THE STABLE TRACE FORMULA, WITH EMPHASIS ON SL_2

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1. Overview

1.1. **The trace formula.** Let G be a connected reductive group, and assume for now that G is *anisotropic*. The *trace formula* is an identity

$$\sum_{\pi} m(\pi) \operatorname{Tr} \pi(f) = \sum_{\gamma \in G(F)/\mathrm{conj}} \mathrm{vol}(\gamma) O_\gamma(f).$$

It was conceived by Selberg, who used it in both directions (deducing information about the RHS when the LHS is simple, or vice versa). Over the years, the point of view has shifted towards viewing the LHS as interesting, and RHS as a way of accessing it.

When G is isotropic, the automorphic side no longer decomposes discretely, but we can restrict our attention to the discrete part:

$$\sum_{\pi \text{ discrete autrep}} m(\pi) \operatorname{Tr} \pi(f) = ???$$

and the other side becomes much more complicated.

There are two ways to use a trace formula.

(1) Use it in isolation – evaluate the geometric side in some explicit sense, and therefore compute the LHS. This is difficult, but has seen some progress, by Langlands, Olivier, Chenevier, Shin-Templier...
(2) Compare two trace formulas, by comparing their geometric sides, and therefore the spectral side. Examples of where this has been applied are: Inner forms of the same group, two different groups, or compare the trace formula with a different kind of trace formula such as point counts on Shimura varieties. This idea was pioneered by Langlands.

1.2. **The problem of stability.** The non-invariant trace formula is an identification of distributions

$$J_{\mathrm{geom}}(f) = J_{\mathrm{spec}}(f).$$

What does non-invariant mean?

Definition 1.1. A distribution I is *invariant* if for any $f \in C_c^\infty(G(F))$ and any element $g \in G(F)$,

$$I(f^g) = I(f)$$

where $f^g(x) := f(gxg^{-1})$.

We'll want invariance, but we'll want an even *stronger* condition, which we'll introduce next. First, we impose the assumption that G_{der} is simply connected.

Definition 1.2. Two elements $\gamma_1, \gamma_2 \in G(F)$ are *stably conjugate* if they are conjugate in $G(\overline{F})$. We write $\gamma_1 \sim_{\mathrm{st}} \gamma_2$.

Example 1.3. For $G = \mathrm{GL}_N$, stable conjugacy is the same as conjugacy (by rational canonical form).

Example 1.4. For $G = \mathrm{SL}_2 / \mathbf{R}$, the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

are conjugate by $\begin{pmatrix} i & \\ & -i \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C})$ but not conjugate in $\mathrm{SL}_2(\mathbf{R})$.

Definition 1.5. The *stable class* of $\gamma \in G(F)$ is the set of all $\gamma' \in G(F)$ stably conjugate to γ.

Definition 1.6. A function $f\colon G(F) \to \mathbf{C}$ is *stably invariant* if $f(\gamma_1) = f(\gamma_2)$ whenever $\gamma_1 \sim_{\mathrm{st}} \gamma_2$.

But what we really want is to define what it means for a distribution to be stably invariant, but this is subtler. The reason is that unlike conjugacy classes, stable conjugacy classes are not orbits for an action. So the definition goes through a detour.

Theorem 1.7 (Harish-Chandra)**.** *Let F be a local field. The following are equivalent.*

(1) I is an invariant distribution.
(2) If $f \in C_c^\infty(G(F))$ is such that $O_\gamma(f) = 0$ for all regular semisimple $\gamma \in G(F)$, then $I(f) = 0$.

We can then make a definition of stable distribution modeling this. We just need to define a *stable* orbital integral.

Definition 1.8. We define

$$\mathrm{SO}_\gamma(f) = \sum_{\gamma' \sim_{\mathrm{st}} \gamma} O_{\gamma'}(f).$$

Definition 1.9. Let F be a local field. We say that a distribution F is *stable* if $I(f) = 0$ for all $f \in C_c^\infty(G(F))$ such that $\mathrm{SO}_\gamma(f) = 0$ for all regular semisimple γ.

Fact 1.10. If I is represented by a function ϕ, then I is stable if and only if ϕ is stably invariant.

We would like to manipulate the trace formula $J_{\text{geom}} = J_{\text{spec}}$ into an equality of stably invariant distributions.

2. The regular elliptic part for SL_2

Let $G = \mathrm{SL}_2$ and F a global field. The geometric expansion of the *regular elliptic* part of the trace formula for SL_2 is:

$$\sum_{\gamma \in G(F)_{\text{reg. ell.}}/\text{conj}} \mathrm{vol}(T_\gamma(F)\backslash T_\gamma(\mathbf{A}))O_\gamma(f) \tag{2.1}$$

where T_γ is the centralizer of γ in G, and

$$O_\gamma(f) := \int_{T_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\, dg.$$

Remark 2.1. A small miracle is that for $\mathrm{SL}_2(\mathbf{R})$, the stable conjugacy classes actually *are* orbits under a group action, namely the conjugation action of $\mathrm{GL}_2(\mathbf{R})$.

2.1. **The space of stable conjugacy classes.** Let's first consider the situation where F is a *local* field. Let $\gamma \in G(F)_{\text{reg. ell.}}$. Let $\widetilde{T}_\gamma$ be the centralizer of γ in GL_2. In the short exact sequence

$$1 \to \mathrm{SL}_2(F)\widetilde{T}_\gamma(F) \to \mathrm{GL}_2(F) \to ? \to 1$$

the quotient ? parametrizes conjugacy classes in the stable conjugacy class. To describe what it is, recall that $\widetilde{T}_\gamma(F) = E^\times$ for a quadratic extension E/F, generated by the eigenvalues of γ. The determinant restricted to $\widetilde{T}_\gamma(F)$ is the norm for this field extension. (So we can say $T_\gamma(F) = E^1$, the norm-1 subgroup.) Hence we have $? = F^\times/N(E^\times)$, which has order 2.

The same discussion applies to the global situation, but then $F^\times/N(E^\times)$ is infinite.

We say $\gamma_1 \sim_{\text{st}} \gamma_2$ if there exist $g \in \mathrm{GL}_2(F)$ such that

$$\mathrm{Ad}(g)\gamma_1 = \gamma_2$$

hence $\mathrm{Ad}(g)$ induces an isomorphism $T_{\gamma_1} \xrightarrow{\sim} T_{\gamma_2}$ over F.

Hence the volume terms are constant in the stable conjugacy classes, so we can group things and rewrite (2.1) as

$$\sum_{\gamma_0 \in G(F)_{\text{reg. ell.}}} \mathrm{vol}(T_{\gamma_0}(F)\backslash T_{\gamma_0}(\mathbf{A})) \sum_{\gamma \sim_{\text{st}} \gamma_0} O_\gamma(f).$$

Is this stable? **No**, because the definition of stability is local. So for it to be stable we need to sum over adelic $\gamma \sim_{\text{st}} \gamma_0$, but we're only summing over the rational classes.

The rational conjugacy classes inside a stable one are parametrized by $F^\times/N(E^\times)$. The adelic conjugacy classes inside a stable one are parametrized by $\mathbf{A}_F^\times/N(\mathbf{A}_E^\times)$. What's the difference between them? It's measured by a short exact sequence

$$1 \underbrace{\to}_{\text{Hasse Norm thm}} F^\times/N_{E/F}(E^\times) \to \mathbf{A}_F^\times/N_{E/F}(\mathbf{A}_E^\times) \to \frac{\mathbf{A}_F^\times/F^\times}{N_{E/F}(\mathbf{A}_E^\times/E^\times)} \to 1. \tag{2.2}$$

We'd like to sum over the middle thing, whereas we were summing over the lelft thing. The difference between them is measured by $\frac{\mathbf{A}_F^\times/F^\times}{N_{E/F}(\mathbf{A}_E^\times/E^\times)}$ which has order 2.

2.2. **Stabilizing the trace formula.** The function $\gamma \mapsto O_\gamma(f)$ is perfectly well defined for all $\gamma \in \mathrm{SL}_2(\mathbf{A}_F)$, even though we were only considering it for $\gamma \in \mathrm{SL}_2(F)$. Write

$$\mathbf{A}_F^\times/N_{E/F}(\mathbf{A}_E^\times) = B_0 \cup B_1$$

lying over the 2 elements of $\frac{\mathbf{A}_F^\times/F^\times}{N_{E/F}(\mathbf{A}_E^\times/E^\times)}$.

We rewrite

$$\sum_{\gamma\in B_0} O_\gamma(f) = \frac{1}{2}\left(\sum_{\gamma\in B_0} O_\gamma(f) + \sum_{\gamma\in B_1} O_\gamma(f)\right) + \frac{1}{2}\left(\sum_{\gamma\in B_0} O_\gamma(f) + \sum_{\gamma\in B_1} O_\gamma(f)\right). \tag{2.3}$$

We define

- $SO_\gamma(f) = O_\gamma(f) + O_{\gamma'}(f)$ where γ' is the element of B_1 corresponding to γ.
- $O^\kappa_\gamma(f) = O_\gamma(f) - O_{\gamma'}(f)$ where γ' is the element of B_1 corresponding to γ.

Then we can rewrite (2.3) as

$$\frac{1}{2}\sum_{\gamma\in B_0} SO_\gamma(f) + O^\kappa_\gamma(f).$$

To summarize, the regular elliptic part of the trace formula can be written as

$$\begin{aligned}\mathrm{TF}_{\text{reg.ell.}}(f) &= \frac{1}{2}\sum_{\gamma_0\in G(F)/\sim_{\text{st}}} \mathrm{vol}(T_{\gamma_0}(F)\backslash T_{\gamma_0}(\mathbf{A}_F))\left(SO_\gamma(f) + O^\kappa_\gamma(f)\right)\\ &= \frac{1}{2}\mathrm{STF}_{\text{reg.ell.}}(f) + \frac{1}{4}\sum_{E/F\text{quad}}\sum_{\gamma\in E,\gamma\neq\pm1} O^\kappa_\gamma(f). \end{aligned} \tag{2.4}$$

(The factor 1/4 is 1/2 divided by 2 because $\gamma^{\pm1}$ give the same stable class.) We will write the second term in (2.4) as

$$\frac{1}{4}\sum_{E/F}\mathrm{STF}_{G-\text{reg}}(f^{T_E})$$

where $f^{T_E}(\gamma) := O^\kappa_\gamma(f)$.

Conclusion: we wrote the trace formula for G as a sum of a stable trace formula for G plus stable trace formulae for smaller groups. Note that we removed $\gamma = \pm1$ though; we'll not see until much later how to put them back in (they come from unipotent conjugacy classes).

2.3. **Extension of test functions.** The set E/F parametrizes

(1) The anisotropic tori.
(2) The endoscopic groups of SL_2.

