NOTES ON DELIGNE'S "LA CONJECTURE DE WEIL. I"

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1. Introduction

1.1. **Weil's conjectures.** Let X be a smooth projective variety of dimension n over $\mathbf{F}_q$.

Definition 1.1. The *zeta function* of X is

$$\zeta(X,s) := \prod_{x\in X}\left(1-\frac{1}{q_x^s}\right)^{-1}.$$

This is in obvious analogy to the Riemann zeta function, but it will be more convenient for us to work with the function

$$Z(X,t) = \prod_{x\in X}\left(1-t^{-\deg x}\right)^{-1}.$$

We clearly have

$$\zeta(X,s) = Z(X,q^s).$$

Now we can state Weil's conjectures.

Conjecture 1.2 (Weil)**.**

(1) $Z(X,t)$ is a rational function of t, i.e $Z(X,t)\in\mathbf{Q}(t)$, with factorization of the form

$$Z(X,t) = \frac{P_1(t)\dots P_{2n-1}(t)}{P_0(t)\dots P_{2n}(t)}.$$

(2) $Z(X,t)$ satisfies a functional equation.
(3) The roots of $P_i(X,t)$ have absolute value $q^{-i/2}$.

Weil envisioned these conjectures as a consequence of an appropriate cohomology theory for $X/\mathbf{F}_q$ which would behave analogously to singular cohomology. In particular, (1) should follow from a "Lefschetz trace formula" in $\overline{X}$, with $X(\mathbf{F}_q)$ interpreted as the "fixed points" of Frobenius. The functional equation predicted in (2) should follow from Poincaré duality. The condition (3) is an analogue of Riemann's hypothesis.

This hypothetical cohomology theory was eventually constructed by Grothendieck, and is now called étale cohomology. The purpose of these notes is to explain the main ideas going into the proof of (3). Everything here comes from Deligne's article [1], but I have reorganized the presentation, and focused on the simplest cases in order to highlight the key ideas.

2. ÉTALE COHOMOLOGY

The P_i in Weil's conjecture are basically characteristic polynomials of Frobenius acting on étale cohomology. The intuition to keep in mind is that étale cohomology with coefficients in a *constant* (torsion) sheaf (or more generally, a torsion local system) behaves "like singular cohomology". As we will shortly see, the familiar fundamental results of classical singular cohomology, once phrased invariantly enough, become theorems in étale cohomology.

Remark 2.1. For *quasi-coherent* sheaves, étale cohomology coincides with coherent cohomology. These won't come up in our discussion.

2.1. The orientation sheaf.

Here's an example of what I mean. It's commonly said that complex manifolds are canonically oriented, but from an algebraic perspective that's not quite true - you have to choose an orientation for $\mathbf{C}$. This amounts to a choice of $\pm i$, which can be thought of as a choice of embedding of $\mathbf{Q}/\mathbf{Z}$ into the roots of unity.

We're going to be talking about $\mathbf{Q}_\ell$, the ℓ-adic numbers. The orientation sheaf for $\mathbf{Q}_\ell$ involves a choice of the ℓ-power roots of unity. Such a choice is equivalent to a choice of trivialization

$$\varprojlim \mu_{\ell^n} \simeq \varprojlim \mathbf{Z}/\ell^n \simeq \mathbf{Z}_\ell.$$

In any case $\mathbf{Z}_\ell$ acts on $\varprojlim \mu_{\ell^n}$, and we define

$$\mathbf{Q}_\ell(1) = \mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell} \varprojlim \mu_{\ell^n}.$$

For any n, we define $\mathbf{Q}_\ell(n) = \mathbf{Q}_\ell(1)^{\otimes n}$. For negative n, this is defined by

$$\mathbf{Q}_\ell(n) := \mathbf{Q}_\ell(-n)^\vee.$$

Remark 2.2. For varieties over finite fields, you can think of this in the following way. $\mathbf{Q}_\ell(n)$ is a $\mathbf{Q}_\ell$-vector space with a natural action of $\mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$, where Frobenius acts as multiplication by q. However, my Frobenius F will always be the *geometric* Frobenius $x \mapsto x^{q^{-1}}$, which acts as multiplication by q^{-1}.

2.2. Properties of étale cohomology.

Let X be a smooth variety of pure dimension n over an algebraically closed field.

