbb{Z}$.

Now what about $\mathrm{Pic}(\mathrm{Gr}_{X^2} / X^2)$? If we restrict to $U = X^2 \setminus \Delta$, we have

$$\mathrm{Pic}(\mathrm{Gr}_X \times \mathrm{Gr}_X / X^2) = \mathbb{Z}_X \times \mathbb{Z}_X|_U$$

by the Seesaw principle. Namely, for a given line bundle $\mathcal{L}$ on $\mathrm{Gr}_X \times \mathrm{Gr}_X$ it must be the case that $\mathcal{L}|_{p \times \mathrm{Gr}_X}$ is constant in p because $\mathrm{Pic}\,\mathrm{Gr}_X$ is discrete. So after tensoring with line bundles from $p_1^* \mathrm{Pic}(\mathrm{Gr}_X)$ and $p_2^* \mathrm{Pic}(\mathrm{Gr}_X)$, we may assume that $\mathcal{L}$ is trivial on all vertical and horizontal fibers, which implies that it is trivial by the Seesaw Theorem. (Of course, in general the Picard group of a product is not the product of Picard groups).

Because we are dealing with étale sheaves, there is a "cospecialization map"

$$\mathit{Pic}(\mathrm{Gr}_{X^2} / X^2)|_\Delta \to \mathit{Pic}(\mathrm{Gr}_{X^2} / X^2)|_U.$$

Lemma 3.27. *The map*

$$\begin{array}{ccc} \mathit{Pic}(\mathrm{Gr}_{X^2} / X^2)|_\Delta & \longrightarrow & \mathit{Pic}(\mathrm{Gr}_{X^2} / X^2)|_U \\ \cong\downarrow & & \cong\downarrow \\ \mathbb{Z}_X & \longrightarrow & \mathbb{Z}_X \times \mathbb{Z}_X \end{array}$$

is the diagonal embedding.

To prove this, we may assume that $X = \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$.

This comes from another fact. Recall from Theorem 3.7 that the map $\mathrm{Gr}_{(\mathbb{P}^1)^2} \to \mathrm{Bun}_G(\mathbb{P}^1)$, induces an isomorphism of Picard groups where we have $\mathcal{L}(\Theta)$. Consider the diagram

$$\begin{array}{ccc} \mathrm{Gr}_{\mathbb{P}^1} & \searrow & \\ \downarrow & & \\ \mathrm{Gr}_{(\mathbb{P}^1)^2} & \longrightarrow & \mathrm{Bun}_G(\mathbb{P}^1) \\ \uparrow & & \\ \mathrm{Gr}_{\mathbb{P}^1} \times \mathrm{Gr}_{\mathbb{P}^1} & & \end{array}$$

The bundle $\mathcal{O}(\Theta)$ on $\mathrm{Bun}_G(\mathbb{P}^1)$ pulls back to the generator of $\mathrm{Pic}(\mathrm{Gr}_{\mathbb{P}^1})$, which we also denote by $\mathcal{O}(\Theta)$. Its pullback to $\mathrm{Gr}_{\mathbb{P}^1} \times \mathrm{Gr}_{\mathbb{P}^1}$ via the two possible projections must coincide.

In other words, we have

$$0 \to Pic(\mathrm{Gr}_{X^2}/X^2) \to \mathbb{Z}_X \times \mathbb{Z}_X \xrightarrow{\text{subtract}} \Delta_* \mathbb{Z}_X \to 0.$$

We can continue this argument in the obvious way, showing that the $\Delta(\phi)$ maps induce

$$\begin{array}{ccc} \mathrm{Pic}(\mathrm{Gr}_{X^I}/X^I) & \xleftarrow[\Delta(\phi)^*]{} & \mathrm{Pic}(\mathrm{Gr}_{X^J}/X^J) \\ \cong\downarrow & & \cong\downarrow \\ \mathbb{Z} & = & \mathbb{Z} \end{array}$$

□

4 Geometric Satake

4.1 Perverse sheaves on the affine Grassmannian

Recall that the affine Grassmannian can be presented as an inductive limit of projective schemes under closed embeddings:

$$\mathrm{Gr} = \varinjlim X_i.$$

Definition 4.1. We define $\mathcal{P}(\mathrm{Gr}) := \varinjlim \mathbf{Perv}(X_i; \mathbb{Q}_\ell)$. Thus while Gr is "infinite," we restrict our definition of perverse sheaves to be supported on finite type subvarieties.

