EXAMPLES

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Contents

1. Correlation functions

There are two examples: "free boson" and "free fermion".

We will explain these by writing down their correlation functions. Then we'll talk about how to extract a factorization algebra from the correlation function.

From this one can further extract an associated ("chiral") Lie algebra, and representation.

Furthermore, there is a "boson-fermion correspondence" which maps from the boson situation to the fermion situation.

Finally, there is a story which as to do with determinants of line bundles. The factorization algebras will come from factorizable line bundles, in the way Jacob described.

2. Free boson

2.1. **Partition function.** A CFT is a way of attaching a number to a Riemann surface Σ, which is given by integrating over the space of maps $\Sigma \to \mathbf{R}$. Let's call it

$$Z(\Sigma) = \int_{\varphi \in \mathrm{Maps}(\Sigma, \mathbf{R})} (\dots)$$

Once we have this, we can try to weight it by observables: given $z_1, \dots, z_n \in \Sigma$, define a correlation function

$$\langle \varphi(z_1), \dots, \varphi(z_n) \rangle = \frac{\int_{\mathrm{Maps}(\Sigma, \mathbf{R})} \varphi(z_1) \dots \varphi(z_n)}{Z(\Sigma)}$$

Actually it's better to put in derivatives, $\phi = \partial\varphi$, because they are translation invariant.

2.2. **Correlation functions on $\mathbb{P}^1$.** We take $\Sigma = \mathbb{P}^1$. We have

$$\langle \phi(z_1)\phi(z_2)\rangle = \frac{1}{(z_1 - z_2)^2} \tag{2.1}$$

and more generally

$$\langle \phi(z_1), \dots, \phi(z_{2n})\rangle = \sum_{\substack{\text{pairings } i \leftrightarrow i' \\ \text{of } \{1,2,\dots,2n\}}} \prod_{i \leftrightarrow i'} \frac{1}{(z_i - z_{i'})^2} \tag{2.2}$$

Physicists have a very useful notation

$$\phi(x)\phi(y) \sim \frac{1}{(x-y)^2}. \tag{2.3}$$

The $\sim$ means "up to functions that are regular as $x - y \to 0$". Also it means we can make this substitution into any correlator function, e.g.

$$\langle \phi(w_1) \dots \phi(w_n)\phi(x)\phi(y)\rangle = \frac{\langle \phi(w_1) \dots \phi(w_n)\rangle}{(x-y)^2} + (\text{regular as } x \to y)$$

So the $\sim$ in (3.1) tracks the "singular part" of the correlation function, and miraculously this contains all that you need to do computations in practice.

Definition 2.1. We define $B := \mathbf{C}[x_1, x_2, \dots]$. Regard "$x_1 \leftrightarrow \phi$, $x_2 \leftrightarrow \partial\phi$, $x_3 \leftrightarrow \partial^2\phi$." So for example $x_1^2 x_2 \leftrightarrow \phi^2(\partial\phi)$.

2.3. **Observables.** We will explain how each element $b \in B$ gives an *observable*. For us "observable" means "something which has a meaning in a correlator". Each, an observable "$b(z)$ at z" means we assigned meaning to all expressions of the form

$$\langle \phi(z)\phi(w_1) \dots \phi(w_n)\rangle$$

which are interpreted as meromorphic functions in the w_i, holomorphic away from $w_i = z, w_i = w_j$.

Example 2.2. By (2.1) ϕ gives an observable, with $\langle \phi(z)\phi(w)\rangle = \frac{1}{(z-w)^2}$.

We will explain how any expression in B can be interpreted as an observable.