(1) (*Fundamental class*) There is a fundamental class

$$\mathrm{Tr}\colon \mathrm{H}_c^{2n}(X, \mathbf{Q}_\ell(n)) \xrightarrow{\sim} \mathbf{Q}_\ell.$$

Equivalently, you can think of this as $\mathrm{Tr}\colon \mathrm{H}_c^{2n}(X, \mathbf{Q}) \xrightarrow{\sim} \mathbf{Q}_\ell(-n)$.

(2) (*Cohomological dimension*) X has cohomological dimension $2n$:

$$\mathrm{H}^i(X, \mathbf{Q}_\ell) = 0 \text{ if } i > 2n.$$

(3) (*Poincaré duality*) There is a cup product

$$\mathrm{H}^i(X, \mathbf{Q}_\ell) \otimes \mathrm{H}_c^{2n-i}(X, \mathbf{Q}_\ell) \to \mathrm{H}_c^{2n}(X, \mathbf{Q}_\ell) \xrightarrow{\sim} \mathbf{Q}_\ell(-n).$$

which induces a perfect pairing.

(4) (*Lefschetz trace formula*) There's a Lefschetz trace formula

$$\mathrm{Fix}(F) = \#X(\mathbf{F}_q) = \sum_i (-1)^i \operatorname{Tr}(F, \mathrm{H}^i_c(X, \mathbf{Q}_\ell)).$$

Everything generalizes to a version with coefficients in a more general local system. It may not be clear how to do that for the last one now, but it should become clear later.

2.3. **Rationality of the zeta function.** Because it will actually be important for us later, we derive the rationality of the zeta function from the above properties. Consider

$$\begin{aligned} t\frac{d}{dt}\log Z(X,t) &= t\frac{d}{dt}\sum_x -\log(1-t^{-\deg x}) \\ &= t\frac{d}{dt}\sum_{n\ge 1}\frac{x t^{-n\deg x}}{n} \\ &= \sum_{n\ge 1} t^{-n}\sum_{\deg x|n}\deg x \end{aligned}$$

Observe that $\sum \deg x \mid n = \#X(\mathbf{F}_{q^n})$, since points of X can be thought of as orbits in $\#X(\mathbf{F}_{q^n})$, of size equal to the their degree. Substituting in the Lefschetz trace formula, we find that this is

$$\sum_{n\ge 1} t^{-n}\sum_i (-1)^i \operatorname{Tr}(F, \mathrm{H}^i_c(X, \mathbf{Q}_\ell) = \sum_i (-1)^i \sum_{n\ge 1}\operatorname{Tr}(F^n, \mathrm{H}^i_c(X, \mathbf{Q}_\ell).$$

Now, recall that for an operator F on a vector space V,

$$t\frac{d}{dt}\log\det(1-tF, V)^{-1} = \sum_{n\ge 1}\operatorname{Tr}(F^n)t^n.$$

Proof: write $\det(1-tF) = \prod(1-t\alpha_i)$, so that this becomes

$$t\frac{d}{dt}\sum_n\sum_i\frac{\alpha_i^n t^n}{n} = \sum_n t^n\sum_i \alpha_i^n.$$

So that tells us that

$$\sum_i (-1)^i \sum_{n\ge 1}\operatorname{Tr}(F^n, \mathrm{H}^i_c(X, \mathbf{Q}_\ell)) = t\frac{d}{dt}\log\det(1-Ft, \mathrm{H}^i_c(X, \mathbf{Q}_\ell))^{-1}.$$

Substituting this above, we obtain

$$\prod_x (1-t^{-\deg x}) = \prod_i \det(1-Ft, \mathrm{H}^i_c(X, \mathbf{Q}_\ell))^{(-1)^{i+1}}.$$

The right hand side predicts the polynomials appearing in Weil's conjectures.

2.4. **Overview of the proof.** By étale cohomology, the statement reduces to bounding the eigenvalues of Frobenius on étale cohomology. By simple reductions, one quickly reduces to checking the eigenvalues of Frobenius on the *middle*-dimensional cohomology. To analyze this, one chooses a Lefschetz pencil $f: X \to \mathbf{P}^1$, which always exists after possibly blowing up X (and it is easy to see that blowing up doesn't affect the problem).