We can choose X_i to be L^+G-invariant (e.g. by taking $X_i = \mathrm{Gr}_{\leq\mu}$) such that the action of L^+G factors through a finite-dimensional quotient. Then we can define a category of *equivariant* perverse sheaves:

Definition 4.2. We define the *Satake category* $\mathbf{Sat}_G$ to be

$$\mathbf{Sat}_G = \mathbf{Perv}_{L^+G}(\mathrm{Gr}_G) := \varinjlim \mathbf{Perv}_{L^+G}(X_i).$$

Properties. We list some properties (which depend crucially on taking $\mathbb{Q}_\ell$ coefficients).

- $\mathcal{P}_{L^+}(\mathrm{Gr})$ is semisimple. We know that it should be artinian by generalities on perverse sheaves, but this is even saying that there are no non-trivial extensions.

  Why is this the case? More generally, the affine Grassmannian is an instance of a "partial flag variety." It's a general fact that the IC sheaves of Schubert varieties in partial flag varieties are pointwise pure.

  Also, on each connected component the $\dim \mathrm{Gr}_{\leq\mu}$ has the same parity, so the codimension between any two strata is at least two, which implies that there are no extensions of IC sheaves.

- There is a convolution product on $\mathbf{Sat}_G$ making it a monoidal category:

$$\begin{aligned} *\colon \mathbf{Sat}_G \times \mathbf{Sat}_G &\to \mathbf{Sat}_G \\ \mathcal{F}, \mathcal{G} &\mapsto \mathcal{F} * \mathcal{G}. \end{aligned}$$

  More generally, the K-equivariant derived category $D_K(G/K)$ always has a convolution product

$$D_K(G/K) \times D_K(G/K) \to D_K(G/K)$$

  obtained from the diagram

$$G/K \times G/K \xleftarrow{\pi \times 1} G \times G/K \xrightarrow{q} G \times^K G/K \xrightarrow{m} G/K$$

$$\mathcal{F}_1, \mathcal{F}_2 \longmapsto \pi^*\mathcal{F}_1 \boxtimes \mathcal{F}_2 \longmapsto q^*(\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2) \longmapsto m_! q^*(\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2)$$

So $\mathcal{F}_1 * \mathcal{F}_2 := m_1(\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2)$, where $\widetilde{\boxtimes}$ is a twisted tensor product coming from the K-equivariant structure.

In general, the convolution of perverse sheaves need not be perverse. However, we have:

Theorem 4.3. *For $G = LG$ and $K = L^+G$ the convolution is perverse, hence descends to a monoidal structure on $\mathbf{Sat}_G$.*

Theorem 4.4 (Geometric Satake Correspondence)**.** *The cohomology functor $H^\bullet \colon \mathbf{Sat}_G \xrightarrow{\sim} \mathrm{Rep}(\widehat{G})$ is an equivalence of tensor categories, and $H^\bullet(IC_\mu) = V_\mu$.*

4.2 Discussion of proof

There is a map

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{q} LG \times \mathrm{Gr}_G \xrightarrow{p} LG \times^{L^+G} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G$$

The fibered product $LG \times^{L^+G} \mathrm{Gr}_G$ can be thought of as a twisted product $\mathrm{Gr}_G \widetilde{\times} \mathrm{Gr}_G$ (since $LG/L^+G \cong \mathrm{Gr}_G$). Given $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{Sat}_G$, there exists a unique $\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2$, such that

$$q^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \cong p^*(\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2)$$

by the L^+-equivariance of $\mathcal{F}$ and $\mathcal{G}$ (i.e. the sheaf $q^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ descends through p), and we define

$$\boxed{\mathcal{F}_1 * \mathcal{F}_2 = m_!(\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2).}$$

