Intersection numbers of cycles on the moduli of Shtukas

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1 Motivation

The main motivation is the Birch and Swinnerton-Dyer conjecture for elliptic curves E/F =global field. Let's recall the first part of the conjecture. This says that

$$\text{rank } E(F) = \text{ord}_{s=1} L(s, E).$$

To me the biggest mystery is how to access the order of vanishing, or more generally the derivatives of $L(s, E)$. For example, if it vanishes to order 2 then how can we relate the second derivative to anything geometric?

1.1 Classical Gross-Zagier

The main progress on this front is the groundbreaking work of Gross-Zagier in the 1980s, which gives a geometric interpretation of the leading term in the case that the rank is 0 or 1.

Let me review their setup. If you have an $E/\mathbb{Q}$, then there exists a *modular parametrization* $\varphi \colon X_0(N) \to E$ over $\mathbb{Q}$. Choose an imaginary quadratic field $K/\mathbb{Q}$. To this K we can attach a point $P_K \in X_0(N)(K)$, which is called a *Heegner point*. This is constructed using the moduli-theoretic interpretation of $X_0(N)$, and P_K corresponds to the elliptic curve with complex multiplication by an order of $\mathcal{O}_K$.

We can project P_K via φ to obtain a point $\varphi(P_K) \in E(\mathbb{Q})$. The Gross-Zagier formula says that

$$c \cdot \langle \varphi(P_K), \varphi(P_K) \rangle = L'(1, E/K)$$

where $\langle \varphi(P_K), \varphi(P_K) \rangle$ is the height pairing, which can be thought of geometrically as a self-intersection, and $c \neq 0$.

This proves certain cases of BSD.

2 Function field setup

For the rest of the talk, $F = k(X)$ is the function field of some (smooth, projective, geometrically connected) curve X, where $k = \mathbb{F}_q$. Let E/F be an elliptic curve over F. We can try to find a model $\mathcal{E} \to X$ such that the generic fiber is E:

$$\begin{array}{ccc} E & \hookrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \operatorname{Spec} F & \hookrightarrow & X. \end{array}$$

What is the L-function in this case? Consider $H^1(X_{\overline{k}}, R^1 f_* \mathbb{Q}_\ell)$. This is a finite-dimensional $\mathbb{Q}_\ell$-vector space equipped with a Frobenius action, and the L-function is

$$L(s, E/F) = \det(1 - q^{-s} \operatorname{Frob} | H^1(X_{\overline{k}}, R^1 f_* \mathbb{Q}_\ell)).$$

This is a polynomial in q^{-s}.

Dream. Access $L^{(r)}(s, E)$ through intersection numbers.

The protypical example is the Gross-Zagier formula

$$c \cdot \langle \varphi(P_K), \varphi(P_K) \rangle = L'(1, E/K).$$

For the first derivative, you can really mimic what Gross-Zagier did for elliptic curves over $\mathbb{Q}$.

- $X_0(N)$ should be replaced with Drinfeld modular varieties (in this case, a curve over F). This was done by Rück-Tipp.

In the function field setting, we actually have more modular varieties. For each r, there is an analogue of Shimura varieties in the function field setting:

$$\mathrm{Sht}^r \to X^r$$

The $r = 1$ case is analogous to a modular curve. $\mathrm{Sht}^1 \to X^1$ has a generic fiber basically the Drinfeld modular variety. But we get this for *any* r. The slogan of our main result is that *one can access* $L^{(r)}(s, E)$ *through intersection numbers of cycles on* Sht^r.

Main result. What we can do is modest in comparison to the dream. What we can do now is only applicable to "elliptic curves" with good reduction everywhere. There aren't so many interesting examples of honest elliptic curves with good reduction everywhere. What we really mean is to enlarge our category from elliptic curves to "cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_F)$."

There is a similar modularity result: to every elliptic curve up to isogeny, one can attach a cuspidal automorphic representations of $\mathrm{PGL}(2, \mathbb{A}_F)$. Although there aren't interesting examples of elliptic curves with everywhere good reduction, there are interesting examples of everywhere unramified automorphic representations of $\mathrm{GL}(2, \mathbb{A}_F)$.

Fix π to be a cuspidal automorphism. We're going to state a formula similar to Gross-Zagier. It will have a "geometric side" and an "analytic side," and the formula says

$$\text{geometric side} = \text{analytic side}.$$

3 Geometric side

3.1 Shtukas

Let $G = \mathrm{PGL}(2)/F$ and $r \geq 0$ an even integer. Then Sht_r parametrizes chains of vector bundles of rank 2:

$$\mathrm{Sht}_r = \{\mathcal{E}_0 \xrightarrow{x_1} \mathcal{E}_1 \xrightarrow{x_2} \ldots \to \mathcal{E}_{r/2} \leftarrow \mathcal{E}_{r/2+1} \xleftarrow{x_{r/2+1}} \ldots \xleftarrow{x_r} \mathcal{E}_r \xrightarrow{\sim} {}^\tau\mathcal{E}_0\}/\,\mathrm{Pic}_X(k)$$

such that

- $\mathcal{E}_i$ is a vector bundle of rank 2 over X,
- $\mathcal{E}_1/\mathcal{E}_0$ is a skyscraper sheaf of length 1 at x_1, etc.
- ${}^\tau(-)$ is the Frobenius pullback.
- We need to mod out by $\mathrm{Pic}_X(k)$ because we are considering PGL(2) rather than GL(2),

The map $\mathrm{Sht}_r \to X^r$ is the obvious forgetful map.