The idea is then to study the cohomology of $R^n f_* \mathbf{Q}_\ell$ on $\mathbf{P}^1$. This sheaf will be a local system on a dense open subset of $\mathbf{P}^1$, for general reasons of constructibility of proper pushforwards. There are three main ingredients:

(1) A "big image" result on monodromy for a Lefschetz pencil.
(2) An algebraicity result, showing that the eigenvalues in question are algebraic over $\mathbf{Q}$ (being a priori in $\overline{\mathbf{Q}}_\ell$). This is achieved by an extremely clever "gcd argument", which is quintessentially Deligne.
(3) A very clever analytic estimate, finally establishing the desired bound (in view of the previous two ingredients). This is inspired by the Rankin-Selberg method.

We will actually present (3) first, even though it relies on the first two points, because it is the crux of the argument. Then we will go back and indicate how to verify (1) and (2).

3. Some reductions

Let X be a smooth proper variety of dimension n. Let $RH(\mathrm{H}^i(X))$ denote the statement that the eignvalues of F^* on $\mathrm{H}^i(X, \mathbf{Q}_\ell)$ are algebraic with absolute value $q^{i/2}$ under all complex embeddings. We would like to prove $RH(\mathrm{H}^i(X))$ for $0 \le i \le 2n$.

3.1. **Formalities.** If we have an embedding

$$\mathrm{H}^i(X) \hookrightarrow \mathrm{H}^i(X')$$

then $RH(\mathrm{H}^i(X')) \implies RH(\mathrm{H}^i(X))$.

Example 3.1. If $X' \to X$ the blowup along a closed subvariety $Z \subset X$, then we get such an embedding. We will use the special case where Z is the section by a codimension-2 plane.

If we have a surjection

$$\mathrm{H}^i(X'') \to \mathrm{H}^i(X)$$

then $RH(\mathrm{H}^i(X')) \implies RH(\mathrm{H}^i(X))$.

3.2. **Poincaré duality.** Thanks to the perfect pairing

$$\mathrm{H}^i(X, \mathbf{Q}_\ell) \times \mathrm{H}^{n-i}(X, \mathbf{Q}_\ell) \to \mathbf{Q}_\ell(-n)$$

furnished by Poincaré duality, we automatically know that the $P_i(T) = T^{???} P_{2n-i}(q^n/T)$. In particular, if α is an eigenvalue for F^* on $\mathrm{H}^i(X, \mathbf{Q}_\ell)$ then q^n/α is an eigenvalue for F^* on $\mathrm{H}^i(X, \mathbf{Q}_\ell)$. Therefore,

$$RH(\mathrm{H}^i) \implies RH(\mathrm{H}^{n-i}).$$

The upshot is that it suffices to prove $RH(\mathrm{H}^i)$ for $i = 0, \ldots, n$.

3.3. **Weak Lefschetz.** Let $Y \subset X$ be a general (smooth) hyperplane section. (Since we're over a finite field, this might not exist a priori. But a smooth *hypersurface* section always exists, so we're okay after passing to some large Veronese embedding first.)

Theorem 3.2 (Lefschetz Hyperplane)**.** *The restriction map* $\mathrm{H}^i(X) \to \mathrm{H}^i(Y)$ *is an isomorphism for* $i < n-1$ *and an injection for* $i = n-1$.

This will be useful for an inductive proof of the theorem. By the preceding reductions, we get for free that the we only need to worry about the *middle* dimension.

4. Cohomology of Lefschetz pencils

4.1. **Introduction to Lefschetz pencils.** Most of what we can do for general varieties is bootstrapped from curves, so it is natural to adopt an inductive approach. We've already seen that a hyperplane section of X captures "most" of its cohomology (everything except the middle). To get the rest we'll put X in the"cookie cutter" to get many hyperplane sections. By induction we "know" the cohomology of the hyperplane sections, and then the task is to assemble them together.

A *pencil* of hyperplanes is the set of hyperplanes passing through some codimension-2 plane A, which we call the *axis* of the pencil. This set has a natural structure of a $\mathbf{P}^1$. We have a natural rational map $X \dashrightarrow \mathbf{P}^1$ sending x to the hyperplane spanned by x and A. This is defined away from $A \cap X$. The fibers of this map are points which lie in a common hyperplane through A, i.e. hyperplane sections of X.