This also works at finite level. We have maps

$$\mathrm{Gr}_{\leq\mu} \times \mathrm{Gr}_{\leq\nu} \leftarrow (LG)_{\leq\mu} \times \mathrm{Gr}_{\leq\nu} \to \mathrm{Gr}_{\leq\mu} \widetilde{\times} \mathrm{Gr}_{\leq\nu} \xrightarrow{m_{\mu,\nu}} \mathrm{Gr}_{\leq\mu+\nu}\,.$$

In particular, we claim that $IC_\mu \widetilde{\boxtimes} IC_\nu = IC(\mathrm{Gr}_{\leq\mu} \widetilde{\times} \mathrm{Gr}_{\leq\nu}, \mathbb{Q}_\ell)$ and $m_! IC(\mathrm{Gr}_{\leq\mu} \widetilde{\times} \mathrm{Gr}_{\leq\nu}, \mathbb{Q}_\ell)$ is perverse. This latter statement is equivalent to $m_{\mu,\nu}$ being a semi-small map, which follows from analyzing the geometry of these Schubert varieties.

We want to show that this satisfies $\mathcal{F}_1 * \mathcal{F}_2 \cong \mathcal{F}_2 * \mathcal{F}_1$, and

$$H^\bullet(\mathcal{F}_1 * \mathcal{F}_2) \cong H^\bullet(\mathcal{F}_1) \otimes H^\bullet(\mathcal{F}_2).$$

These are key ingredients in the proof, and their justification will come out simultaneously from a further study of the construction.

Consider the second isomorphism. We seek a canonical isomorphism

$$H^\bullet(IC(\mathrm{Gr}_{\leq\mu} \widetilde{\times} \mathrm{Gr}_{\leq\nu})) \cong H^\bullet(IC_{\leq\mu}) \otimes H^\bullet(IC_{\leq\nu}).$$

Of course, there is always such an isomorphism non-canonically, by the decomposition theorem. Indeed, the twisted product $\mathrm{Gr}_{\leq\mu} \widetilde{\times} \mathrm{Gr}_{\leq\nu} \to \mathrm{Gr}_{\leq\mu}$ is a fibration with fibers $\mathrm{Gr}_{\leq\nu}$.

However, if we were working with an actual product rather than a twisted one, then we could get a canonical isomorphism. So the idea is to reduce to this case by deforming $\mathrm{Gr}_{\leq\mu}\widetilde{\times}\mathrm{Gr}_{\leq\nu} \rightsquigarrow \mathrm{Gr}_{\leq\mu}\times\mathrm{Gr}_{\leq\nu}$ using the Beilinson-Drinfeld Grassmannian.

Let's see this for the special case $X = \mathbb{A}^1$ (though it's important to take an arbitrary curve in general). We saw that by the Mirkovic-Vilonen trick, $\mathrm{Gr}_{\mathbb{A}^1} \cong \mathrm{Gr}\times\mathbb{A}^1$. So in this case we have an equivalence of categories

$$\mathbf{Sat}_G \cong \mathcal{P}_{L^+G\times\mathbb{G}_a}(\mathrm{Gr}_{\mathbb{A}^1})$$

sending $\mathcal{F} \mapsto \pi^*\mathcal{F}[-1]$, where π is the projection $\mathrm{Gr}\times\mathbb{A}^1 \to \mathrm{Gr}$.

We have diagrams

$$\begin{array}{ccccc} \mathrm{Gr}_X & \hookrightarrow & \mathrm{Gr}_{X^2} & \hookleftarrow & \mathrm{Gr}_X\times\mathrm{Gr}_X|_U \\ \downarrow & & \downarrow & & \downarrow \\ \Delta & \hookrightarrow & X^2 & \hookleftarrow & U = X^2\setminus\Delta \end{array}$$

which corresponds to the diagram at the level of Schubert varieties

$$\begin{array}{ccccc} \mathrm{Gr}_{\leq\mu+\nu,X} & \hookrightarrow & \mathrm{Gr}_Z & \hookleftarrow & (\mathrm{Gr}_{\leq\mu,X}\times\mathrm{Gr}_{\leq\nu,X})|_U \\ \downarrow & & \downarrow & & \downarrow \\ \Delta & \hookrightarrow & X^2 & \hookleftarrow & U = X^2\setminus\Delta \end{array}$$