This space was introduced by Drinfeld for $r = 2$ and for general G (and r) by Yakov Varshavsky.

Geometric properties.

- Sht_r is a smooth, equidimensional DM stack of dimension $2r$, but it is *not* of finite type. In fact, $f\colon \mathrm{Sht}_r \to X^r$ is smooth of relative dimension r.
- $Rf_*\overline{\mathbb{Q}_\ell}$ is a complex, inside of which we expect to find all SL(2) local systems on X to appear. So this is the "universal host" of two-dimensional local systems.

3.2 Drinfeld-Heegner cycle

Let F'/F be a quadratic extension which is everywhere unramified. We can choose an embedding

$$T = \mathrm{Res}_F^{F'} \mathbb{G}_m/\mathbb{G}_m \hookrightarrow \mathrm{PGL}_2 = G.$$

We can consider a similar moduli space for T instead of G. We have the torus T, which can be though of as coming from F'. Take the étale double cover $\nu\colon X' \to X$. Define $\mathrm{Sht}_{T,r}$ to be the moduli space parametrizing

$$\{\mathcal{L}_0 \xrightarrow{x_1'} \mathcal{L}_1 \to \ldots \xleftarrow{x_r'} \mathcal{L}_r \cong {}^\tau\mathcal{L}_0\}$$

where each $\mathcal{L}_i$ is a line bundle over X'. We claim that there is a natural map

$$\begin{array}{ccc} \mathrm{Sht}_{T,r} & \longrightarrow & \mathrm{Sht}_r \\ & \searrow & \downarrow \\ & (X')^r \longrightarrow & X^r \end{array}$$

sending $\mathcal{L}_\bullet \mapsto \nu_* \mathcal{L}_\bullet$.

We can insert the base change

$$\begin{tikzcd} \mathrm{Sht}_{T,r} \arrow[r] \arrow[dr] & \mathrm{Sht}'_r \arrow[r] \arrow[d] & \mathrm{Sht}_r \arrow[d] \\ & (X')^r \arrow[r] & X^r \end{tikzcd}$$

Now $\mathrm{Sht}_{T,r}$ is a smooth DM stack of dimension r. The map $\mathrm{Sht}_{T,r} \to (X')^r$ is a torsor under $\mathrm{Pic}_{X'}(k)/\mathrm{Pic}_X(k)$. The key point is that Sht'_r has dimension $2r$, so $\mathrm{Sht}_{T,r}$ is a middle-dimensional cycle. Moreover, $\mathrm{Sht}_{T,r}$ is proper, so it makes sense to talk about its self-intersection inside Sht'_r (even though the latter is not proper).

3.3 Hecke symmetry

We need one more ingredient: the *Hecke algebra*

$$\mathcal{H} = \bigotimes_{x \in |X|} \mathcal{H}_x \quad \mathcal{H}_x \cong \mathbb{Q}[T_x].$$

Then $\mathcal{H}$ acts on Sht_r by correspondences, so $\mathcal{H}$ acts on $Ch_r(\mathrm{Sht}'_r)_{\overline{\mathbb{Q}}}$. Let $Z := [\mathrm{Sht}_{T,r}] \in CH_r(\mathrm{Sht}'_r)_{\overline{\mathbb{Q}}}$. We can consider the subspace $\mathcal{H} \cdot Z$, and then pass to a quotient on which the intersection pairing is perfect by modding out by the kernel of the intersection pairing: $\mathcal{H} \cdot Z / \ker(\cdot, \cdot)$. This is a $\overline{\mathbb{Q}}$-vector space.

Fact: there is a decomposition

$$V = \bigoplus_{\pi \text{ cusp}} V_\pi \oplus V_{\text{Eisenstein}}.$$

Then $\mathcal{H}$ acts on V_π through the character $\chi_\pi \colon \mathcal{H} \to \overline{\mathbb{Q}}$, where χ_π is describes the action of $\mathcal{H}$ on the unramified part π^K.

Denote by $Z_\pi \in V_\pi$ the π-part of Z. Now we can finally state the geometric side:

$$\text{geometric side } = (Z_\pi, Z_\pi)_{\mathrm{Sht}'_r}.$$

4 Analytic side

From π a cupsidal automorphic representation of $G(\mathbb{A}_F)$, we get an automorphic representation $\pi_{F'}$ of $G(\mathbb{A}_{F'})$ and hence an L-function $L(s, \pi_{F'})$.

For reasons we won't go into, one should introduce a modification

$$\mathcal{L}(s, \pi_{F'}) = \epsilon 9 s, \pi_{F'})^{1/2} L(s, \pi_{F'}).$$

The important things is $\mathcal{L}(s) = \mathcal{L}(1-s)$, the central value now being $s = 1/2$.

Theorem 4.1 (Yun-Zhang)**.** *We have*

$$(Z_\pi, Z_\pi)_{\mathrm{Sht}'_r} = c \cdot \mathcal{L}^{(r)}(1/2, \pi_{F'})$$

where c is a non-zero constant mildly dependent on π.