We can resolve the indeterminacy of the map by blowing up at the locus $A \cap X$, giving an honest fibration

$$\widetilde{X} \to \mathbf{P}^1.$$

Furthermore,

$$\mathrm{H}^i(X) \hookrightarrow \mathrm{H}^i(\widetilde{X}) = \mathrm{H}^i(X) \oplus \mathrm{H}^{i-2}(X \cap A)(-1)$$

(the last equality by the Thom isomorphism theorem), so by one of reductions it suffices to prove $RH(\mathrm{H}^i(\widetilde{X}))$.

There's an additional technical point in the definition of Lefschetz pencil. The map $\widetilde{X} \to \mathbf{P}^1$ is not smooth, since hyperplane sections can be singular (exactly when the hyperplane becomes tangent to X). I'll want to choose A generally, so that these singularities are as mild as possible, i.e. simple points. You can think of this as asking that the function $f : \widetilde{X} \to \mathbf{P}^1$ be a "morse function". A *Lefschetz pencil* is by definition a fibration $\widetilde{X} \to \mathbf{P}^1$, with singularities as mild as possible. As more precise definition will be given when it is needed, in §6.

4.2. **Monodromy and the spectral sequence.** We're going to try to "fit together" the cohomologies of the different hyperplane sections and see what they tell us about the cohomology of the whole thing. This is an obvious setting for a spectral sequence.

$$E_2^{iq} = \mathrm{H}^i(\mathbf{P}^1, R^q f_* \mathbf{Q}_\ell) \implies \mathrm{H}^{i+q}(X, \mathbf{Q}_\ell).$$

Now, since $\mathbf{P}^1$ is a curve we have that $\mathrm{H}^i(\mathbf{P}^1, R^q f_* \mathbf{Q}_\ell)$ vanishes for $i > 2$. Therefore, there are only three groups that we need to worry about, corresponding to $(i,q) = (0,n), (1, n-$

1), and $(2, n-2)$. However, it is clear that in order to analyze them we need to understand $R^q f_* \mathbf{Q}_\ell$.

Basically, you should think of this "constructible sheaf" $R^q f_* \mathbf{Q}_\ell$ as being assembled together from its stalks $(R^q f_* \mathbf{Q}_\ell)_u = \mathrm{H}^q(X_u, \mathbf{Q}_\ell)$ using monodromy. Let me explain.

Let $j \colon U \hookrightarrow \mathbf{P}^1$ be the inclusion of the open set where f is smooth. Over U, $R^q f_* \mathbf{Q}_\ell$ restricts to a local system. This means that it is a locally constant $\mathbf{Q}_\ell$ sheaf for the étale topology (with some finiteness assumptions). There is a monodromy action of $\pi_1(U, u)$ on the fibers which determines the local system - in fact, a $\mathbf{Q}_\ell$-local system is equivalent to the data of a finite-dimensional $\mathbf{Q}_\ell$-representation of $\pi_1(U, u)$.

The key is to understand this monodromy action. Its precise nature will be elaborated upon later, but for now it's enough to emphasize that *the monodromy is only non-trivial on the middle-dimensional groups* $\mathrm{H}^{n-1}(X_{\text{ét}}, \mathbf{Q}_\ell)$. In other words, the local systems $R^i f_* \mathbf{Q}_\ell|_U$ are *trivial* except when $i = n-1$. This fact will be part of the "Picard-Lefschetz" formula for the monodromy to be discussed in the future.

Armed with this knowledge, we can immediately dispose of a couple terms of the spectral sequence. One of them was

$$\mathrm{H}^0(\mathbf{P}^1, R^{n+1} f_*(X_u, \mathbf{Q}_\ell) = (\mathrm{H}^{n+1}(X_u, \mathbf{Q}_\ell))^{\pi_1} = \mathrm{H}^{n+1}(X_u, \mathbf{Q}_\ell).$$

Now, the result follows from induction on the dimension of X. Actually, it turns out that we need to induct on *even* dimension (for reasons having to do with the Picard-Lefschetz description of monodromy), so technically we need to take another hyperplane section of X_u, but that's okay: the cohomology group is not in middle dimension, and so is "detected" by a hyperplane section.