Exercise 4.5. Show that the twisted product has the moduli-theoretic interpretation

$$\begin{aligned} \mathrm{Gr}_G\widetilde{\times}\mathrm{Gr}_G &:= LG\times^{L^+G}\mathrm{Gr}_G \\ &\cong \left\{(E_1,E_2,\beta_1,\beta_2) \mid {}^{\beta_1\colon E_1|_{X^*}\cong E_0|_{X^*}}_{\beta_2\colon E_2|_{X^*}\cong E_1|_{X^*}}\right\} \end{aligned}$$

Motivated by this, we define

$$\mathrm{Gr}^{\mathrm{conv}}_{X^2} = \left\{(x_1,x_2,E_1,E_2,\beta_1,\beta_2) \mid {}^{\beta_1\colon E_1|_{X-x_1}\cong E_0|_{X-x_1}}_{\beta_2\colon E_2|_{X-x_2}\cong E_1|_{X-x_2}}\right\}$$

which whose fibers over X^2 will "deform" from the twisted product $\mathrm{Gr}_G\,\widetilde{times}\,\mathrm{Gr}_G$ over the diagonal to $\mathrm{Gr}_G\times\mathrm{Gr}_G$ over its complement.

This fits into a large commutative diagram

$$\begin{array}{ccccc} (\mathrm{Gr}\widetilde{\times}\mathrm{Gr})_X & \hookrightarrow & \mathrm{Gr}^{\mathrm{conv}}_{X^2} & \overset{\tilde{j}}{\hookleftarrow} & \mathrm{Gr}_X\times\mathrm{Gr}_X|_U \\ \downarrow{\scriptstyle m} & & \downarrow & & \| \\ \mathrm{Gr}_X & \overset{i}{\hookrightarrow} & \mathrm{Gr}_{X^2} & \overset{j}{\hookleftarrow} & \mathrm{Gr}_X\times\mathrm{Gr}_X|_U \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ \Delta & \overset{\Delta}{\hookrightarrow} & X^2 & \overset{j}{\hookleftarrow} & U = X^2\setminus\Delta \end{array} \tag{2}$$

What happens if we restrict this picture to the Schubert varieties?

$$\begin{array}{ccccc}
(\mathrm{Gr}_{\leq\mu}\widetilde{\times}\mathrm{Gr}_{\leq\nu})_X & \hookrightarrow & \widetilde{Z} & \hookleftarrow & \mathrm{Gr}_{\leq\mu,X}\times\mathrm{Gr}_{\leq\nu,X} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Gr}_{\leq\mu+\nu,X} & \hookrightarrow & \mathrm{Gr}_Z & \hookleftarrow & (\mathrm{Gr}_{\leq\mu,X}\times\mathrm{Gr}_{\leq\nu,X)}\,|_U \\
\downarrow & & \downarrow & & \downarrow \\
\Delta & \hookrightarrow & X^2 & \hookleftarrow & U = X^2\setminus\Delta
\end{array}$$

This is nice because it witnesses the desired deformation from the twisted product to the ordinary product, which is precisely what we wanted.

Lemma 4.6. *For $\mathcal{F}_{1X}, \mathcal{F}_{2X} \in \mathbf{Sat}_{\mathrm{Gr}}$, we have a canonical isomorphism*

$$(j_{!*}(\mathcal{F}_{1X}\boxtimes\mathcal{F}_{2X})_U)|_\Delta \cong (\mathcal{F}_1 * \mathcal{F}_2)_X.$$

Proof. Recall that the convolution product is *semi-small* (this is a general fact) but since here everything is over a curve, the global convolution product $(\mathrm{Gr}_{\leq\mu}\widetilde{\times}\mathrm{Gr}_{\leq\nu})_X \to \mathrm{Gr}_{\leq\mu+\nu,X}$ is even small (since the fiber dimension doesn't change, but the codimension increases).