This looks great at first, but there is a big catch. The test functions that you put into the trace formula are smooth, compactly supported functions. So we need to know that $f^{T_E} \in C_c^\infty(T_E(\mathbf{A}))$. But f^{T_E} is not even defined on all of $T_E(\mathbf{A})$; it's only defined on regular elements. So we need to extend it in a smooth way.

Consider again the last piece of (2.2):

$$\mathbf{A}_F^\times / N_{E/F}(\mathbf{A}_E^\times) \twoheadrightarrow \frac{\mathbf{A}_F^\times / F^\times}{N_{E/F}(\mathbf{A}_E^\times / E^\times)}$$

Really κ is a character of the second term, and the κ-orbital integral is

$$O_\gamma^\kappa(f) = \sum_{\gamma' \sim_{\mathrm{st}} \gamma} \kappa(\gamma') O_{\gamma'}(f)$$

Now, we can write

$$\mathbf{A}_F^\times / N_{E/F}(\mathbf{A}_E^\times) = \bigoplus_v F_v^\times / N_{E_v/F_v}(F_v^\times).$$

The summand

$$F_v^\times / N_{E_v/F_v}(F_v^\times) = \begin{cases} \mathbf{Z}/2\mathbf{Z} & v \text{ does not split} \\ 0 & v \text{ splits} \end{cases}$$

The pullback of κ to each summand is the canonical local character associated to the quadratic extension E_v/F_v. So

$$O_\gamma^\kappa(f) = \prod_v O_{\gamma_v}^{\kappa_v}(f_v).$$

This means that it suffices to solve a local problem, of extending f_v smoothly.

2.4. **Local harmonic analysis.** We want to know: if F is a local field, E/F a quadratic extension corresponding to an anisotropic torus $T \subset G$, the map

$$T(F)_{\mathrm{reg}} \to \mathbf{C}$$

sending

$$\gamma \mapsto O_\gamma^\kappa(f) =: f^{T_E}_{\text{naïve}}$$

extends smoothly to $T(F)$. But unfortunately it does *not*. It blows up as $\gamma \to 1$ or $\gamma \to -1$. But it blows up in a controlled way, so we can try to renormalize it and hope that works out.

Supposing $\gamma \leftrightarrow e \in E$, replace $f^{T_E}_{\text{naïve}}$ by

$$f_1^{T_E} := |e - \overline{e}|^{1/2} O_\gamma^\kappa(f).$$

This does not blow up. However, it is still not what we want. Why? This is *odd* under swapping $e \mapsto \overline{e}$ because this realizes stable conjugacy, at least when -1 is not a norm. An odd function can only extend smoothly to $e = 1$ if it vanishes at $e = 1$, but one can easily show that this doesn't happen. So $f_1^{T_E}$ is discontinuous.

To remedy this, we need to put in a *sign*. That is, we replace $f_1^{T_E}$ by

$$\kappa\left(\frac{e - \overline{e}}{\eta}\right) |e - \overline{e}|^{1/2} O_\gamma^\kappa(f)$$

where $\eta \in E$ is any element of trace 0. That guarantees $\frac{e-\overline{e}}{\eta} \in F^\times$, so it makes sense to apply η, and if we switch e and $\overline{e}$ then the sign changes.

We are still not quite done, but we suppress some of the details. Defining

$$f^{T_E}(\gamma) := \lambda(E/F,\psi)\cdot |e-\overline{e}|^{1/2}\left(\frac{e-\overline{e}}{\eta}\right) O_\gamma^\kappa(f),$$

this turns out to extend smoothly. Here ψ is an additive character of ψ, and $\lambda(E/F,\psi)$ is basically an ϵ-factor.

Any character $\theta\colon E^1 \to \mathbf{C}^\times$ gives rise to a pair of representations $\pi_\theta = \{\pi_+(\theta), \pi_-(\theta)\}$ of $\mathrm{SL}_2(F)$. The set π_θ depends only on θ, but labelling $\pi_\pm(\theta)$ depends also on $\psi\colon F \to \mathbf{C}^\times$. If we define

$$\theta(f^{T_E}) := \int_{T(F)} \theta(t) f^{T_E}(t)\, dt$$

then

$$\theta(f^{T_E}) = \pi_+(\theta)(f) - \pi_-(\theta)(f)$$

This is an example of an endoscopic character identity.

Final observation: the difference between what we wanted and what we ended up with is

$$\prod_v \lambda(E_v/F_v,\psi_v)\kappa\left(\frac{e-\overline{e}}{\eta}\right)|e_v-\overline{e_v}|_v^{1/2}$$

but this is $=1$ provided ψ_v is the local component of an additive character $\psi\colon \mathbf{A}_F/F \to \mathbf{C}$, and η_v is the local component of $\eta \in E$ of trace 0, and $e \in E^1$ – in other words, if the local choices we made come from a global choice.

(The λ-constants are $\epsilon(1/2, \mathrm{sgn}_{\Gamma_{E_v/F_v}}, \psi)$, so this is the root number of an orthogonal Galois representation.)

One final technicality: to get global smoothness, we also need the fundamental lemma.

3. Endoscopy for quasi-split groups

3.1. **Overview.** For SL_2, we wrote

$$\mathrm{TF}_{\text{reg.ell.}}(f) = \frac{1}{2}\mathrm{STF}_{\text{reg.ell.}}(f) + \frac{1}{4}\sum_{E/F \text{ quad}} \mathrm{STF}^{T_E}_{G-\mathrm{reg}}(f^{T_E}).$$

We are going to try to do something similar in general, but there are several complications.

(1) Stable conjugacy is not the same as conjugacy for a bigger group $\widetilde{G}$ in general.
(2) The unstable part doesn't just come from tori, but come from a general class of groups called *endoscopic groups*.
(3) An endoscopic group H is not necessarily a subgroup of G. So how we can we relate $\gamma \in H$ to $\delta \in G$?
(4) The transferred function f^H is not given (or suggested) to us, since we are only given its orbital integrals (which do not determine the function once H is non-abelian).

The solutions to these problems will go as follows:

(1) Galois cohomology.
(2) Define endoscopic groups.
(3) Admissible isomorphisms.
(4) Transfer factors.

3.2. **Stable conjugacy.**

Definition 3.1. We say $\delta, \delta' \in G(F)$ are *stably conjugate* if

(1) There exists $g \in G(\overline{F})$ such that $\mathrm{Ad}(g)\delta = \delta'$,
(2) The element $g^{-1}\sigma(g) \in G_\delta$ lies in $G^\circ_{\delta_{\mathrm{ss}}}$ for all $\sigma \in \Gamma := \mathrm{Gal}(\overline{F}/F)$. In other words, $g^{-1}\sigma(g) \in G_\delta \cap G^\circ_{\delta_{\mathrm{ss}}} =: G^*_\delta$.

Remark 3.2. If G_{der} is simply-connected, then a theorem of Steinberg implies that $G_{\delta_{\mathrm{ss}}}$ is connected, so stable conjugacy is equivalent to $\overline{F}$-conjugacy.

Definition 3.3. A semisimple $\delta \in G$ is *strongly regular* if G_δ is a torus. (For regular elements, we only ask that G°_δ is a torus.) So for strongly regular elements, stable conjugacy is equivalent to $\overline{F}$-conjugacy.

Example 3.4. The element $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in \mathrm{PGL}_2$ is regular but not strongly regular.

Pick $g \in G(\overline{F})$ such that $\mathrm{Ad}(g)\delta = \delta'$. Then $\mathrm{Ad}(g)\colon T_\delta \xrightarrow{\sim} T_{\delta'}$.

Fact 3.5. If δ is strongly regular, then we have the following facts.

(1) The isomorphism $\mathrm{Ad}(g)\colon T_\delta \xrightarrow{\sim} T_{\delta'}$ depends only on δ, δ' (not on the choice of g). Hence it is defined over F (as it must be stable under σ) We denote it by $\varphi_{\delta,\delta'}$.
(2) The map $\sigma \mapsto g^{-1}\sigma(g)$ is an element $Z^1(\Gamma, T_\delta)$ and its class $\mathrm{inv}(\delta, \delta') \in H^1(\Gamma, T_\delta)$ is independent of g.
(3) The map $\delta' \mapsto \mathrm{inv}(\delta, \delta')$ induces a bijection between the rational classes in the stable conjugacy class of δ and $\ker(H^1(\Gamma, T_\delta) \to H^1(\Gamma, G))$.

Example 3.6. This replaces the phenomenon for SL_2 that rational classes in the stable class are orbits of $\mathrm{SL}_2 \subset \mathrm{GL}_2$. There we had

$$1 \to T_\delta \to \widetilde{T}_\delta \xrightarrow{\det} \mathbf{G}_m \to 1$$

which induces

$$1 \to E^1 \to E^\times \xrightarrow{N_{E/F}} F^\times \to H^1(\Gamma, T_\delta) \to H^1(\Gamma, \widetilde{T}_\delta).$$

Exercise 3.7. Show that $H^1(\Gamma, \widetilde{T}_\delta) = 0$. [Hint: use that tori of GL_n are induced.]

This identifies $\ker(H^1(\Gamma, T_\delta) \to H^1(\Gamma, G)) \cong F^\times / N_{E/F}(E^\times)$.

Definition 3.8. Let G, H be algebraic groups. An isomorphism $\xi\colon G \to H$ over $\overline{F}$ is called an *inner twist* if

$$\xi^{-1} \circ \sigma_H \circ \xi \circ \sigma_G^{-1} \in \mathrm{Inn}(G)$$

for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$.

Fact 3.9. We have the following facts.

(1) An isomorphism $\mathrm{Ad}(g)\colon G^*_\delta \to G^*_{\delta'}$ is an inner twist. It depends on g only up to an element of $(G_\delta/G^*_\delta)(F)$. (So if G_{der} is simply connected, so that $G_\delta = G^*_\delta$, then it independent of (g).)
(2) $g^{-1}\sigma(g) \in Z^1(G, G^*_\delta)$ and its image in $H^1(G, G_\delta)$ is independent of g.
(3) Rational classes in the stable class of δ are in bijection with the image of $\ker(H^1(\Gamma, G^*_\delta) \to H^1(\Gamma, G))$ in $\ker(H^1(\Gamma, G_\delta) \to H^1(\Gamma, G))$.

3.3. **Langlands dual group.** Let G be quasi-split. Fix a Borel pair (T_0, B_0). For $X = X^*(T_0) \supset R \supset \Delta$ and $Y = X_*(T_0) \supset R^\vee \supset \Delta^\vee$, we get a *based root datum*

$$(X, \Delta, Y, \Delta^\vee).$$

Then $(Y, \Delta^\vee, X, \Delta)$ is another based root datum, which gives a split reductive group $(\widehat{G}, \widehat{B}_0, \widehat{T}_0)$ that we take over $\mathbf{C}$ (but it is popular nowadays to take it over $\overline{\mathbf{Q}}_\ell$). If we extend $(\widehat{T}_0, \widehat{B}_0)$ to a pinning $(\widehat{T}_0, \widehat{B}_0, \{\widehat{X}_\alpha\}_{\alpha\in\Delta^\vee})$ with $\widehat{X}_\alpha \in \widehat{\mathfrak{g}}_\alpha - 0$ then we get an action of Γ on $\widehat{G}$, and we define ${}^LG := \widehat{G} \rtimes \Gamma$.