There is a difference between $\mathrm{H}^i(U, R^{n+1} f_* \mathbf{Q}_\ell)$ and $\mathrm{H}^i(\mathbf{P}^1, R_{n+1} f_* \mathbf{Q}_\ell)$ and it can happen that $R_{n-1} f_* \mathbf{Q}_\ell$ is not a local system, whileits restriction to U is. But that's not really an issue, because we always have a short exact sequence

$$0 \to j_! \mathscr{F} \to j_* \mathscr{F} \to j_* \mathscr{F} / j_! \mathscr{F} \to 0$$

which induces (because the last thing is torsion)

$$\mathrm{H}^1_c(U, \mathscr{F}) \to \mathrm{H}^1(\mathbf{P}^1, j_* \mathscr{F}) \to 0.$$

Therefore, for our purposes is really is enough to consider the restriction to U.

The other term $\mathrm{H}^2(\mathbf{P}^1, R^{n-1} f_* \mathbf{Q}_\ell)$ is basically dual to the one just discussed, and in fact the complication above never even arises.

The last case $\mathrm{H}^1(\mathbf{P}^1, R^n f_* \mathbf{Q}_\ell)$ is the most subtle. For now we'll just say that there is a sequence

$$0 \to j_* \mathscr{E} \to R^n f_* \mathbf{Q}_\ell \to (\text{constant sheaf}) \to 0$$

and so it suffices to analyze $\mathrm{H}^1(U, \mathscr{E})$. The local system $\mathscr{E}$ contains the "vanishing cycles", which are the cohomology classes that vanish in restriction to some special (singular) fiber. The monodromy action is unipotent, deforming the cohomology by vanishing cycles, so the quotient is constant. For now please just accept the above plus the fact that the restriction of the symplectic form to $\mathscr{E}$ is *non-degenerate*. (This is true, but only by deduction a posteriori; the actual argument requires a further filtration by $\mathscr{E} \cap \mathscr{E}^\perp$.)

5. THE FUNDAMENTAL ESTIMATE

5.1. **Theorem on weights.** Recall that a local system $\mathscr{F}$ on X is said to have *weight* β if F_x^* acting on $\mathscr{F}_x$ has eigenvalues which are algebraic with absolute value $q_x^{\beta/2}$ under every complex embedding. In particular, $\mathbf{Q}_\ell(r)$ has weight $-2r$.

Theorem 5.1. *Suppose $\mathscr{E}_0$ is a sheaf on U_0 satisfying the following conditions:*

(1) $\mathscr{E}_0$ is equipped with an alternating, non-degenerate bilinear form

$$\psi\colon \mathscr{E}_0 \otimes \mathscr{E}_0 \to \mathbf{Q}_\ell(-\beta).$$

(2) The image of $\pi_1(U,u)$ in $\mathrm{GL}(\mathscr{E}_u)$ is an open subgroup of $\mathrm{Sp}(\mathscr{E}_u,\psi_u)$.

(3) For all $x \in U_0$, the polynomial $\det(1-F_x t,\mathscr{E}_0)$ has rational coefficients.

Then $\mathscr{E}_0$ has weight β.

The inspiration from the following argument is said to come from ideas of Rankin attacking the Ramanujan conjecture (one of the consequences of Deligne's work).

Recall that

$$t\frac{d}{dt}\log\det(1-F_x t,\mathscr{E}_0)=\sum_{n\ge 1}\mathrm{Tr}(F_x^n)t^n.$$

In particular, since $\mathrm{Tr}(F_x,\bigotimes^{2k}\mathscr{E}_0)=\mathrm{Tr}(F_x,\mathscr{E}_0)^{2k}$ we have that $t\frac{d}{dt}\log\det(1-F_x t,\mathscr{E}_0)$ has positive rational coefficients (the positivity would make no sense without knowing that they were rational!). Therefore, the same holds for

$$\det(1-F_x t,\otimes^{2k}\mathscr{E}_0).$$

Now,

$$Z(U,\otimes^{2k}\mathscr{E}_0,t)=\prod_u \det(1-F_u t,\otimes^{2u}\mathscr{E}_0).$$

A product of power series with *positive* coefficients has radius of convergence at most that of any of its factors, since this is just some statement about the size of the coefficients.