The upshot is that the pushforward via small maps commutes with forming intermediate extensions. Therefore, in the diagram (2), we can start with an outer product of sheaves on the top right copy of $\mathrm{Gr}_X\times\mathrm{Gr}_X\,|_U$ and take the middle extension sheaf via $\widetilde{j}_{!*}$ and then restrict to Δ and push down via $m_!$, and that will agree with taking the middle extension sheaf via $j_{!*}$ and then restricting to the diagonal.

$$\begin{array}{ccccc}
(\widetilde{j}_{!*}(\mathcal{F}_{1X}\boxtimes\mathcal{F}_{2X}|_U))|_\Delta & \longleftarrow & \widetilde{j}_{!*}(\mathcal{F}_{1X}\boxtimes\mathcal{F}_{2X}|_U) & \longleftarrow & \mathcal{F}_{1X}\boxtimes\mathcal{F}_{2X}|_U \\
\downarrow & & \downarrow & & \| \\
 & \longleftarrow & j_{!*}(\mathcal{F}_{1X}\boxtimes\mathcal{F}_{2X}|_U) & \longleftarrow & \mathcal{F}_{1X}\boxtimes\mathcal{F}_{2X}|_U
\end{array}$$

Therefore, we conclude that

$$(j_{!*}(\mathcal{F}_{1X}\boxtimes\mathcal{F}_{2X})_U)|_\Delta \cong m!(\widetilde{j}_{!*}(\mathcal{F}_{1X}\boxtimes\mathcal{F}_{2X})|_U)|_\Delta).$$

But if you look over the diagonal, then the right hand side is precisely $m_!(\mathcal{F}_1\widetilde{\boxtimes}\mathcal{F}_2)_X = (\mathcal{F}_1 * \mathcal{F}_2)_X$. ☐

One can then conclude that after pushing forward, one gets a constant sheaf, since you have invariance in both directions (a trick observed by Markovic-Vilonen), i.e.

$$\mathcal{H} := f_!(j_{!*}((\mathcal{F}_{1X}\widetilde{\boxtimes}\mathcal{F}_{2X})|_U)) \text{ is a constant sheaf on } X^2.$$

Then the stalk at a point (x, y) is

$$\mathcal{H}_{(x,y)} = \begin{cases} H^\bullet(\mathcal{F}_1)\otimes H^\bullet(\mathcal{F}_2) & x \neq y \\ H^\bullet(\mathcal{F}_1 * \mathcal{F}_2) & x = y \end{cases}$$

Switching the two factors of X^2 gives the canonical isomorphism $\mathcal{F}_1 * \mathcal{F}_2 \cong \mathcal{F}_2 * \mathcal{F}_1$.

4.3 Applications

Example 4.7. You can use this to compute the cohomology of smooth projective quadratic hypersurfaces (see the problem sheet), or Grassmannians: $H^\bullet(\mathrm{Gr}(i,n)) \cong \bigwedge^i \mathbb{C}^n$.

Another application is to computing a "Clebsch-Gordon" type decomposition for the highest weight representations. Consider

$$\mathrm{Hom}_{\widehat{G}}(V_\mu, V_\lambda \otimes V_\nu) \cong \mathrm{Hom}(IC_\mu, IC_\lambda * IC_\nu).$$

In order to compute the right hand side, we should remind ourselves about the convolution of the IC sheaves. This was defined via the diagram

$$\begin{array}{ccc} & & \mathrm{Gr}_{\leq\lambda} \widetilde{\times} \mathrm{Gr}_{\leq\nu} \\ & & \downarrow m \\ \mathrm{Gr}_{\leq\mu} & \hookrightarrow & \mathrm{Gr}_{\leq\lambda+\nu} \end{array}$$

so that $IC_\lambda * IC_\mu = m_!(IC_\lambda \boxtimes IC_\mu) \cong m_! IC_{\leq(\lambda,\mu)}$. Then

$$\mathrm{Hom}(IC_\mu, IC_\lambda * IC_\mu) \cong \mathrm{Hom}(m^* IC_\mu, IC_{\leq(\lambda,\mu)}).$$