Let $T \subset G$ be a maximal torus. Then we get a canonical Γ-invariant $\widehat{G}$-conjugacy class of embeddings $\widehat{T} \hookrightarrow \widehat{G}$. In particular, $\sigma \in \mathrm{Gal}(\overline{F}/F)$ sends $\eta\colon \widehat{T} \hookrightarrow \widehat{G}$ is sent to $\sigma(\eta) = \widehat{\sigma}_G \circ \eta \circ \widehat{\sigma}_T^{-1}$. (A particular embedding won't be Galois-invariant, but the $\widehat{G}$-conjugacy class is.)

This is how we construct η. Choose $g \in G(\overline{F})$ giving $\mathrm{Ad}(g)\colon T_0 \xrightarrow{\sim} T$. Compose $\widehat{\mathrm{Ad}(g)}\colon \widehat{T} \to \widehat{T}_0$ (the map of dual tori) with the canonical identification of $\widehat{T}_0$ (viewed as the dual group of T_0) with $\widehat{T}_0$ (viewed as a subgoup of $\widehat{G}_0$).

To see the canonicity: changing g amounts to translating by a Weyl element. This is the same as applying an element of the Weyl group to $\widehat{T}_0$. The Weyl group of a root system is canonically identified with the Weyl group of its dual (they are generated by the same simple reflections).

3.4. **Endoscopic data.** Let F be a local or global field.

Definition 3.1. An *endoscopic triple* (H, s, η) consists of

(1) A quasi-split connected reductive group H,
(2) an embedding $\eta\colon \widehat{H} \hookrightarrow \widehat{G}$,
(3) an element $s \in [Z(\widehat{H})/Z(\widehat{G})]^\Gamma$,

subject to the conditions:

(i) $\eta\colon \widehat{H} \xrightarrow{\sim} (\widehat{G}^\Gamma_{\eta(s)})^\circ$,
(ii) The $\widehat{G}$-conjugacy class of η is Γ-stable,
(iii) (a) If F is local, then s lifts to $Z(\widehat{H})^\Gamma$.
(b) If F is global, then s lifts to $Z(\widehat{H})^{\Gamma_v}$ for all v.

There is a notion of isomorphism for endoscopic data, but I will skip the definition.

Example 3.2. Levi subgroups are endoscopic subgroups.

Definition 3.3. The datum (H, s, η) is *elliptic* if $Z(\widehat{H})^{\Gamma,\circ} = Z(\widehat{G})^{\Gamma,\circ}$.

Example 3.4. Let $G = \mathrm{SL}_2$, elliptic endoscopic data have the following form: $H = T$ is an anisotropic 1-dimensional torus, and $s = -1 \in (\widehat{T} = \mathbf{C}^{\times}_{(-1)})^{\Gamma}$ (Galois acts by inversion, since it must be non-trivial action; $s = 1$ would violate condition (1).

In general H is not naturally a subgroup of G, and there may be no way to put H into G at all. The best we can do is to put a maximal torus of H into G, which is the theory of admissible embeddings.

3.5. **Admissible embeddings.** We're going to define "admissible embeddings" that generalize $\varphi_{\delta,\delta'} \colon T_\delta \to T_{\delta'}$ from Fact 3.5 to the case when one elements in in H and one is in G.

Definition 3.1. Let $T^H \subset H$ and $T^G \subset G$ be maximal tori. An *admissible isomorphism* $T^H \to T^G$ is one whose dual is given as the composition of:

(1) $\widehat{T}^H \hookrightarrow \widehat{H}$ from the construction of §3.3,
(2) $\eta \colon \widehat{H} \hookrightarrow \widehat{G}$,
(3) The inverse of $\widehat{T}^G \hookrightarrow \widehat{G}$ (on the image of η) by the construction of §3.3.

Definition 3.2. Strongly regular semisimple elements $\gamma \in H, \delta \in G$ are *related* if there exists an admissible isomorphism

$$\varphi \colon T^H \to T^G$$

such that $\varphi(\gamma) = \delta$.

Remark 3.3. In the notation above, φ is uniquely determined by γ and δ, and will be called $\varphi_{\gamma,\delta}$.

Remark 3.4. If $\gamma \in H(F)$ and $\delta \in G(F)$ then $\varphi_{\gamma,\delta}$ is defined over F, by the canonicity.

Definition 3.5. Let $T^H \subset H$. An *admissible embedding* $T^H \hookrightarrow G$ is the composition of an admissible isomorphism $T^H \xrightarrow{\sim} T^G$ with the inclusion $T^G \hookrightarrow G$.

Theorem 3.6 (Steinberg)**.** *For every maximal torus $T^H \subset H$, there exists an admissible embedding $T^H \hookrightarrow G$ defined over F.*

Remark 3.7. This is the first crucial use of the hypothesis that G is quasi-split. It is simply not true without that hypothesis.

3.6. **Transfer factors.** Recall that for $G = \mathrm{SL}_2$, we had

$$f^T_{\text{naïve}}(\gamma) = O^{\kappa}_{\gamma}(f)$$

and we had to modified this to

$$f^T(\gamma) = \lambda \cdot \kappa \cdot |\ldots|^{1/2} f^T_{\text{naïve}}(\gamma).$$

The extra stuff $\lambda \cdot \kappa \cdot |\ldots|^{1/2}$ are the transfer factors. But now we don't even have $f^T_{\text{naïve}}(\gamma)$ because the endoscopic group can be non-abelian, and we've only specified its orbital integrals.

Definition 3.1. A *Whittaker datum* of G is a $G(F)$-conjugacy class of (B, ψ) where $B \subset G$ is a Borel subgroup over F, and $\psi \colon B_U(F) \to \mathbf{C}^{\times}$ is a generic character, meaning its restriction to any simple root subgroup is non-trivial.

We need to improve our notion of endoscopic datum.

Definition 3.2. An *extended endoscopic datum* is a triple $(H, s, {}^L\eta)$ where (H, s, η) is an endoscopic datum and

$$ {}^L\eta \colon {}^LH \to {}^LG $$

extends η.

Given $\gamma \in H(F)_{\text{str.reg.}}$ and $\delta \in G(F)_{\text{str.reg}}$ which are related, there is $\Delta(\gamma, \delta) \in \mathbf{C}$ expressing the relation. We don't have time to give the details, so we'll just explain one crucial property.

Proposition 3.3. *If δ, δ' are stably conjugate, then*

$$ \Delta(r, \delta') = \Delta(r, \delta)\langle \underbrace{\mathrm{inv}(\delta, \delta')}_{H^1(\Gamma, T_\delta)}, \widehat{\varphi}_{\delta,\delta'}(s)\rangle \tag{3.1} $$

where $s \in (Z(\widehat{H})/Z(\widehat{G}))^\Gamma \hookrightarrow (\widehat{T}_\gamma/Z(\widehat{G}))^\Gamma \xrightarrow{\varphi_{\gamma,\delta}} (\widehat{T}_\delta/Z(\widehat{G}))^\Gamma$.

The pairing in (3.1) coming Tate-Nakayama duality. A priori this is a pairing between $H^1(\Gamma, T_\delta)$ and $H^1(\Gamma, X^*(T_\delta)) \cong (T_\delta)^\Gamma$. Because we've modded out by $Z(\widehat{G})$, we need to choose a lift of $\mathrm{inv}(\delta, \delta')$ to make this pairing defined. Does the pairing then depend on the lift? No, because we assume s lifts.

Definition 3.4. Test functions $f \in C_c^\infty(G), f^H \in C_c^\infty(H)$ are called *matching* if

$$ \sum_\delta \Delta(\gamma, \delta) O_\delta(f) = SO_\gamma(f^H) $$

for any $\gamma \in H(F)_{\text{rs}}$.

Example 3.5. For SL_2, there are two terms with opposite signs, hence why the κ-orbital integral was the difference was orbital integrals.

Theorem 3.6 (Fundamental Lemma, Langlands-Shelstad-Kottwitz-Waldspurger-Ngô)**.**

(1) For any f there is a matching f^H.
(2) If $f = \mathbf{I}_{G(\mathcal{O}_F)}$, then f^H can be taken to be $\mathbf{I}_{H(\mathcal{O}_F)}$.

Conjecture 3.7. *Let $\varphi^H \colon \mathrm{WD}_F \to {}^LH$ be tempered. Let $\varphi = {}^L\eta \circ \varphi^H$. Then we have*

$$ \sum_{\pi \in \Pi_\varphi} \langle \pi, s\rangle \Theta_\pi(f) = \sum_{\pi^H \in \Pi_{\varphi^H}} \langle \pi^H, 1\rangle \Theta_{\pi^H}(f^H). $$

4. Endoscopy for general groups

4.1. **Recap of the quasi-split case.** Let's review what we've discussed so far.

(1) The issue of *stable conjugacy.* For a strongly regular $\delta \in G(F)$, with centralizer T, rational classes in the stable class of δ are in bijection with $\ker(H^1(\Gamma, T) \to H^1(\Gamma, G))$. An element δ' maps to a cohomology class $\mathrm{inv}(\delta, \delta') \in H^1(\Gamma, T)$.

(2) The notion of *endoscopic triples* (H, s, η). In particular, $s \in (Z(\widehat{H})/Z(\widehat{G}))^\Gamma$ and we assumed that it was liftable to $Z(\widehat{G})^\Gamma$, although we do not specify a lift.

(3) The notion of *admissible embeddings* of maximal tori $T^H \subset H$ into G. This gave a notion of *related elements*: strongly regular semisimple elements $\gamma \in H$ and $\delta \in G$ which are related induce a well-defined $\varphi_{\gamma,\delta} \colon T_\gamma \xrightarrow{\sim} T_\delta$.

(4) *For quasi-split G*, we introduced the *transfer factor*. Given an *extended endoscopic triple* $(H, s, {}^L\eta)$, a transfer factor is a function

$$\Delta[\mathfrak{w}] \colon H(F)_{\text{str.reg.}} \times G(F)_{\text{str.reg.}} \to \mathbf{C}.$$

(5) The notion of *matching functions*, which was defined by matching orbital integrals: f^H on H and f on G match if

$$SO_\gamma(f^H) = \sum_{\delta \in G(F)/\sim_{\text{conj}}} \Delta[\mathfrak{w}](\gamma, \delta) O_\delta(f).$$

(6) The *spectral side* of endoscopy, expressed by the following conjecture.