Now let's consider the Grothendieck-Lefschetz formula for the zeta function:

$$Z(U,\otimes^{2k}\mathscr{E}_0,t)=\frac{P_1(t)}{P_0(t)P_2(t)}.$$

Here $P_0(t)=\det(1-F^*t,\mathrm{H}^0_c(U,\mathscr{E}))$. But a local system on an affine variety has no compactly supported global sections, so $P_0(t)=1$. What about H^2_c? By duality,

$$\mathrm{H}^2_c(\mathscr{E}_0)\simeq \mathrm{H}^0(\mathscr{E}_0^\vee)^\vee(-1)=((\mathscr{E}_u^\vee)^{\pi_1})^\vee=(\mathscr{E}_u)_{\pi_1}(-1)$$

Now, since $\pi_1(U,u)$ is open in $\mathrm{Sp}(\mathscr{E}_u)$ it has the same coinvariants. Then $\mathscr{E}_u$ is just the "standard representation" of the symplectic group. This become a classical question about the coinvariants of tensor powers of the standard representation. It is a theorem that the ring of invariants is generated by the tensor symbols $[x,y]$ corresponding to the symplectic form, and so we find that

$$\left(\otimes^{2k}\mathscr{E}_u\right)_{\pi_1}\simeq \mathbf{Q}_\ell(-k\beta)^{\mathscr{P}'}$$

where $\mathscr{P}'$ is a set of partitions of $[1,2k]$ into pairs, corresponding to $[x_i,x_j]$.

The upshot is that $\mathrm{H}^2_c(U, \otimes^{2k}\mathscr{E}) \simeq Q_\ell(-k\beta-1)^N$. So

$$Z(U_0, \otimes^{2k}\mathscr{E}, t) = \frac{P_1}{(1-q^{k\beta+1}t)^N}.$$

In particular, the only pole is at $t = q^{-k\beta-1}$. So there are no zeros of the $\det(1-F_x t, \otimes^{2k}\mathscr{E}_0)$ with absolute value less than $q^{-k\beta-1}$. The zeros are the inverses of the eigenvalues of Frobenius raised to $2k$, so

$$|\alpha|^{-2k} \geq q^{-k\beta-1}.$$

Rearranging we get

$$|\alpha| \leq q^{\frac{\beta}{2}+\frac{1}{2k}}.$$

Now we just take $2k \to \infty$ to get the desired upper bound. By Poincaré duality q^β/α is also an eigenvalue, so

$$|q^\beta/\alpha| \leq q^{\beta/2}$$

implies the opposite inequality.

5.2. **Frobenius eigenvalues.** Recall that what we actually wanted was the eigenvalues of Frobenius on $\mathrm{H}^1_c(\mathbf{P}^1, \mathscr{E}_0)$. The zeta function is

$$Z(U, \mathscr{E}_0, t) = P_1(t).$$

The zeros of $P_1(t)$ are the inverses of the Frobenius eigenvalues. Now, this is manifestly an ℓ-adic polynomial, but also a power series with *rational* coefficients by our assumptions, hence a rational polynomial. This shows that the eigenvalues are rational.

Since Z has an Euler product expansion, we should be able to use that to see that the location of the zeros. The factors have, as showed, eigenvalues with absolute value $q^{\beta/2}$ under every complex embedding. The problem is that we do not necessarily know that the Euler product converges at $t = q^{-\beta/2}$. In fact it doesnt, so we will show that it converges for $|t| < q^{-\beta/2-1}$.

We have $\det(1-F_u^* t, \mathscr{E}_u) = \prod(1-\alpha_{i,u}t)$. Therefore, it suffices to analyze when

$$\sum_{i,u} \alpha_{i,u} t$$

converges. We know that $|\alpha_{i,u}| = q^{\beta \deg u/2}$, so we can regroup the sum as

$$\sum_u \sum_n q^{n\beta/2} \# U(\mathbf{F}_{q^n}) t^n.$$

What is $\#U(\mathbf{F}_{q^n})$? Well U is off from $\mathbf{A}^1$ by just a few points, so $\#U(\mathbf{F}_{q^n}) \leq \mathbf{A}^1(\mathbf{F}_{q^n}) = q^n$. So the conclusion is that the sum is

$$\sum_n q^{n(1+\beta/2)} t^n$$

and thus converges for $|t| < q^{-(1+\beta/2)}$.