Example 2.3. We explain how to interpret $\partial\phi$ as an observable. We try to set

$$\langle \partial\phi(z_1)\phi(w_1) \dots \phi(w_n)\rangle := \partial_z \langle \phi(z)\phi(w_1) \dots \phi(w_n)\rangle.$$

Let's compute this (for $\mathbb{P}^1$). We sum over pairings, and we distinguish what z gets paired with, say w_i.

$$\langle \partial\phi(z)\phi(w_1) \dots \phi(w_n)\rangle = \sum_i \frac{-2}{(z-w_i)^3} \langle \phi(w_1) \dots \widehat{\phi(w_i)} \dots \phi(w_n)\rangle.$$

Going forward we will write $\langle \widehat{i} \rangle = \langle \phi(w_1) \dots \widehat{\phi(w_i)} \dots \phi(w_n)\rangle$. Similarly abbreviate $\langle \widehat{ij} \rangle = \langle \phi(w_1) \dots \widehat{\phi(w_i)} \dots \widehat{\phi(w_j)} \dots \phi(w_n)\rangle$, etc.

Similarly, $\partial\phi, \partial^2\phi, \dots$ are observables.

Remark 2.4. We can also think of $\partial\phi(z_1), \partial\phi(z_2)$ as an observable at (z_1, z_2).

Example 2.5. Next we would like to make sense of ϕ^2 as an observable. Naively, we try to define

$$\langle \phi^2(z)\phi(w_1)\dots\phi(w_n)\rangle = \lim_{z_1\to z_2}\langle \phi(z_1)\phi(z_2)\phi(w_1)\dots\phi(w_n)\rangle$$

We will try to regularize this. We will interpret this limit as the constant term of the Laurent expansion in z_1 as $z_1 \to z_2$. Note that this is not symmetric in z_1, z_2. This regularization scheme is called "normal ordering".

So we've decided to set

$$\langle \phi^2(z)\phi(w_1)\dots\phi(w_n)\rangle = \lim_{z_1\to z_2}\langle \phi(z_1)\phi(z_2)\phi(w_1)\dots\phi(w_n)\rangle$$

with limit understood in the above sense. When you expand this out, you get various terms. One is

$$\frac{1}{(z_1-z_2)^2}\langle \phi(w_1)\dots\phi(w_n)\rangle$$

and this vanishes in our limiting sense, because there is no constant term in the Laurent expansion.

The rest is a sum over i and j (pairing z_1 with w_i and z_2 with w_j) of

$$\sum_{i,j}\frac{\langle \widehat{ij}\rangle}{(z_1-w_1)^2(z_2-w_2)^2}.$$

Now this is regular as $z_1 \to z_2$. So

$$\langle \phi^2(z)\phi(w_1)\dots\phi(w_n)\rangle \sim \sum_{i,j}\frac{\langle \widehat{ij}\rangle}{(z_1-w_1)^2(z_2-w_2)^2}. \tag{2.4}$$

Example 2.6. Let's try to define $\phi(\partial\phi)(\partial\phi)$ as an observable. The convention that works is to regularize according to the above scheme from *right to left*.

$$\phi\cdot\left(\underbrace{(\partial\phi)(\partial\phi)}_{\text{regularize}}\right).$$

Example 2.7. Let $T = \frac{1}{2}\phi^2$ be the "stress-energy tensor".

We will write down the operator product expansion of $T(z_1)\phi(z_2)$. We claim that

$$T(z_1)\phi(z_2) \sim \frac{\phi(z_2)}{(z_1-z_2)^2} + \frac{\partial\phi(z_2)}{(z_1-z_2)}. \tag{2.5}$$

We have to figure out

$$\langle T(z_1)\phi(z_2)\phi(w_1)\dots\phi(w_n)\rangle.$$

According to (2.4), to calculate the above we have to pair $T(z_1)$ with two other variables in $\{z_2, w_1, \dots, w_n\}$. The terms where it is not paired with z_2 are regular as $z_2 \to z_1$, so we can ignore them. In the other terms, it is paired z_2 and something w_i. Such terms contribute

$$\frac{1}{(z_1-z_2)^2}\sum_i \frac{\langle \widehat{i}\rangle}{(z_1-w_i)^2} + (\text{regular as } z_1\to z_2).$$

This is the same as

$$\frac{1}{(z_1-z_2)^2}\langle \phi(z_1)\phi(w_1)\dots\phi(w_n)\rangle + (\text{regular as } z_1\to z_2).$$

This says that $T(z_1)\phi(z_2) \sim \frac{\phi(z_1)}{(z_1-z_2)^2}$. To get (2.5), we then rewrite this by expanding around z_2.