Conjecture 4.1. *Let $\varphi \colon L_F \to {}^L G$ be a tempered parameter. (L_F is the local Langlands group, which is W_F if F is archimedean and $W_F \times \mathrm{SL}_2(\mathbf{C})$ if F is non-archimedean.)*

(a) There exists an L-packet $\Pi_\varphi(G)$, i.e. a finite set of tempered representations corresponding to φ.

(b) There exists a map

$$\Pi_\varphi(G) \to \mathrm{Irr}(\mathcal{S}_\varphi), \tag{4.1}$$

where $S_\varphi := Z_{{}^L G}(\varphi)$, $\overline{S_\varphi} := S_\varphi / Z(\widehat{G})^\Gamma$, and

$$\mathcal{S}_\varphi := \pi_0(\overline{S_\varphi}),$$

and (4.1) *injective when F is archimedean and bijective when F is non-archimedean.*

(c) (Shahidi's conjecture) There exists a unique $\mathfrak{w}$-generic *representation in each L-packet, say $\pi_{\mathfrak{w}}$, which corresponds to the trivial representation of $\mathcal{S}_\varphi$.*

(d) Assume that φ came from an endoscopic datum, i.e. factors though

$$\begin{array}{ccc} {}^L H & \xrightarrow{{}^L\eta} & {}^L G \\ \varphi^H \uparrow & \nearrow_{\varphi} & \\ L_F & & \end{array}$$

then $s \in S_\varphi$ lies in S_{φ^H}, and if $f \leftrightarrow f^H$ then we should have the following character identity:

$$\sum_{\pi \in \Pi_\varphi(G)} \langle \pi, s \rangle \Theta_\pi(f) = \sum_{\pi^H \in \Pi_{\varphi^H}(H)} \langle \pi^H, s \sim 1 \rangle \Theta_{\pi^H}(f^H)$$

where $\langle \pi, s \rangle = \mathrm{Tr}\, \rho(s)$, and $\pi \leftrightarrow \rho$ under (4.1).

Remark 4.2. The pairing $\langle \pi, s\rangle$ depends on the Whittaker datum, because the parametrization does. Since s is central in ${}^L H$ (hence 1 in $\mathcal{S}_\varphi$), and changing the Whittaker datum has the effect of tensoring by a character, the RHS is independent of the Whittaker datum.

4.2. **Notions of inner forms.** We now want to move to describe what happens when G is not quasi-split. What are the problems? (The immediate problem is that you can't define a transfer factor, because it involved a choice of Whittaker datum to normalize it.)

Recall that G and H are *inner forms* if there exists an isomorphism $\xi \colon G_{\overline{F}} \to H_{\overline{F}}$ such that

$$\xi^{-1}\sigma(\xi) \in \mathrm{Inn}(G).$$

Inner forms are an equivalence relation, and the equivalence classes are called inner classes. A fundamental idea of Adams-Barbasch-Vogan is that you should treat the inner classes together.

The thing to notice is that for archimedean F, you only have an injection from the L-packet to $\mathrm{Irr}(\mathcal{S}_\varphi)$, rather than a bijection. (In the non-archimedean case you do get a bijection, but then there are other issues.) Considering all inner forms corrects this.

Example 4.1. Consider the unitary groups $U(p, q)$ over $\mathbf{R}$. All such groups, for $p + q = n$, constitute an inner class. Let φ be a discrete parameter. Then

$$|\Pi_\varphi(U(p, q))| = \binom{n}{p} = \binom{n}{q}.$$

On the other hand, $|\mathcal{S}_\varphi| = 2^{n-1}$ (and $\mathcal{S}_\varphi$ is abelian, so this is the same as $|\mathrm{Irr}(\mathcal{S}_\varphi)| = 2^{n-1}$). Note that

$$\sum_{p+q=n} \binom{n}{p} = 2^n = |S_\varphi|.$$

We have to say when we think of two inner forms of being the same, and when we think of them as being different. In other words, we have to define a notion of isomorphism for inner forms. This will have to satisfy the following condition: the set of automorphisms of an inner form G' has to preserve conjugacy classes and representations of $G'(F)$.

4.2.1. *Attempt 1.* Let's build up. The simplest thing to do would be to say that inner forms G', G'' of G are isomorphic if and only if $G' \cong G''$ as reductive groups over F.

Is this a good notion?

Exercise 4.2.

(1) Given an inner twist $\xi \colon G \to G'$, we can form $\xi^{-1}\sigma(\xi) \in Z^1(\Gamma, G_{\mathrm{ad}})$. This cocycle depends on ξ, but its image in $H^1(\Gamma, \mathrm{Aut}(G))$ does not depend on ξ – it only depends on G'. Hence we get a bijection between isomorphism classes of G', in the above sense, with the image of $H^1(\Gamma, G_{\mathrm{ad}}) \to H^1(\Gamma, \mathrm{Aut}(G))$.

(2) Under this definition, the group of automorphisms of an inner twist G' are just $\mathrm{Aut}(G')(F)$. This is bad, because e.g. transpose inverse on GL_n does not preserve conjugacy classes or representations (it sends a representation to its contragredient), but is an automorphism under our definition.

4.2.2. *Attempt 2.* Suppose we next try to refine the notion of isomorphism to be sensitive to the structure of the inner twist. Say an *isomorphism* between $\xi_1 \colon G \to G_1$ and $\xi_2 \colon G \to G_2$ is $f \colon G_1 \to G_2$ over F such that $\xi_2^{-1} \circ f \circ \xi_1$ is an inner automorphism.

Exercise 4.3.

(1) The map $(\xi, G') \mapsto \xi^{-1}\sigma(\xi)$ induces a bijection between isomorphism classes of inner twists and $H^1(\Gamma, G_{\mathrm{ad}})$.
(2) $\mathrm{Aut}(\xi, G') = G'_{\mathrm{ad}}(F) = H^0(\Gamma, G'_{\mathrm{ad}})$.

This is better because it disallows outer automorphisms. However, there is a subtlety: there is a difference between the notion of "outer automorphism" for the algebraic group versus the topological group of F-points. So we disallowed the first kind, but not the second kind.

Example 4.4. Consider $\mathrm{PGL}_2(\mathbf{R})$ acting on $\mathrm{SL}_2(\mathbf{R})$. Then conjugation by $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ swaps the holomorphic and anti-holomorphic discrete series.

4.2.3. *Attempt 3.* So we're still not happy. To correct this, Vogan suggested the notion of *pure inner twist*: this is a triple (G', ξ, z) where $\xi \colon G \to G'$ is an inner twist and $z \in Z^1(\Gamma, G)$ such that $\xi^{-1}\sigma(\xi) = \mathrm{Ad}(z)$.

An isomorphism between pure inner twists is a pair $(f, g) \colon (G_1, \xi_1, z_1) \to (G_2, \xi_2, z_2)$ where $f \colon G_1 \xrightarrow{\sim} G_2$ is an isomorphism over F, and $g \in G(\overline{F})$ is such that $\mathrm{Ad}(g) = \xi_2^{-1} \circ f \circ \xi_1$ and $z_2(\sigma) = g z_1(\sigma) \sigma(g)^{-1}$.

Exercise 4.5.

(1) $(G', \xi, z) \mapsto [z]$ is a bijection between isomorphism classes of pure inner twists and $H^1(\Gamma, G)$.
(2) $\mathrm{Aut}(G, \xi, z) = G'(F) = H^0(\Gamma, G)$. This obviously preserves conjugacy classes and representations.

But there is a serious problem here: usually the map $H^1(\Gamma, G) \to H^1(\Gamma, G_{\mathrm{ad}})$ is not surjecive, so not all inner forms can be extended to a pure inner form. We'll ignore this for a while, and first focus on pure inner forms.

4.3. **Endoscopy for pure inner forms.** We first make some definitions in the spirit of treating all the inner forms as if they were part of the same group.

Definition 4.1.

(1) *An element* of a pure inner twist is a quadruple (G', ξ, z, δ') where (G', ξ, z) is a pure inner twist and $\delta' \in G'(F)$.
(2) We say $(G_1, \xi_1, z_1, \delta_1)$ is *rationally conjugate* to $(G_2, \xi_2, z_2, \delta_2)$ if there exists $(f, g) \colon (G_1, \xi_1, z_1) \xrightarrow{\sim} (G_2, \xi_2, z_2)$ such that $f(\delta_1) = \delta_2$.

(3) A *representation* of a pure inner twist is a datum (G', ξ, z, π) where (G', ξ, z) is a pure inner twist and π is a representation of $G'(F)$.
(4) Two representations (G_1, ξ_1, z_1, π_1) and (G_2, ξ_2, z_2, π_2) are *equivalent* if there exists $(f, g) \colon (G_1, \xi_1, z_1, \pi_1) \xrightarrow{\sim} (G_2, \xi_2, z_2, \pi_2)$ such that $\pi_2 = \pi_1 \circ f$.
(5) $(G_i, \xi_i, z_i, \delta_i)_{i=1,2}$ are *stably conjugate* if $\xi_1^{-1}(\delta_1)$ and $\xi_2^{-1}(\delta_2)$ are conjugate in $G(\overline{F})$.

Exercise 4.2. Show that the set of rational classes in the stable class of $\delta \in G(F)_{\text{str.reg}}$ is in bijection with $H^1(\Gamma, T)$. (Note that we don't ask for this to be in the kernel of the map to $H^1(\Gamma, G)$. The image of $H^1(\Gamma, T) \to H^1(\Gamma, G)$ tells you which pure inner form the class lives in.)

We can now define a transfer factor. The theory of reductive groups says that in every inner class, there is a unique quasi-split form. So we fix G_0 to be a quasi-split group. Let $(H, s, {}^L\eta)$ be an endoscopic triple for G_0. Let $\mathfrak{w}$ be a Whittaker datum for G_0. We will get a transfer factor $\Delta[\mathfrak{w}] \colon H(F)_{\text{str.reg.}} \times G(F)_{\text{str.reg.}} \to \mathbf{C}$ for any pure inner twist (G, ξ, z).

Proposition 4.3. *There exists $\delta_0 \in G_0(F)_{str.reg.}$ stably conjugate to (G, ξ, z, δ) for any $\delta \in G(F)_{str.reg.}$.*

Remark 4.4. This is a consequence of a theorem of Steinberg.

Definition 4.5. We set $\Delta[\mathfrak{w}](\gamma, \delta) = \Delta[\mathfrak{w}](\gamma, \delta_0) \cdot \langle \operatorname{inv}(\delta_0, \delta), \widehat{\varphi}_{\gamma,\delta_0}^{-1}(s) \rangle$.

Note that the definition of the invariant moved across inner forms.