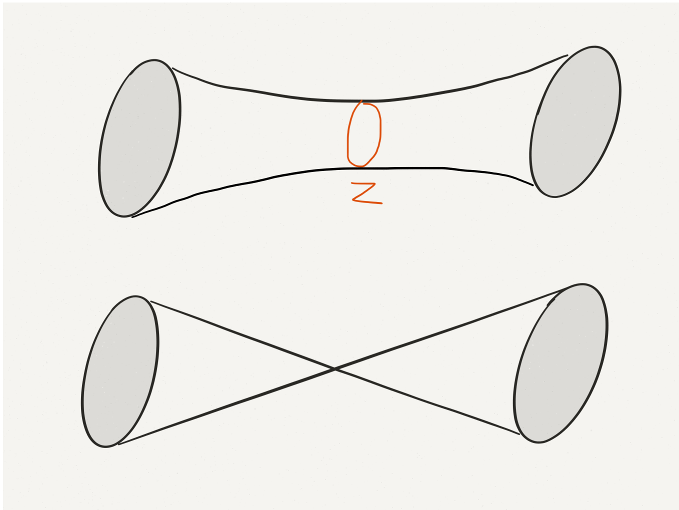
But because this is m is semi-small, that is isomorphic to $\mathbb{Q}_\ell$ generated the irreducible components of $m^{-1}(\mathrm{Gr}_\mu)$ of dimension $(\rho, \mu + \lambda + \nu)$.

Example 4.8. Let $G = \mathrm{GL}_2$ and ω_1 the fundamental weight. Then consider

$$\begin{array}{ccc} & & \mathrm{Gr}_{\leq\omega_1} \widetilde{\times} \mathrm{Gr}_{\leq\omega_1^*} \\ & & \downarrow \\ \mathrm{pt} = \mathrm{Gr}_0 & \hookrightarrow & \mathrm{Gr}_{\leq\omega_1+\omega_1^*} \end{array}$$

which is a resolution of the singular quadric surface. You might know from before that this is the Hirzebruch surface $\mathbb{P}^1(\mathcal{O} \oplus \mathcal{O}(-2))$, but let's see how this falls out of the Geometric Satake correspondence.

Let Z be the exceptional divisor.



Then we claim that the maps of IC sheaves correspond under Geometric Satake to

$$\mathbb{Q}_\ell = V_0 \xrightarrow{[Z]} V_{\omega_1} \otimes V_{\omega_1}^* \cong V_{\omega_1}^* \otimes V_{\omega_1} \xrightarrow{[Z]} \mathbb{Q}_\ell.$$

where $[Z]$ denotes cupping with the class of Z in cohomology. So what should you get from the composition? One the other hand, it should be multiplication by the self-intersection $[Z]\cdot[Z]$. On the other hand, if we identify $V^* \otimes V$ with $\mathrm{End}(V, V)$ then the above composition is just the inclusion of the identity map follows by trace, which is multiplication by 2. However, we have accounted for that fact that isomorphism $V_{\omega_1} \otimes V_{\omega_1}^* \cong V_{\omega_1}^* \otimes V_{\omega_1}$ introduces a sign, so the punchline is that $[Z] \cdot [Z]$ is -2 curve.

If $G \supset B \supset U$ (the unipotent radical), then one can consider the orbit

$$S_\lambda = LU \cdot t_\lambda L^+G/L^+G.$$

This is a "semi-infinite orbit" because it is of infinite dimension and codimension. However, $S_\lambda \cap \mathrm{Gr}_{\le\mu}$ is finite-dimensional.

Theorem 4.9 (Mirkovic-Vilonen)**.** *We have:*

1. *$S_\lambda \cap G_{\le\mu}$ is equi-dimensional of dimension $(\rho, \lambda + \mu)$.*
2. *If $\mathcal{F} \in \mathbf{Sat}_G \leftrightarrow V$, then*

$$H_c^\bullet(S_\lambda, \mathcal{F}) \cong \bigoplus_\lambda H^{2\rho(\lambda)} V(\lambda)$$

where $V(\lambda)$ is the λ weight space of V.

Corollary 4.10. *The irreducible components of $S_\lambda \cap \mathrm{Gr}_{\le\mu}$ provide a **canonical** basis of $V_\mu(\lambda)$.*