Exercise 2.8. Compute that

$$T(z_1)T(z_2) \sim \frac{1/2}{(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{\partial T(z_2)}{(z_1-z_2)}.$$

3. Factorization algebras associated to correlation functions

In our definition of factorization algebra, we need to give $\mathcal{F}_1$ on $\mathbb{A}^1$, $\mathcal{F}_2$ on $\mathbb{A}^2$, $\mathcal{F}_3$ on $\mathbb{A}^3$, etc. with the property that when you restrict away from diagonals they factorize, and when you restrict to the diagonals there are isomorphisms between the $\mathcal{F}_i$.

To construct the factorization algebra associated to the free boson, we will take $\mathcal{F}_1 = B \otimes \mathcal{O}_{\mathbb{A}^1}$, $\mathcal{F}_2$ to be the subsheaf of $\mathcal{F}_1 \boxtimes \mathcal{F}_1$ such that all correlations are regular, $\mathcal{F}_3$ the subsheaf of $\mathcal{F}_1 \boxtimes \mathcal{F}_1 \boxtimes \mathcal{F}_1$ such that all correlations are regular, etc.

This is not imposing any condition away from the diagonals. What about on the diagonals? Fix a small $U_0 \subset \mathbf{C}$. Take sections a_1, a_2 of $\mathcal{F}_1$ on U_0. From this we get a correlation function $\langle a_1(z_1)a_2(z_2)\phi(w_1)\dots\phi(w_n)\rangle$. For $a_1 \boxtimes a_2$ to be in $\mathcal{F}_2$, this should be regular as $z_1 \to z_2$.

For any $b_1, b_2 \in B$ we have an OPE

$$b_1(z_1)b_2(z_2) \sim \sum_{k\in\mathbf{Z}} c_k(z_2)(z_1-z_2)^k. \tag{3.1}$$

for some $c_k \in B$. This is equivalent to:

(1) LHS $\sim$ RHS in any correlator.
(2) In a unital factorization algebra, $b_1(z_1) \boxtimes b_2(z_2)$ is a meromorphic section of $\mathcal{F}_2$ which agrees with the section $\sum_{k\in\mathbf{Z}} c_k(z_2)(z_1-z_2)^k$ of $\mathcal{O}\otimes\mathcal{F}_1$ in a neighborhood of the formal completion.

4. Relation to vertex algebras

In the language of vertex algebras, (3.1) would say:

$$Y(b_1, z)b_2 = \sum c_k z^k.$$

We have seen some OPEs. Let's try to go further:

$$\phi(z)\phi(w) = \frac{1}{(z-w)^2} + A_0(w) + A_1(w)(z-w) + \dots \tag{4.1}$$

By *definition*, $A_0(w) =: \phi^2$. (Our definition of ϕ^2 w as as the constant term in the Laurent expansion of $\phi(z)\phi(w)$).

The next term can be computed by differentiating with respect to z. So it's another tautology that $A_1(w) = (\phi\phi')(w)$. The next term would be $\frac{\phi\phi''}{2}(w)(z-w)^2$. So the OPE of $\phi(z)\phi(w)$ is

$$\phi(z)\phi(w) = \frac{1}{(z-w)^2} + \phi^2(w) + (\phi\phi')(w)(z-w) + \frac{\phi\phi''}{2}(w)(z-w)^2 + \dots$$

The is equivalent to the identity in the vertex algebra incarnation:

$$Y(\phi, z)\phi = z^{-2} + (\phi^2)z^0 + (\phi\phi')z^1 + \frac{(\phi\phi'')}{2}z^2 + \dots$$

Next let's consider $T(z)\phi(z)$. We have

$$T(z)\phi(w) = \frac{\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)} + (T \cdot \phi) + \dots \tag{4.2}$$

The term $T \cdot \phi$ is there *by definition* – we defined it to be the constant term in this OPE. We emphasize that although $T = \frac{1}{2}\phi^2$, $T \cdot \phi$ is *not* $\frac{1}{2}\phi^3$, because we chose a regularization scheme that goes from right to left. In fact $(T \cdot \phi) = \frac{1}{2}\phi^3 + \frac{1}{2}\phi''$.