There is an issue: $\widehat{\varphi}_{\gamma,\delta_0}^{-1}(s) \in \pi_0((\widehat{T}/Z(\widehat{G}))^\Gamma)$. On the other hand, $\operatorname{inv}(\delta_0, \delta) \in H^1(\Gamma, T)$. In the quasi-split case, we argued that the invariant lifted across $H^1(\Gamma, T_{\text{sc}}) \to H^1(\Gamma, T)$. This made the pairing well-defined. But now the invariant cannot be lifted, so we can only pair with an element in $\pi_0(\widehat{T}^\Gamma)$. So we need to refine the concept of endoscopic data.

Definition 4.6. A *pure refined endoscopic triple* is $(H, \dot{S}, \eta)$ where (H, s, η) is as before and $\dot{s} \in Z(\widehat{H})^\Gamma$ lifts s.

An isomorphism of such triples must preserve the lifts. (So there is a finer notion of isomorphism, which is related to the refined notion of isomorphism for inner twists.)

Conjecture 4.7. *Let $\varphi \colon L_F \to {}^L G$ be a tempered parameter.*

(1) There exists an L-packet $\Pi_\varphi(G)$ of representations of pure inner forms, up to equivalence.
(2) There exists a bijection $\pi_\varphi(G) \leftrightarrow \operatorname{Irr}(\pi_0(S_\varphi))$, where $S_\varphi = Z_{{}^L G}(\varphi)$. The base-point map $\Pi_\varphi \to H^1(\Gamma, G)$ giving the class of the pure inner twist matches the central character: $\operatorname{Irr}(\pi_0(S_\varphi)) \to \pi_0(Z(\widehat{G})^\Gamma)^*$ under the Kottwitz map.

In other words, the following diagram commutes:

$$\begin{array}{ccc} \Pi_\varphi & \longrightarrow & \operatorname{Irr}(\pi_0(S_\varphi)) \\ \downarrow & & \downarrow \\ H^1(\Gamma, G_0) & \xrightarrow{\textit{Kottwitz}} & \pi_0(Z(\widehat{G_0})^\Gamma)^* \end{array}$$

(3) Assume that φ came from an endoscopic datum, i.e. factors though

$$\begin{array}{ccc} {}^L L & \xrightarrow{{}^L\eta} & {}^L G \\ \varphi^H \uparrow & \nearrow_{\varphi} & \\ L_F & & \end{array}$$

then $s \in S_\varphi$ lies in S_φ, and if $f \leftrightarrow f^H$ then we should have the following character identity:

$$\sum_{\pi \in \Pi_\varphi(G)} \langle \pi, s \rangle \Theta_\pi(f) = \sum_{\pi^H \in \Pi_{\varphi^H}(H)} \langle \pi^H, s \sim 1 \rangle \Theta_{\pi^H}(f^H)$$

where $f \leftrightarrow f^H$ under the above definition, and $\langle \pi, s \rangle = \operatorname{Tr} \rho(\dot{s})$ where $\rho \in \operatorname{Irr}(\pi_0(S_\varphi))$.

4.4. **Extending the notion of pure inner forms.** Finally, let me briefly describe what you do with inner forms that cannot be purified (the issue described at the end of §4.2.3). The solution is to replace $H^1(\Gamma, G)$ by a different kind of cohomology $H^1(?, G)$. This will have the properties:

(1) $H^1(\Gamma, G) \hookrightarrow H^1(?, G) \twoheadrightarrow H^1(\Gamma, G_{\mathrm{ad}})$.
(2) There is a Tate-Nakayama duality on $H^1(?, G)$ that allows us to form the pairing in the transfer factor.
(3) This should work uniformly for all local fields.
(4) There should be a global analog, with localization maps.

5. Stabilization of the elliptic regular term for general groups

5.1. **First steps.** Now we will discuss stabilization of the strongly regular elliptic part of the trace formula in general.

What is the strongly regular elliptic part? It will is the sum

$$\sum_{\delta \in G(F)_{\mathrm{reg.ell.}}/\sim_{\mathrm{conj}}} \mathrm{vol}(T_\gamma(F) \backslash T_\gamma(\mathbf{A})) O_\delta(f)$$

where

- $T_\gamma = Z_G(\gamma)$,
- G is a connected reductive group over F,
- F is a number field.

Thinking to the case $G = \mathrm{SL}_2$ which we discussed earlier, the first thing we did was group terms into stable conjugacy classes. If $\delta, \delta' \in G(F)_{\mathrm{str.reg}}$ are stably conjugate then we have an isomorphism $\varphi_{\delta,\delta'} \colon T_\delta \xrightarrow{\sim} T_{\delta'}$ over F. Hence T_δ and $T_{\delta'}$ have the same volume, and we can group them together, writing the regular elliptic part as

$$\sum_{\delta \in G(F)_{\mathrm{reg.ell.}}/\sim_{\mathrm{st}}} \mathrm{vol}(T_\gamma(F) \backslash T_\gamma(\mathbf{A})) \sum_{\delta' \sim_{\mathrm{st}} \delta} O_{\delta'}(f). \tag{5.1}$$

However, the inner sum is not a stable distribution, because stability is in terms of adelic conjugacy classes instead of rational ones.

Since we're dealing with adeles, we introduce some notation. Letting $\Gamma := \mathrm{Gal}(\overline{F}/F)$, we define

$$H^i(F,T) = H^i(\Gamma, T(\overline{F})),$$
$$H^i(\mathbf{A},T) = H^i(\Gamma, T(\overline{\mathbf{A}})),$$
$$H^i(\mathbf{A}/F,T) = H^i(\Gamma, T(\overline{\mathbf{A}})/T(\overline{F})).$$

We know that $\ker(H^1(\Gamma,T) \to H^1(\Gamma,G))$ parametrizes rational conjugacy classes inside the stable one. The *adelic* rational conjugacy classes inside the stable one are parametrized by $\ker(H^1(\mathbf{A},T) \to H^1(\mathbf{A},G))$.

For $G = \mathrm{SL}_2$, two special things happened:

- $H^1(F,G) = H^1(\mathbf{A},G) = 0$.
- The map $H^1(F,T) \to H^1(\mathbf{A},T)$ was injective.

What happens when two elements in $H^1(F,T)$ become the same in $H^1(\mathbf{A},T)$? The orbital integrals will be equal, so we can factor that out. Call the map α

$$\begin{array}{c} \ker(H^1(\Gamma,T) \to H^1(\Gamma,G)) \\ \big\downarrow \alpha \\ \ker(H^1(\mathbf{A},T) \to H^1(\mathbf{A},G)) \end{array}$$

Hence we can rewrite (5.1) as

$$= \sum_{\gamma} (\dots) \ker(\alpha) \sum_{a \in \mathrm{Im}\,(\alpha)} \mathcal{O}_{a\delta}(f)$$

where $a\delta$ is the stable conjugacy class δ' such that $\mathrm{inv}(\delta,\delta') = a$.

5.2. **Pre-stabilization, when G satisfies the Hasse principle.** For SL_2, the difference between rational stable conjugacy and adelic stable conjugacy was measured by the short exact sequence

$$1 \to H^1(F,T) \xrightarrow{\alpha} H^1(\mathbf{A},T) \to \underbrace{H^1(\mathbf{A}/F,T)}_{\mathbf{Z}/2\mathbf{Z}} \to 1 \tag{5.2}$$

What do we do in general? We need a pre-stabilization step. This is easier when G satisfies the Hasse principle, so we'll discuss that first.

For any reductive group H let's define $\ker^1(F,H) := \ker(H^1(F,H) \to H^1(\mathbf{A},H))$, so $\alpha\colon \ker^1(F,T) \to k^1(F,G)$.

Definition 5.1. We say that G *satisfies the Hasse principle* if $\ker^1(F,G) = 0$.

Remark 5.2. Many groups satisfy the Hasse principle, e.g. simply connected groups, adjoint groups, and classical groups. It is easy to find reductive groups, and even tori, that fail to satisfy the Hasse principle. It is harder to find semisimple groups that fail to satisfy the Hasse principle, but they do exist, even in type A.

In this case, we can do something similar to what we did for SL_2. And what was that? We partitioned $H^1(\mathbf{A},T)$ into two cosets $B_0 \cup B_1$. We summed over both and then subtracted the part from B_1. This generalizes to taking the Fourier transform.

Exercise 5.3. Let $1 \to A \to B \to C \to 1$ be a short exact sequence of abelian groups, with C finite, and let $y\colon B \to \mathbf{C}$ be finitely supported. Then

$$\sum_{a\in A} y(a) = |C|^{-1} \sum_{\kappa\in C^*} \sum_{b\in B} \kappa(b)y(b).$$

What is C in our case? It needs to satisfy the

Condition: an element $a \in \ker(H^1(\mathbf{A},T) \to H^1(\mathbf{A},G))$ maps to $0 \in C$ if and only if it lifts to $\ker(H^1(F,T) \to H^1(F,G))$.

Exercise 5.4. If G satisfies the Hasse principle, then we can take $C = H^1(\mathbf{A}/F,T)$. In particular, $H^1(\mathbf{A}/F,T)$ is finite.

Using Exercise 5.3, we rewrite (5.1) as

$$\sum_{\delta_0\in G(F)_{\text{reg.ell.}}/\sim_{\text{st}}} \operatorname{vol}(T(F)\backslash T(\mathbf{A}))\ker(\alpha)|H^1(\mathbf{A}/F,T)|^{-1} \cdot \sum_{\kappa\in H^1(\mathbf{A}/F,T)^*} \underbrace{\sum_{a\in\ker(H^1(\mathbf{A},T)\to H^1(\mathbf{A},G))} \kappa(a)O_{a\delta_0}(f)}_{O^{\kappa}_{\delta_0}(f)}. \tag{5.3}$$

For $\kappa = 1$, $O^{\kappa}_{\delta_0} = \mathrm{SO}_{\delta_0}$ is stable. The other κ-orbital integrals will be related to stable orbital integrals on endoscopic groups. But we will first explain what to do when G doesn't satisfy the Hasse principle.

5.3. Pre-stabilization, when G doesn't satisfy the Hasse principle.

When G doesn't satisfy the Hasse principle, we have to modify $H^1(\mathbf{A}/F,T)$. The first observation is that a related group does satisfy the Hasse principle.

Theorem 5.1 (Harder – Kneser – Chernousov)**.** *The simply-connected cover of the derived group of G, denoted G_{sc}, satisfies the Hasse principle.*

We also need:

Fact 5.2. Two strongly regular elements $\delta,\delta' \in G(\mathbf{A})$ are $G(\overline{\mathbf{A}})$-conjugate if and only if they are conjugate under $G_{\mathrm{sc}}(\overline{\mathbf{A}})$.

