PROBLEM SET III: ORBITAL INTEGRALS

In these exercises, $k = \mathbb{F}_q$ denotes a finite field with $\mathrm{char}(k) \neq 2$, and $F = k((t))$.

0.1. Let $a \in k^\times - (k^\times)^2$ and $G = \mathrm{SL}_2$. Consider the matrix $\gamma = \begin{pmatrix} 0 & at \\ t & 0 \end{pmatrix}$ and $\gamma' = \begin{pmatrix} 0 & at^2 \\ 1 & 0 \end{pmatrix}$.

(1) What are the centralizers G_γ and $G_{\gamma'}$?
(2) Are γ and γ' conjugate in $\mathrm{SL}_2(F)$?
(3) Calculate $O_\gamma(\mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)})$ and $O_{\gamma'}(\mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)})$ (using the Haar measure at your choice).
(4) We have the affine Springer fiber $\mathscr{X}_\gamma$ with the action of $L_\gamma \cong \mathbb{Z}$, both defined by passing to $\overline{k}$. What is the $\mathrm{Gal}(\overline{k}/k)$-action on $\mathscr{X}_\gamma(\overline{k})$; how is it compatible with the L_γ-action?
(5) The quotient $L_\gamma \backslash \mathscr{X}_\gamma$ is a variety over k. Calculate the number of k-points of the quotient $L_\gamma \backslash \mathscr{X}_\gamma$, and compare your answer with the orbital integrals in (3).

0.2. Let $G = \mathrm{SL}_2$ over F and let $\gamma = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}$. What is the centralizer $G_\gamma(F)$? Is the loop group LG_γ (as a group ind-scheme over k) connected?

0.3. Let $G = \mathrm{GL}_n$ over F and let $\gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_m)$ be a block diagonal matrix in $\mathfrak{g}(F)^{\mathrm{rs}}$, where $\gamma_i \in \mathfrak{gl}_{n_i}(F)^{\mathrm{rs}}$. Consider the asymptotic behavior of the orbital integral $O_\gamma(\mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)})$ as $q = \#k$ tends to ∞. Find the smallest integer d such that

$$O_\gamma(\mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)}) = O(q^d)$$

as $q \to \infty$. Note that the $O(\cdot)$ on the right side is analysts' O while on the left side it means orbital integral. Can you interpret d in terms of the characteristic polynomials of the γ_i's?

0.4. Let $G = \mathrm{SL}_3$ and $\gamma = \begin{pmatrix} 0 & 0 & t^4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Compute $O_\gamma(\mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)})$.

Hint: use the cell decomposition introduced in the lecture.

0.5. Let $G = \mathrm{SL}_2$. Let f be the characteristic function of elements $X \in \mathfrak{g}(\mathcal{O}_F)$ such that the reduction $\overline{X}$ in $\mathfrak{g}(k)$ is regular nilpotent (i.e., nilpotent but nonzero in this case). Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element.

(1) Show that $O_\gamma(f) \neq 0$ if and only if $\det(\gamma) \in t\mathcal{O}_F$.
(2) When $\det(\gamma) \in t\mathcal{O}_F$, show that

$$O_\gamma(f) = O_\gamma(\mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)}) - O_{t^{-1}\gamma}(\mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)}).$$