PROBLEM SET I: SPRINGER FIBERS

0.1. For $G = \mathrm{SL}_2$ and SL_3, calculate the coordinate ring of the *non-reduced* Springer fiber $\widetilde{\mathscr{B}}_e$ for a regular nilpotent element e. Show also that the Springer fiber $\mathscr{B}_e$ is indeed a reduced point. Recall here that $\widetilde{\mathscr{B}}_e$ is the fiber of the map $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ over e while $\mathscr{B}_e$ is the fiber of $\widetilde{\mathscr{N}} \to \mathscr{N}$ over e.

Hint: if you write e as an upper triangular matrix, then $\widetilde{\mathscr{B}}_e$ lies in the big Bruhat cell of the flag variety $\mathscr{B}$, and you get coordinates for your calculation.

0.2. Let $G = \mathrm{SL}_3$ and let $e \in \mathscr{N}$ be a subregular element. Calculate the action of S_3 on the two dimensional $\mathrm{H}^2(\mathscr{B}_e)$ in terms of the basis given by the fundamental classes of the two $\mathbb{P}^1$'s.

0.3. Describe the Springer fibers for Sp_4. Calculate the Springer correspondence for $G = \mathrm{Sp}_4$ explicitly.

Hint: in doing Exercise 0.2 and 0.3, you may find Exercise 0.4 useful. You may take the statement in Exercise 0.4 for granted.

0.4. Let $e \in \mathscr{N}$. Let $B \subset G$ be a Borel subgroup.

(1) Let α be a simple root. Let $P_\alpha \supset B$ be a parabolic subgroup whose Levi factor has semisimple rank one with roots $\pm\alpha$. Let $\mathscr{P}_\alpha \cong G/P_\alpha$ be the partial flag variety of G classifying parabolics conjugate to P_α. Restricting the projection $\mathscr{B} \to \mathscr{P}_\alpha$ to $\mathscr{B}_e$, we get a map
$$\nu_\alpha : \mathscr{B}_e \to \pi_\alpha(\mathscr{B}_e).$$
Show that the pullback ν_α^* gives an isomorphism
$$\mathrm{H}^*(\nu_\alpha(\mathscr{B}_e)) \cong \mathrm{H}^*(\mathscr{B}_e)^{s_\alpha}$$
where $s_\alpha \in W$ is the simple reflection associated with α, which acts on $\mathrm{H}^*(\mathscr{B}_e)$ via Springer's action.

(2) Can you generalize the above statement to other partial flag varieties?

The following exercises involve semismall maps and perverse sheaves that will be covered in Mark de Cataldo's lectures. You may revisit these problems later during the summer school.

0.5. Let $k = \mathbb{C}$. Show that $\mathscr{N}$ is rationally smooth; i.e., its intersection cohomology complex is isomorphic to the shifted constant sheaf $\mathbb{Q}[\dim \mathscr{N}]$. Note that $\mathscr{N}$ is never smooth.

0.6. Show that the Springer fibers $\mathscr{B}_e$ are connected.

Hint: the informtation on the connected components of the Springer fibers are encoded in a certain constructible sheaf.

0.7. Using the dimension formula for $\mathscr{B}_e$, verify that the Springer resolution $\pi : \widetilde{\mathscr{N}} \to \mathscr{N}$ is semismall and that the Grothendieck alteration $\pi_{\mathfrak{g}} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is small.