Remark 5.3. This is obvious for field-valued points, but not for adelic points, because the natural map

$$G_{\mathrm{sc}}(\overline{\mathbf{A}})Z(\overline{\mathbf{A}}) \to G(\overline{\mathbf{A}})$$

is not surjective. But, a Lemma of Kottwitz says that for almost all places w, we have $G_{\mathrm{sc}}(\mathcal{O}_{E,w})T(\mathcal{O}_{E,w}) = G(\mathcal{O}_{E,w})$ for some finite extension E/F.

This gives a refinement of the cohomological invariant $\operatorname{inv}(\delta_0,\delta) \in H^1(\mathbf{A},T)$. Recall that this was represented by $g^{-1}\sigma(g)$ for $g \in G(\overline{A})$ with $g\delta_0 g^{-1} = \delta$. The fact that we find such a g in G_{sc} gives a refined invariant $\operatorname{inv}_{\mathrm{sc}}(\delta_0,\delta) \in H^1(\mathbf{A},T_{\mathrm{sc}})$. This lifts $\operatorname{inv}(\delta_0,\delta)$ under the map $H^1(\mathbf{A},T_{\mathrm{sc}}) \to H^1(\mathbf{A},T)$. This refined invariant will be used to get the appropriate finite group C.

Definition 5.4. Define Δ to be the image of

$$\ker(H^1(\mathbf{A}, T_{\mathrm{sc}}) \to H^1(\mathbf{A}, T)) \cap \ker(H^1(\mathbf{A}, T_{\mathrm{sc}}) \to H^1(\mathbf{A}, G_{\mathrm{sc}}))$$

under the map $H^1(\mathbf{A}, T_{\mathrm{sc}}) \to H^1(\mathbf{A}/F, T_{\mathrm{sc}})$.

Exercise 5.5. The original invariant $\mathrm{inv}(\delta_0, \delta) \in \ker(H^1(\mathbf{A}, T) \to H^1(\mathbf{A}, G))$ lifts to $\ker(H^1(F, T) \to H^1(F, G))$ if and only if the image of $\mathrm{inv}_{\mathrm{sc}}(\delta_0, \delta)$ in $H^1(\mathbf{A}/F, T_{\mathrm{sc}})$ lies in Δ.

Definition 5.6. Let $\kappa(T/F)^D := H^1(\mathbf{A}/F, T_{\mathrm{sc}})/\Delta$. (This is what we should take as C.) Define $\mathrm{obs}(\delta)$ to be the image of $\mathrm{inv}_{\mathrm{sc}}(\delta_0, \delta)$ in $\kappa(T/F)^D$.

Remark 5.7. In hindsight we could have just used the old invariant. There is a natural map $\ker(H^1(\mathbf{A}, T) \to H^1(\mathbf{A}, G)) \to \kappa(T/F)^D$ sending $\mathrm{inv}(\delta_0, \delta) \mapsto \mathrm{obs}(\delta)$.

Applying Exercise 5.3, we rewrite (5.3) as

$$\sum_{\delta_0 \in G(F)_{\mathrm{str.reg.}}/\sim_{\mathrm{st}}} \mathrm{vol}(T(F)\backslash T(\mathbf{A})) \cdot |\ker(\alpha)| \cdot |\kappa(T/F)^D| \cdot \sum_{\kappa \in \kappa(T/F)^D} \sum_{\delta \in G(\mathbf{A})_{\mathrm{str.reg.ell.}}/\mathbf{A}-\mathrm{conj}, \delta \sim_{\mathrm{str}} \delta_0} \kappa(\mathrm{obs}(\delta)) O_\delta(f) \tag{5.4}$$

Another computation shows that $|\ker(\alpha)| \cdot |\kappa(T/F)^D| = \tau(G) \cdot \tau(T)^{-1}$. This then cancels the volume term, which is also $\tau(T)$.

We're almost done with the prestabilization step. When κ is trivial we get a stable orbital integral, and when κ is non-trivial it should be related to an endoscopic group. But before we do this we have to pass from G to its quasisplit form.

5.4. **Stable classes in G_0.** Let G_0 be the quasi-split form of G and $\xi\colon G_0 \to G$ an inner twist.

Given $\delta_0 \in G(F)$ and $\delta \in G(\mathbf{A})$ which are stably conjugate, the class $\mathrm{obs}(\delta)$ vanishes if and only if the $G(\mathbf{A})$-class of δ has an F-point. We need to refine this statement to apply in the case where $\delta_0 \in G_0(F)$ and $\delta \in G(\mathbf{A})$.

But $\mathrm{inv}(\delta_0, \delta) \in H^1(\mathbf{A}, T)$ doesn't make sense a priori. The purpose of §4.3 was to make sense of this. That is needed in the local setting, but one can skirt around these issues in the global case.

Here is the construction. Choose $g \in G_0(\overline{\mathbf{A}})$ such that $\delta = \xi(g\delta g^{-1})$; such a g exists by the definition of stable conjugacy for pure inner twists . Let $\overline{z}_\sigma = \xi^{-1}\sigma(\xi) \in Z^1(\Gamma, G_{\mathrm{ad}}(\overline{F}))$. Choose a lift $u_\sigma \in C^1(\Gamma, G_{0,\mathrm{sc}}(\overline{F}))$ of $\overline{z}_\sigma$. Then $g^{-1}u_\sigma\sigma(g) \in C^1(\Gamma, T_{\mathrm{sc}}(\overline{A}))$. What is the failure of this to be a cochain? It is something central, and also rational, since u_σ was rational. So the image in $C^1(T_{\mathrm{sc}}(\overline{A})/T_{\mathrm{sc}}(\overline{F}))$ is a cocycle, and we define $\mathrm{obs}(\delta)$ to be its class in $\kappa(T/F)^D$. This construction is independent of the choices of g and u_σ.

Exercise 5.1. Show that $\mathrm{obs}(\delta)$ is trivial if and only if the $G(\mathbf{A})$-conjugacy class of δ has an F-point.

We can now rewrite (5.4) as

$$\sum_{\delta_0\in G_0(F)_{\mathrm{str.reg.}}/\sim_{\mathrm{st}}} \tau(G) \sum_{\kappa\in\kappa(T/F)^D} \sum_{\substack{\delta\in G(\mathbf{A})_{\mathrm{str.reg.ell.}}/\mathbf{A}-\mathrm{conj:}\\ \delta\sim_{\mathrm{str}}\delta_0}} \kappa(\mathrm{obs}(\delta))O_\delta(f).$$

5.5. **Transfer.** We now want to relate κ-orbital integrals to endoscopic data and transfer identities. The transfer identities are defined in terms of transfer factors, so we need to relate these to the obstruction classes.

Fix T,κ. From this we obtain an endoscopic datum as follows. By Tate-Nakayama duality, we have

$$H^1(\mathbf{A}/F, T_{\mathrm{sc}})^* \cong \pi_0((\widehat{T}/Z(\widehat{G}))^\Gamma).$$

As $\kappa(T/F)^D$ was a quotient of $H^1(\mathbf{A}/F,T_{\mathrm{sc}})$, $\kappa(T/F)$ will be a subgroup of $H^1(\mathbf{A}/F,T_{\mathrm{sc}})^*$, and it corresponds to $\ker\left(\pi_0((\widehat{T}/Z(\widehat{G}))^\Gamma)\to\bigoplus_v H^1(\Gamma_v,Z(\widehat{G}))\right)$. So κ corresponds to $s\in\pi_0((\widehat{T}/Z(\widehat{G}))^\Gamma)$, and the condition of lying in the subspace is the condition that s be locally trivial in §3.4.

In the elliptic case, we have $\kappa\in(\widehat{T}/Z(\widehat{G}))^\Gamma$. Recall that we have a canonical $\widehat{G}$-conjugacy class of embeddings $\widehat{T}\hookrightarrow\widehat{G}$. Let $\widehat{H}=Z_{\widehat{G}}(\kappa)$. This gives us an endoscopic triple $(\widehat{H},s,\eta)$.

Let $R^H\subset X^*(T)=X_*(\widehat{T})$ be the pre-image of the coroot system of $\widehat{H}$. (By construction, $\widehat{H}$ and $\widehat{G}$ share maximal tori, so the coroot system for $\widehat{H}$ is a sub of that for $\widehat{G}$.) It is Γ-invariant, because κ is Γ-invariant. We want to use this to make the group H, however, the Γ-action need not preserve a base. But we can force it to do so. Choose a base $\Delta^H\subset R^H$. For each $\sigma\in\Gamma$, there is $w_\sigma^H\in\Omega(\widehat{T},\widehat{H})$ such that $w_\sigma^H\sigma$ preserves Δ^H.

Thus we get a quasi-split reductive group H with root system R_H. There is an embedding $T\hookrightarrow H$ over F, which is well-defined up to stable conjugacy. So we get an endoscopic group *plus* a canonical way to put T in it.

Now we can define the transfer factors. For $\delta_0\in T(F)$, we get $\gamma\in H(F)$ up to stable conjugacy.

Theorem 5.1 (Langlands-Shelstad)**.** *Let $\mathfrak{w}$ be a Whittaker datum for G_0 over F. Then*

$$\prod_v \Delta[\varpi](\gamma_v,\delta_{0v})=1.$$

Recall that $\Delta[\mathfrak{w}](\gamma_v,\delta_v)$ has the property that

$$\Delta[\mathfrak{w}](\gamma_v,\delta_v)=\Delta[\mathfrak{w}](\gamma_0,\delta_{0v})\langle\mathrm{inv}(\delta_{0v},\delta_v),s\rangle.$$

Then

$$\prod_v\Delta[\mathfrak{w}](\gamma_v,\delta_v)=\prod_v\langle\sum\mathrm{inv}(\delta_{0v},\delta_v),s\rangle=\langle\mathrm{obs}(\delta),\kappa\rangle=\kappa(\mathrm{obs}(\delta)).$$

This tells you that

$$O^{\kappa}_{\delta_0}(f) = \prod_v \underbrace{\sum_{\delta_v \in G(F_v)/\mathrm{conj}} \Delta[\mathfrak{w}_v](\gamma_v, \delta_v) O_{\delta_v}(f_v)}_{SO_{\gamma_v}(f_v^{H_v})}.$$

Conclusion: we have written

$$\mathrm{TF}_{\mathrm{reg.ell.}}(f) = \sum_{(H,s,\eta)/\cong} \iota(H,G)\mathrm{STF}_{G-\mathrm{reg}}(f).$$

6. Stabilization of the full trace formula for SL_2

The trace formula is an equality between two different expressions of a distribution:

$$J_{\mathrm{geom}}(f) = J_{\mathrm{spec}}(f). \tag{6.1}$$

The LHS is referred to as the "geometric side" and the RHS is referred to as the "spectral side". These distributions are not stably invariant, and the goal of stabilization is to manipulate (6.1) into an equality of stable distributions.

6.1. **The geometric side.** We break J_{geom} into several different terms, according to the type of the conjugacy class:

6.1.1. *Central.* The central terms are

$$(\mathrm{central}) := f(1) + f(-1)$$

(these should be weighted by volume factors, which are Tamagawa numbers, but these are 1 for SL_2).

