PROBLEM SET II: AFFINE SPRINGER FIBERS

In these exercises, $F = k((t))$, and γ always denotes a regular semisimple element in $\mathfrak{g}(F)$.

0.1. Consider the case $G = \mathrm{SL}_2$ and $\gamma = \begin{pmatrix} 0 & t^n \\ 1 & 0 \end{pmatrix}$.

(1) Describe $\mathscr{X}_\gamma$ and $\mathscr{Y}_\gamma$. Note: separate into two cases according to the parity of n.
(2) Construct a nontrivial $\mathbb{G}_m$-action on both $\mathscr{X}_\gamma$ and $\mathscr{Y}_\gamma$, and determine its fixed points.

0.2. Recall that we have a lattice L_γ in the centralizer $G_\gamma(F)$. Show that the action of L_γ on Gr_G is free, hence in particular its action on $\mathscr{X}_\gamma$ is free. Show also that the permutation action of the lattice L_γ on the set of irreducible components of $\mathscr{X}_\gamma$ is free.

0.3. Consider the case $G = \mathrm{SL}_3$ and $\gamma = \mathrm{diag}(x_1 t, x_2 t, x_3 t)$. Describe the affine Springer fibers $\mathscr{X}_\gamma$ and $\mathscr{Y}_\gamma$.

0.4. Let $G = \mathrm{SL}_2$ and let $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$. What is the action of the affine Weyl group $\widetilde{W} = \langle s_0, s_1 \rangle$ (infinite dihedral group) on $\mathrm{H}_2(\mathscr{Y}_\gamma)$?

0.5. Using the dimension formula for affine Springer fibers, can you come up with some examples of 1-dimensional $\mathscr{X}_\gamma$ and $\mathscr{Y}_\gamma$ for various types of G?