5. Passage to Lie algebras

A vertex algebra is some kind of map $B \to \mathrm{End}(B)((z))$. We will define a Lie algebra homomorphism

$$\mathrm{Op}\colon \frac{B[z, z^{-1}]}{\text{“derivatives”}} \to \mathrm{End}(B) \tag{5.1}$$

Derivatives are what you think, e.g. $D(\phi^2 z^3) = 2\phi\phi' z^3 + 3\phi^2 z^3$.

We will define the map (and explain the Lie algebra structure later). Let $a \otimes z^m \in B[z, z^{-1}]$ for $a \in B$.

$$\mathrm{Op}(a \otimes z^m)b = \int z^m Y(a, z)b$$

where the integration is over a contour in $\mathbf{C}$ containing 0. More algebraically, this extracts the residue of the OPE for $z^m a(z)b$.

Example 5.1. $\mathrm{Op}(\phi \otimes 1)\phi$ is the residue of $\phi(z)\phi(0)$, at $z = 0$. Examine (4.1) with w set to 0 – the residue vanishes, so $\mathrm{Op}(\phi \otimes 1)\phi = 0$.

Similarly, $\mathrm{Op}(\phi \otimes z)\phi = 1$, $\mathrm{Op}(\phi \otimes z^n)\phi = 0$ for $n > 1$.

Example 5.2. We calculate $\mathrm{Op}(T)\phi$. Set $w = 0$ in (4.2) and taking the residue, it gives $\mathrm{Op}(T)\phi = \partial\phi$.

So Op is defined by “integrating the vertex operator”. We put the Lie algebra structure on the LHS by

$$[a \otimes z^m, b \otimes z^n] = \mathrm{Res}_{z=w}(z^m w^n a(z)b(w)).$$

Call $a_m = a \otimes z^m, b_n = b \otimes z^n$.

Example 5.3. Let $E \in$ LHS be the class of $1 \otimes z^{-1}$. We can check that $\mathrm{Op}(E) = \mathrm{Id}_B$.

Then $[\phi_n, \phi_m] = \mathrm{Res}_{w \to z}(w^m z^n \frac{1}{(z-w)^2}) = m z^{m+n-1}$. This vanishes unless $m + n = 0$. If $m + n = 0$ then its mE. So our Lie algebra contains a bunch of commuting Heisenberg algebras $\langle \phi_n, \phi_{-n}, E \rangle$.

How do the ϕ_n act on B? Write $B \cong \mathbf{C}[x_1, x_2, \dots]$ with $x_i \leftrightarrow \partial^{i-1}\phi$.

Up to normalization constants, ϕ_{-n} goes to multiplication by x_m, ϕ_m goes to $\frac{\partial}{\partial x_m}$. So the Heisenberg subalgebras are acting by their standard representations.

Let's sketch why this is a Lie algebra homomorphism. We will compute $[\mathrm{Op}(a \otimes z^m), \mathrm{Op}(b \otimes z^n)]c$. The composition $\mathrm{Op}(a \otimes z^m), \mathrm{Op}(b \otimes z^n)$ is computed by

$$\int_z \int_w Y(a, z) z^m Y(b, w) w^n c$$

This is an integral over a w-contour *contained in* the z-contour. When you do it in the other order, the contours of integration are switched – the z contour is inside the w-contour. Fixing w, the difference of the z integrals picks up the residue at $z = w$.