We're almost done. We proved that $\mathrm{H}^1(U, \mathscr{E}_0)$ has eigenvalues of magnitude

$$q^{\beta/2-1} \leq |\alpha| \leq q^{\beta/2+1}.$$

We want to get $|\alpha| = q^{\beta/2}$ on the nose. But this is just another application of the tensor power trick. Replace X by X^k, and we get (by Künneth)

$$q^{k\beta/2-1} \leq |\alpha|^k \leq q^{k\beta/2+1}.$$

6. Monodromy theory of Lefschetz pencils

We now want to go back and substantiate some of the claims about Lefschetz pencils that we used. The setup of interest is that we have a fibration

$$f: X \to \mathbf{P}^1$$

such that

(1) X is non-singular of dimension $n+1$
(2) f is proper,
(3) f has non-degenerate critical points, i.e. the only singular points of the singular fibers are simple double points.

The third condition is essentially that of being a "morse function".

In such a situation, f will be smooth outside a finite set of points $S \subset \mathbf{P}^1$. If U is the open complement, then $R^i f_* \mathbf{Q}_\ell$ will be a loca system on U, and we want to understand the monodromy action of $\pi_1(U, u)$ on $(R^i f_* \mathbf{Q}_\ell)_u = \mathrm{H}^i(f^{-1}(u), \mathbf{Q}_\ell)$.

6.1. **Existence of Lefschetz pencils.** This situation arose from taking a pencil of hyperplane sections of a smooth projective $X \subset \mathbf{P}^N$ along an axis A, and blowing up along $A \cap X$. Why does a pencil of the desired form exist? The picture is clarified by looking at the *dual variety* $X^\vee \subset (\mathbf{P}^N)^\vee$. The points of $(\mathbf{P}^N)^\vee$ are the hyperplanes of $\mathbf{P}^N$, and $X^\vee$ is the subset of hyperplane tangent to some point of X. In other words, it is the image of the incidence correspondence

$$\{(x, H) \subset X \times (\mathbf{P}^N)^\vee \mid H \supset T_x X\}.$$

By dimension counting, this has dimension $\dim X + (N - \dim X - 1) = N - 1$, so $X^\vee$ has dimension at most $N-1$. A pencil of hyperplanes is the same as a literal pencil $\mathbf{P}^1 \subset (\mathbf{P}^N)^\vee$ (linearly embedded). It turns out that if it avoids the singular locus and intersects $X^\vee$ transversely, then it will be a Lefschetz pencil. This is a local calculation which we leave as an exercise to the reader.

6.2. **The local theory.** Let's considerthe classical case first: suppose we have a map $f: X^{n+1} \to D$ where is an open unit disc in $\mathbf{C}$, which is smooth outside 0 and such that $X_0 := f^{-1}(0)$ has a double point.

It turns out (but is not obvious) that X deformation retracts to X_0, so we have an isomorphism

$$\mathrm{H}^i(X_0, \mathbf{C}) \simeq \mathrm{H}^i(X, \mathbf{C}).$$

On the other hand, if t denotes some generic non-zero point of D then we have a restriction map

$$\mathrm{H}^i(X_0, \mathbf{C}) \simeq \mathrm{H}^i(X, \mathbf{C}) \to \mathrm{H}^i(X_t, \mathbf{C}).$$

The image consists of the "monodromy invariants" under the monodromy action of $\pi_1(D^*, t) \simeq \mathbf{Z}$ on $\mathrm{H}^i(X_t, \mathbf{C})$. Let T be a generator.

Definition 6.1. We define the *vanishing subspace* to be $\mathrm{H}^n(X_0,\mathbf{C})^{\perp}\subset \mathrm{H}^n(X_t,\mathbf{C})$ under the pairing induced by Poincaré duality.

♠♠♠ TONY: [drawing my picture here]

Facts:

- The vanishing cycles form a line generated by δ.
- T acts trivially on $\mathrm{H}^i(X_t,\mathbf{C})$ for $i\neq n$.
- For $x\in \mathrm{H}^n(X_t,\mathbf{C})$, T acts by $x\mapsto x\pm(x,\delta)\delta$.

Remark 6.2. The $\pm$ depends on n mod 4.

Example 6.3. Example: degeneration of elliptic curves.