6.1.2. *Regular elliptic.* The regular elliptic terms are

$$\mathrm{TF}_{\mathrm{reg.ell.}} := \sum_{\gamma \in G(\mathbf{Q})_{\mathrm{reg.ell.}}} \mathrm{vol}(T(F)\backslash T(\mathbf{A})) O_\gamma(f). \tag{6.2}$$

6.1.3. *Regular hyperbolic.* The regular hyperbolic terms (corresponding to the diagonal maximal torus) are

$$(\mathrm{hyperb.}) := \sum_{\gamma \in \mathbf{Q}^\times/\pm 1} \mathrm{vol}(\mathbf{Q}^\times \backslash \mathbf{A}^1) \int_{T(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g) v_T(g)\, dg. \tag{6.3}$$

The $v_T(g)$ is the "weighting factor", and the integral is called a "weighted orbital integral". The weighting factor is itself a sum of local weighting factors, so we can write this as

$$(6.3) = \sum_v \int_{T(F_v)\backslash G(F_v)} f(g_v^{-1}\gamma_v g_v) V_v(g_v) dg_v \prod_{w \neq v} \int_{T(F_w)\backslash G(F_w)} f(g_w^{-1}\gamma_w g_w)\, dg_w. \tag{6.4}$$

6.1.4. *Unipotent.* The unipotent part is, at first attempt:

$$\sum_{\substack{a=\pm 1 \\ u \in \mathbf{Q}^\times/(\mathbf{Q}^\times)^2}} \int_{Z(\mathbf{A})N(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1} n_{a,u} g)\, dg$$

where

$$n_{a,u} := a \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

Unfortunately, these orbital integrals diverge. Each local factor converges, but the orbital integral of the unit in the Hecke algebra is basically the local L-factor at 1, and the global L-function has a pole at 1.

You regularize by inserting a factor $\beta(g)^{-s}$ into the integral, where $\beta(g) = |a|^2$ if a, a^{-1} are the eigenvalues in the toral part in the Iwasawa decomposition of g. So instead we consider

$$\sum_{\substack{a=\pm 1 \\ u \in \mathbf{Q}^\times/(\mathbf{Q}^\times)^2}} \int_{Z(\mathbf{A})N(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1} n_{a,u} g)\beta(g)^{-s}\, dg \tag{6.5}$$

Instead of evaluating this at $s = 0$, where it has a simple pole, you take the *constant coefficient* of the Laurent expansion at $s = 0$.

$$\text{(unip.)} := \text{Constant-coefficient}(6.5). \tag{6.6}$$

Here is another way of expressing this. We define a function

$$\theta(s, f) := \zeta(s+1)^{-1} \int_{Z(\mathbf{A})N(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1} u_{a,x} g)\beta(g)^{-s}\, dg.$$

This is analytic at $s = 0$. Writing

$$\zeta(s+1) = \frac{\lambda_{-1}}{s} + \lambda_0 + \dots$$

then the unipotent term becomes

$$\text{(unip.)} = \sum_{a,u} \lambda_{-1}\theta'(0, f) + \lambda_0 \theta(0, f).$$

6.2. **The spectral side.** Again there are a number of terms, and we take stock of them in turn.

6.2.1. *Discrete spectrum.* The one which we are really interested in is

$$r(f) := \mathrm{Tr}(f \mid R_{\mathrm{disc}}).$$

But there are supplementary terms.

6.2.2. *Discrete contribution from the continuous spectrum.* One is the discrete contribution to the trace formula coming from the continuous spectrum:

$$(\text{cont. } 0) := -\frac{1}{4} \sum_{\substack{\eta \colon F^\times \backslash \mathbf{A}^\times \to \mathbf{C} \\ \eta^2 = 1}} \operatorname{Tr}(M(\eta)\rho(\eta, f)) \tag{6.7}$$

where $\rho(\eta) = \operatorname{Ind}_B^G(\eta)$ and $M(\eta)$ is the standard un-normalized intertwining operator

$$M(\eta) \colon \rho(\eta) \to \rho(\eta^{-1}) = \rho(\eta)$$

defined by

$$(M(\eta)\varphi)(g) := \int_{N(\mathbf{A})} \varphi(wug)\, du.$$

6.2.3. *Continuous contribution from the continuous spectrum.* There are two more terms that contribute continuously, which come from the continuous part of the spectrum. One is

$$(\text{cont. } 1) := \frac{1}{4\pi} \int_{D^0} \frac{L(1, \eta^{-1})}{L(1, \eta)} \operatorname{Tr}(\rho(\eta, f))\, d\eta \tag{6.8}$$

where D^0 is the space of unitary characters η of $F^\times \backslash A^\times$, which is a disjoint union of components that are copies of $\mathbf{R}$. Hence $\mathbf{R}$ acts on D^0 in such a way that $\eta_s = \eta |\cdot|^{2is}$.

The second continuous contribution is

$$\sum_v \frac{1}{4\pi} \int_{D^0} \operatorname{Tr} R^{-1}(\eta_v) R'(\eta_v) \rho(\eta_v, f_v) \prod_{w \neq v} \operatorname{Tr}(\rho(\eta_w, f_w))\, d\eta \tag{6.9}$$

where R is the *normalized* version of the standard intertwining operator:

$$R(\eta_v) := \epsilon(0, \eta_v, \psi_v) \frac{L(1, \eta_v)}{L(0, \eta_v)} M(\eta_v).$$

6.3. **Stabilization: geometric side.** We would like to stabilize (6.1), but we have a lot of work to do. First of all, none of the distributions we just defined are not even invariant under conjugacy. We need to manipulate them to be invariant under stable conjugacy. This will entail putting together geometric and spectral terms, so there is not really a separation into the two sides at the end.

6.3.1. Let's start with the regular elliptic term. In §2 we found that

$$\mathrm{TF}_{\text{reg.ell.}}(f) = \frac{1}{2} \mathrm{STF}_{\text{reg.ell.}}(f) + \frac{1}{4} \sum_T \mathrm{STF}^T_{G-\text{reg}}(f^T).$$

The term $\mathrm{STF}_{\text{reg.ell.}}(f)$ is stable, but $\mathrm{TF}_{\text{reg.ell.}}(f)$ is not stable, so of course $\sum_T \mathrm{STF}^T_{G-\text{reg}}(f^T)$ must also not be stable. It wants to be a stable trace formula for T, but it's missing some terms, so we have to go fish around elsewhere in the trace formula to find them and put them in.

What are the terms that we're missing? The terms $f^T(1) + f^T(-1)$, up to a volume factor (namely, the Tamagawa number of T). Previously we explained that you can modify f^T to extend it smoothly. Now we actually have to understand the value of this extension.

6.3.2. Recall that the smoothened function was

$$f_v^T(\gamma_v) = \lambda(E_V/F_v, \psi_v)\kappa_v\left(\frac{e_v - \overline{e}_v}{\eta_v}\right)|e_v - \overline{e}_v|_v \underbrace{f^T_{\text{naïve}}(\gamma_v)}_{O_{e_v}(f_v) - O_{\overline{e}_v}(f_v)} .$$

We will give an example calculation of this.

Example 6.1. Let $v = \infty$, and $\gamma_v \to 1$. One can compute

$$\lim_{\theta \to 0^+} |e^{i\theta} - e^{-i\theta}| O_{e^{i\theta}}(f) = \int_0^\infty \int_K f\left(k \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} k^{-1}\right) dk du.$$

$$\lim_{\theta \to 0^-} |e^{i\theta} - e^{-i\theta}| O_{e^{i\theta}}(f) = \int_{-\infty}^0 \int_K f\left(k \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} k^{-1}\right) dk du.$$

(This is why we introduced a sign, to extend things smoothly.) This means that

$$\lim_{\theta \to 0^+} |e^{i\theta} - e^{-i\theta}| f^T_{\text{naïve}} = \int_{-\infty}^\infty \operatorname{sgn}(u) \int_K f\left(k \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} k^{-1}\right) dk du. \tag{6.10}$$

Now, $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ can be seen as the orbit of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ under conjugation by the diagonal torus, and together with the integral over K this can be thought of as an integral over G/N. So (6.10) becomes

$$(6.10) = \int_{N(F_v) \backslash \mathrm{PGL}_2(F_v)} \operatorname{sgn}(\det g) f(g^{-1}\gamma g)\, dg. \tag{6.11}$$

The sign character appearing can be interpreted as κ_∞. So we see a unipotent κ-orbital integral.

> Moral: regular elliptic κ-orbital integrals degenerate to unipotent κ-orbital integrals:
>
> $$\lim_{\gamma_v \to a_v := \pm 1} f_v^T(\gamma_v) = L_v(1, \kappa_v) \int_{N(F_v) \backslash \mathrm{PGL}_2(F_v)} \kappa_v(\det g_v) f(g_v^{-1} n_a g_v)\, dg_v.$$
>
> (The L-factor comes from switching between measures.)

6.3.3. We hope to find these terms sitting in the unipotent contribution. By an elementary computation we repackage (6.6) as follows. First define

$$G' := \{g \in \mathrm{GL}_2(\mathbf{A}) \mid \det g \in \mathbf{Q}^\times \cdot (\mathbf{A}^\times)^2\}.$$

An easy manipulation shows that (6.6) can be rewritten as

$$(\text{unip.}) = \int_{Z' N(\mathbf{A}) \backslash G'} f(g^{-1} u_a g) \beta(g)^{-s}\, dg \tag{6.12}$$

where Z' is the center of G'. We wrote this to introduce κ. Note that $\mathbf{A}^\times/\mathbf{Q}^\times\cdot(\mathbf{A}^\times)^2$ is a compact abelian group, and the integrand is defined on all of $\mathrm{GL}_2(\mathbf{A})$. By Fourier analysis, we can then decompose over characters of that compact abelian group.

$$\text{(unip.)} = \sum_{\kappa\in(\mathbf{A}^\times/\mathbf{Q}^\times\mathbf{A}^{\times 2})^*} \int_{N(\mathbf{A})\backslash\mathrm{PGL}_2(\mathbf{A})} \kappa(\det g) f(g^{-1}u_a g)\beta(g)^{-s}\,dg. \tag{6.13}$$

Let's now look at the contribution from an individual, *non-trivial* κ. A very similar computation to the one showing that the adelic unipotent orbital integral diverges (as discussed in §6.1.4) shows that this gives

$$\int_{N(F_v)\backslash\mathrm{PGL}_2(F_v)} \kappa_v(\det g_v) f_v(g_v^{-1}n_a g_v)\beta(g_v)^{-s} dg_v = L_v(1,\kappa_v) f_v(a).$$

The basic observation is that the global $L(s,\kappa)$ has *no* pole at $s=1$.