Example 5.4. Let $L_n := T \otimes z^n \in B[z^{\pm 1}]/\text{derivatives}$. The Lie bracket satisfies

$$[L_n, L_m] = \frac{n(n^2-1)}{12} E\delta_{n+m,0} + (n-m)L_{n+m}.$$

Together with E, these generate the *Virasoro algebra*. It is a central extension of $\langle z^n \partial_z \rangle$ (span of these vector fields on $\mathbf{G}_m$). "T is a conformal vector". The conformality is what allows vertex algebras to be transferred to any Riemann surface.

6. Free fermion

In the case of the free boson, we started with the "basic field" ϕ and we specified its correlation functions.

Now we will have two fields ψ, ψ^*. We specify the correlation function

$$\langle \psi(p_1)\psi(p_2)\dots\psi(p_n)\psi^*(q_1)\dots\psi^*(q_m)\rangle$$

It is 0 if $n \neq m$. If $n = m$, it is given by

$$\langle \psi(p_1)\dots\psi(p_n)\psi^*(q_1)\dots\psi^*(q_n)\rangle = \frac{\prod_{i<j}(p_i - p_j)(q_i - q_j)}{\prod_{i,j}(p_i - q_j)}. \tag{6.1}$$

e.g. $\langle \psi(p)\psi^*(q)\rangle = 1/(p-q)$.

Warning 6.1. if you swap $\psi(p_1), \psi(p_i)$ the sign changes.

It's convenient to extend the definition so that it's anti-symmetric in all arguments. Then more generally,

$$\langle (\psi \text{ or } \psi^*)(x_1)(\psi \text{ or } \psi^*)(x_2)\dots(\psi \text{ or } \psi^*)(x_{2n})\rangle$$

is 0 or $\prod (x_i - x_j)^{\epsilon_i \epsilon_j}$ where $\epsilon_i \in \pm 1$ tracking whether the argument is ψ or ψ^*.

Now we're going to enlarge the algebra of observables in the same way. They will consist of ψ and ψ^*, $\partial\psi, \partial^2\psi$, etc. We can try to make ψ^2 or $(\psi^*)^2$, but they turn out to be 0. But $\psi\psi^*$ is a non-trivial observable. Let's call it A.

Example 6.2. Let's try computing $\langle A(z)\psi(w_1)\psi^*(w_2)\rangle$. How $A(z)$ is standing for $\psi(z_1)\psi^*(z_2)$. The answer is the constant term of

$$\lim_{z_1 \to z_2 = z} -\frac{(z_1 - w_1)(z_2 - w_2)}{(z_1 - w_2)(z_1 - z_2)(w_1 - w_2)(w_1 - z_2)}.$$

Definition 6.3. Let B' be the space of observables, which is the $\mathbf{C}$-span of

$$(\partial^{a_1}\psi)(\partial^{a_2}\psi)\dots(\partial^{a_n}\psi)(\partial^{b_1}\psi^*)\dots(\partial^{b_m}\psi^*)$$

where $a_1 > a_2 > \dots, a_n$ and $b_1 < b_2 < \dots < b_m$.

Regard B' as a $\mathbf{Z}/2$-graded algebra with this expression having parity $n+m$. This gives a (super?) factorization algebra.

Example 6.4. Consider $\langle A(z)A(w)\rangle = \frac{1}{(z-w)^2}$. This is the same as the answer for $\langle \phi(z)\phi(w)\rangle$ in the bosonic case. This is an example of the Boson-Fermion correspondence. There's a map $B \to B'$ carrying $\phi \mapsto A$, preserving all correlations. It maps isomorphically onto the subspace with $m = n$.

As before, we get a map of Lie *super*algebras

$$\frac{B'[z, z^{-1}]}{\text{derivatives}} \to \text{End}(B').$$

Example 6.5. Let $\psi_m = \psi \otimes z^m$ and $\psi_n^* = \psi \otimes z^n$.

$$[\psi_m, \psi_n^*]_+ = \mathrm{Res}_{z=w}(z^m w^n \frac{1}{z-w}) = \mathrm{Res}_z(z^{m+n}) = \begin{cases} 0 & m+n=-1 \\ E & m+n=-1 \end{cases}$$

This looks like a bunch of Clifford algebras,

$$[\psi_m, \psi_{1-m}^*]_+ = E.$$

Again, the representation on B' looks like the tensor product of standard representations.