It is straightforward to write down the algebro-geometric analogue. We replace D by the spectrum of a (strictly henselian) DVR, with special point s and generic point η, so have maps

$$\mathrm{H}^i(X_s,\mathbf{Q}_\ell)\simeq \mathrm{H}^i(X,\mathbf{Q}_\ell)\to \mathrm{H}^i(X_{\overline{\eta}},\mathbf{Q}_\ell).$$

Now the possibilities are a little complicated. First they depend on whether n is odd or even. Fortunately we're only going to discuss the even case, so we can ignore that. It is also possible that there is no vanishing cycle, i.e. $\delta=0$, which makes things easier (no monodromy means everything is a local system). The interesting case is the one where

$$\gamma(x)=x+(x,\delta)\delta.$$

6.3. **The global theory.** We have a Lefschetz pencil

$$f\colon X\to \mathbf{P}^1.$$

This is smooth outside a finite set S. We choose a baseopint $u\notin S$. For each $s\in S$, we get a vanishing cycle δ_s, an da loop γ_s such that for $x\in \mathrm{H}^n(X_u:=f^{-1}(u),\mathbf{Q}_\ell)$

$$\gamma_s(x)=x\pm(x,\delta_s)\delta_s.$$

Definition 6.4. We define the subspace of vanishing cycles $E\subset \mathrm{H}^n(X_u)$ to be the psan of hte vanishing cycles.

Proposition 6.5. *$E^{\perp}$ is the monodromy invariants.*

This obvious from the nature of the Picard-Lefschetz formula. Therefore, we rename $E=E/E^{\perp}$ and forget that $E^{\perp}$ exists.

Theorem 6.6. *The vanishing cycles δ_s are conjugate.*

Proof. I honestly don't understand Deligne's argument. ☐

Corollary 6.7. *The representation of $\pi_1(U,u)$ on E is irreducible.*

Proof. Note that $\gamma_s x=x\pm(x,\delta_s)\delta_s$. Take some non-zero $x\in F$. Then $(x,\delta_s)\neq 0$ for some s, so

$$\gamma_s x-x=\pm(x,\delta_s)\delta_s.$$

Therefore, $\delta_s\in F$. But since the δ_s are all conjugate, they must then all lie in F. ☐

Theorem 6.8. *The image of $\rho\colon \pi_1(U,u)\to \mathrm{Sp}(E)$ is open.*

Proof. The image is some compact ℓ-adic Lie group. It suffices to show that its Lie algebra $\mathfrak{L}$ is open. Note that the $\mathfrak{L}$ is generated by automorphisms of the form

$$d(x \mapsto x \mp (x, \delta_s)\delta_s) = x \mapsto \pm(x, \delta_s)\delta_s.$$

In slightly more generality, we claim that if V is an irreducible representation of $\mathfrak{L}$ which is irreducible, and $\mathfrak{L}$ is generated by endomorphisms of the form $x \mapsto \psi(x, \delta)\delta$ then $\mathfrak{L} = \mathrm{Sp}(V, \psi)$.

To see this, we want to try to define a subrepresentation. Let W be the set of $\delta \in V$ such that $N(\delta) := x \mapsto (x, \delta)\delta \in \mathscr{L}$. This has no evident linear structure. Well, it is obviously closed under scaling. But what about addition?

Notice that since $N(\delta)$ has square 0, the automorphism $\exp(N(\delta))$ makes sense. Moreover, this preserves ψ and $\mathfrak{L}$ if $\delta \in W$:

$$\exp(\lambda N(\delta))\delta' \exp(-\lambda N(\delta))$$

acts by

$$\begin{aligned} x &\mapsto (1 + \lambda N(\delta))N(\delta')(1 - \lambda N(\delta))x \\ &= \delta' + \lambda N(\delta)N(\delta') - \lambda N(\delta')N(\delta) + \lambda^2 \psi(x, \delta)\psi(\delta, \delta')\psi(\delta', \delta)\delta. \end{aligned}$$

But $\exp(\lambda N(\delta'))\delta'' = \delta'' + \lambda\psi(\delta'', \delta')\delta') \in W$, so if $\psi(\delta', \delta'') \neq 0$ then one gets that the subspace spanned by δ' and δ'' are in W.

This shows that W is the union of its maximal linear subspaces, which are pairwise mutually orthogonal. Any such is stable under $N(\delta)$, by the above considerations. Since it not 0, it must be everything.

□

References

[1] Deligne, P. *La Conjecture de Weil. I.*