Upshot: for $\kappa\neq 1$, we get

$$\text{(unip.)}_{\kappa\neq 1} = \underbrace{L(1,\kappa)\prod_v L(1,\kappa_v)^{-1}}_{=1} \int_{N(F_v)\backslash\mathrm{PGL}_2(F_v)} \kappa_v(\det g_v) f_v(g_v^{-1}u_a g_v)\,dg_v,$$

which is exactly what we were missing from the regular elliptic part. (We weren't careful about how the L-factors arose; you may ignore them.)

6.3.4. *Recap.* We have now rewritten the geometric side of the trace formula as

$$\text{(central)} + \frac{1}{2}\mathrm{STF}^G_{\text{reg.ell.}}(f) + \frac{1}{4}\sum_T \mathrm{STF}^T(f) + \text{(hyperbolic)} + \text{(unip)}_{\kappa=1}. \tag{6.14}$$

At the moment we have not changed the spectral side of the trace formula.

6.4. **Stabilization: entering the spectral side.** The term $\frac{1}{4}\sum_T \mathrm{STF}^T(f)$ in (6.14) is still not stable. But since it is the full trace formula, it has itself a geometric and a spectral side. So we can write

$$\mathrm{STF}_T(f^T) = \sum_{\theta\colon T(F)\backslash T(\mathbf{A})\to\mathbf{C}^\times} \theta(f^T). \tag{6.15}$$

We are going to move this to the spectral side and rework it there.

What are we going to do with it? Recall that the discrete contribution of the continuous spectrum on the spectral side was a sum over sign characters (§6.2.2). We're going to compare that part with this.

6.4.1. *Treating the unstable part of cts. 0.* Recall from (6.7) that

$$\text{(cts. 0)} = \sum_{\eta\text{ quad.}} -\frac{1}{4}\operatorname{Tr}(M(\eta)\rho(\eta,f)).$$

Let's think about a particular summand.

Remark 6.1. Now, $\rho(\eta)$ is the restriction to $\mathrm{SL}_2(\mathbf{A})$ of a representation of $\mathrm{GL}_2(\mathbf{A})$, so $\mathrm{Tr}(\rho(\eta, f))$ is stable. However, ρ_η is reducible in general and $M(\eta)$ acts on the different pieces in different ways, which makes this unstable. This is what we will try to understand.

We replace the un-normalized intertwining operator by the *normalized* one, which is:

$$-\frac{1}{4}\frac{L(1,\eta^{-1})}{L(1,\eta)}\,\mathrm{Tr}(R(\eta)\rho(\eta, f)).$$

We have two cases:

(1) If η is trivial, then $R(\eta)$ acts as Id. (This is a local computation by Labesse–Langlands.) Hence, by Remark 6.1, the entire summand is stable.

(2) Consider $\eta \neq 1$. By class field theory, η gives a quadratic extension E/F, and then an endoscopic torus T. Consider $\theta\colon T(F)\backslash T(\mathbf{A}) \to \mathbf{C}^\times$, the trivial character. We have ${}^LT \hookrightarrow {}^LG$. The "trivial" Langlands parameter $W_F \to {}^LT$ composed with the canonical and *non-trivial* L-embedding gives a *non-trivial* Langlands parameter $W_F \to {}^LG$. Hence you get an L-packet for G. Which one?

At each place v, the L-packet $\Pi({}^L\theta_v)$ consists of the irreducible constituents of $\rho(\eta_v)$, which are $\{\pi_+(\eta_v), \pi_-(\eta_v)\}$. (When $\rho(\eta_v 0$ is irreducible, our convention is that $\pi_-(\eta_v) = 0$.) Then $R(\eta_v)$ acts on $\pi_+(\eta_v)$ by $+1$ and on $\pi_-(\eta_v)$ by -1. Hence

$$\mathrm{Tr}(R(\eta_v)\rho(\eta_v, f_v)) = \pi_+(\eta_v)(f_v) - \pi_-(\eta_v)(f_v).$$

This is one side of the endoscopic character identity for T. So by the endoscopic character identities this is equal to $\theta_v(f_v^T)$. This matches what we saw earlier in (6.15).

6.4.2. *Summary.* At this point, we have an equality of the distributions

$$(\text{central}) + \mathrm{STF}^G_{\text{reg.ell.}}(f) + (\text{hyperb.}) + (\text{unip.})_{\kappa=1}$$

and

$$r(f) + (\text{cts. }0)_{\eta=1} - \frac{1}{4}\sum_T \mathrm{STF}^T(f^T)_{\theta\neq 1} + (\text{cts. }1) + (\text{cts. }2).$$

6.4.3. *Final manipulation.* Due to time constraints, we won't be able to go through everything.

Lemma 6.2. *The linear combination*

$$(\textit{hyperb.}) + (\textit{unip.})_{\kappa=1} - (\textit{cts. }2)$$

is a stable distribution.

This is the part where you start mixing geometric and spectral terms. Let's accept this for a moment and see how we conclude.

Theorem 6.3 (Stable Trace Formula for SL_2)**.** *We have the following:*

(1) The distribution

$$r(f) - \frac{1}{4}\sum_{T} \mathrm{STF}^T(f^T)_{\theta\neq 1} \tag{6.16}$$

is stable.

(2) The (stable) distribution (6.16) *is equal to*

$$\begin{aligned}&\textit{(central)} + \mathrm{STF}^G_{\textit{reg.ell.}}(f) + \\ &\quad [\textit{(hyperbolic)} + \textit{(unipotent)}_{\kappa=1} - \textit{(cts. 2)}] \\ &\qquad - \textit{(cts. 1)} - \textit{(cts. 0)}_{\eta=1}\end{aligned}$$

where all summands are stable.

Remark 6.4. (1) is not obvious, but it is expected from the conjectures. Why? You expect the discrete spectrum to break into packets, which have either

- 1 element locally, in which case its character is conjectured to be stable, or
- more than 1 element locally, which means that the centralizer of the parameter is non-trivial. Then elements of this centralizer give endoscopic groups, and then the unstable κ-orbital integral cancels with something in $\sum_T \mathrm{STF}^T(f^T)_{\theta\neq 1}$.

6.5. **Proof of Lemma 6.2.** Lastly, how do you prove the lemma? We need to analyze:

$$(\text{hyp.}) - (\text{hyp.})^g + (\text{unip.})_{\kappa=0} - (\text{unip.})^g_{\kappa=0} - (\text{cts. } 2) - (\text{cts. } 2)^g \tag{6.17}$$

where

- conjugation by $g \in \mathrm{GL}_2(\mathbf{A})$ effects stable conjugacy in $\mathrm{SL}_2(\mathbf{A})$, and
- $(\ldots)^g(f) = (\ldots)(f^g)$ with $f^g(\gamma) = f(g\gamma g^{-1})$.

Showing stability means that showing (6.17) is 0. The point is that none of the individual terms in (6.17) is 0, but they combine to be 0.

6.5.1. *Continuous term.* Let's look at

$$(\text{cts. } 2) = \sum_v \frac{1}{4\pi}\int_{D^0} \mathrm{Tr}(R^{-1}(\eta_v)R'(\eta_v)\rho(\eta_v, f_v)) \prod_{w\neq v} \mathrm{Tr}\,\rho(\eta_w, f_w)d\eta.$$

Since $\mathrm{Tr}\,\rho(\eta_w, f_w)$ is stable by Remark 6.1, the instability comes from the intertwining operators. More precisely,

$$(\text{cts. } 2) - (\text{cts. } 2)^g = \sum_v \frac{1}{4\pi}\int_{D_0} [(*)(f_v) - (*)(f_v^g)] \prod_{w\neq v} \mathrm{Tr}\,\rho(\eta_w, f_w)d\eta. \tag{6.18}$$

Here

$$[(*)(f_v) - (*)(f_v^g)] = \mathrm{Tr}\,\rho(\eta_v, f_v)N(g_v) - \mathrm{Tr}(\eta_v^{-1}, f_v)N(g_v),$$

where $N(g_v)$ is an operator on $\rho(\eta_v)$ given by multiplication by $\ln\beta(g_v)$.

Since we integrate over D_0, we can combine η and η^{-1}. This lets us rewrite (6.18) as

$$(\text{cts. } 2) - (\text{cts. } 2)^g = \sum_v \frac{1}{2\pi} \int_{D^0} \underbrace{\operatorname{Tr}(\rho(\eta_v, f_v) N(g_v))}_{=:H_v(\eta_v)} \prod_{w \neq v} \underbrace{\operatorname{Tr} \rho(\eta_w, f_w)}_{=:I_w(\eta_w)} \, d\eta$$

i.e.

$$(\text{cts. } 2) - (\text{cts. } 2)^g = \frac{1}{2\pi} \sum_v \int_{D^0} H_v(\eta_v) \prod_{w \neq v} I_w(\eta_w) \, d\eta. \tag{6.19}$$

This ends the analysis of the first term. What about the others?

6.5.2. *Hyperbolic and unipotent terms.* Let me just tell you the result.

$$(\text{hyp.}) - (\text{hyp.})^g = \sum_{\gamma \in F^\times - \{\pm 1\}} \sum_v \frac{\widehat{H}(\gamma_v)}{L(1, F_v)} \prod_{w \neq v} \frac{\widehat{I}_w(\gamma_w)}{L(1, F_w)}.$$

Here $\widehat{H}, \widehat{I}$ are the Fourier transforms.

Next,

$$(\text{unip.})_{\kappa=0} - (\text{unip.})^g_{\kappa=0} = \sum_{a = \pm 1} \sum_v \frac{\widehat{H}(\gamma_v)}{L(1, F_v)} \prod_{w \neq v} \frac{\widehat{I}_w(\gamma_w)}{L(1, F_w)}.$$

Hence the combination of the hyperbolic and unipotent terms is

$$(\text{hyp.}) - (\text{hyp.})^g + (\text{unip.})_{\kappa=0} - (\text{unip.})^g_{\kappa=0} = \sum_{\gamma \in F^\times} \sum_v \frac{\widehat{H}(\gamma_v)}{L(1, F_v)} \prod_{w \neq v} \frac{\widehat{I}_w(\gamma_w)}{L(1, F_w)}.$$

We want this to be equal to (6.19).

Now look at the exact sequence

$$1 \to F^\times \to \mathbf{A}^\times \to \mathbf{A}^\times / F^\times \to 1.$$

By definition, $\mathbf{A}^\times / F^\times$ is Pontrjagin dual to D^0. We can apply the following version of Poisson summation: If

$$1 \to A \to B \to C \to 1$$

is an exact sequence of locally compact abelian groups, and $f \colon B \to \mathbf{C}^\times$ is a pleasant function, then

$$\int_A f(a) \, da = \int_{C^\vee} \widehat{f}(\xi) \, d\xi.$$

Applying this, we finally win.