7. Determinant bundles

We want to explain how the correlation formulas

$$\frac{\prod(p_i - p_j)(q_i - q_j)}{\prod(p_i - q_j)} = \det\left(\frac{1}{p_i - q_j}\right)_{ij}$$

arises from determinants. This is parallel to the discussion of how factorization algebras come from the determinant line bundle on the Beilinson-Drinfeld Grassmannian.

Let Σ be a Riemann surface and L be a line bundle on Σ. We define $[L]$ to be the determinant of the cohomology of L, which is $\det H^0(\Sigma, L) \otimes \det H^1(\Sigma, L)^\vee$. This construction has nice properties.

- $[L]$ defines a line bundle on $\mathrm{Pic}(\Sigma)$.
- L has a factorization property: if s is a rational section of L, we get an isomorphism

$$[L]/[\mathcal{O}] \xrightarrow{\sim} \otimes_{p \in \mathrm{Div}(s)}(\text{fiber of local det bundle on } \mathrm{Gr}_1 \text{ at } p).$$

Example 7.1. Consider $L = \mathcal{O}(\sum p_i - \sum q_j)$, for p_i, q_j distinct. This has a tautological rational section. From this tautological section we get an isomorphism

$$\frac{[\mathcal{O}(\sum p_i - \sum q_j)]}{[\mathcal{O}]} \cong \bigotimes_i \frac{[\mathcal{O}(p_i)]}{[\mathcal{O}]} \otimes \frac{[\mathcal{O}(-q_i)]}{[\mathcal{O}]}$$

We compute the local terms as follows. We have a short exact sequence

$$0 \to \mathcal{O}(-q_j) \to \mathcal{O} \to \mathcal{O}_{q_i} \to 0 \tag{7.1}$$

which exhibits a canonical trivialization of $\frac{[\mathcal{O}]}{[\mathcal{O}(-q_i)]}$. Applying $\mathcal{RH}om(-, \mathcal{O})$ to (7.1) and using Serre duality, we get a short exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(p_i) \to K_{p_i}^{-1} \to 0 \tag{7.2}$$

where K is the dual to canonical bundle, which exhibits an isomorphism $\frac{[\mathcal{O}(p_i)]}{[\mathcal{O}]} \cong K_{p_i}^{-1}$.

Now let $K^{1/2}$ be an even spin structure. It has degree $g-1$. On $\mathrm{Pic}^{g-1}(\Sigma)$, there's a distinguished section of $[L]$. Generically both H^0 and H^1 are trivial. That gives a generic section. The zero locus of this canonical section is the theta divisor.

Now we will construct the correlators from this. There's a variant where you replace the reference bundle $[\mathcal{O}]$ by $[K^{1/2}]$.

$$\frac{[K^{1/2}(\sum p_i - \sum q_j)]}{[K^{1/2}]} \cong \bigotimes_i \frac{[K^{1/2}(p_i)]}{[K^{1/2}]} \otimes \frac{[K^{1/2}(-q_i)]}{[K^{1/2}]}.$$

Now the right hand side is $\bigotimes K_{p_i}^{-1/2} \otimes \bigotimes K_{q_i}^{-1/2}$ and we have a canonical section of this. The inverse of this section gives a meromorphic section on Σ^{2n} of $K^{1/2} \boxtimes \ldots \boxtimes K^{1/2}$, which is

$$\langle \psi(p_1) \ldots \psi(p_n) \psi^*(q_1) \ldots \psi^*(q_n) \rangle$$

Example 7.2. In the previous example, wrote down a correlator function

$$\frac{\prod_{i,j}(p_i - p_j)(q_i - q_j)}{\prod_{i,j}(p_i - q_j)}$$

Think of $p_i - p_j$ as being $x_{p_i} y_{p_j} - x_{p_j} y_{p_i} \in \Gamma(\mathcal{O}(1) \boxtimes \mathcal{O}(1))$.