GEOMETRY OF THE MOMENT MAP AND THE RELATIVE TRACE FORMULA

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Today I want to highlight the theory of the moment map $T^*X \to \mathfrak{g}^*$ for a G-space X, which is due to Friedrich Knop. This has not previously played much role in automorphic forms, but it seems to be important, because the dual group (more precisely, its elements) seem to have a geometric (rather than combinatorial) meaning.

1. Introduction

Langlands functoriality predicts that when you have a map ${}^L G_1 \to {}^L G_2$, one should (roughly) get a transfer between irreducible representations of G_1 to irreducible representations of G_2. By "representations" we mean:

- local case: representations of $G(F)$, where F is a local field
- global case: automorphic representations.

We now know that Langlands functoriality holds more generally than the setting of groups. It extends to *spherical* G-spaces X, i.e. the Borel subgroup acts with an open orbit.

Example 1.1. Toric varieties, flag varieties, reductive groups H (with $G = H \times H$ acting by left and right multiplication), and more generally symmetric spaces such as $\mathrm{GL}_{2n} / \mathrm{Sp}_{2n}, \mathrm{SO}_{n+1} / \mathrm{SO}_n$, ...

To such a space, we can attach an L-group ${}^L X$ with a map ${}^L X \to {}^L G$ [Gaitsgory-Nadler, S.-Venkatesh, Knop-Schalke].

A map ${}^L X_1 \to {}^L X_2$ should induce a transfer between irreducible representations in the spectrum of X_1 to irreducible representations in the spectrum of X_2. What does "in the spectrum" mean?

- Locally, you have the Plancherel formula

$$\mathcal{S}(X \times X) \ni \Phi_1 \otimes \Phi_2 \mapsto \langle \Phi_1, \overline{\Phi}_2 \rangle_{L^2(X)} = \int_{\widehat{G}} \mathcal{J}_\pi(\Phi_1 \otimes \Phi_2) \mu_X(\pi).$$

  The representations π on which μ_X is supported are what I call the "spectrum on X". This is closely related to the question of distinction by X, i.e. $\pi \hookrightarrow C^\infty(X)$.

- Globally, you have the relative trace formula: let k be a global field and $\mathbf{A}$ its ring of adeles. Then you consider

$$\underbrace{\Phi_1 \otimes \Phi_2}_{\in \mathcal{S}(X \times X(\mathbf{A}))} \mapsto \underbrace{\sum_{(\gamma_1,\gamma_2) \in X^2(k)} \Phi_1(\gamma_1, -) \otimes \Phi_2(\gamma_2, -)}_{\in C^\infty(G(k)\backslash G(\mathbf{A}) \times G(k) \backslash G(\mathbf{A}))} \xrightarrow{\langle \cdot, \cdot \rangle_{G(k)\backslash G(\mathbf{A})}} \mathbf{C}.$$

Example 1.2. If $X = H$ is a group, then $\sum_{X(k)} \Phi = K_\Phi$ (it's a function on $G = H \times H$). Then you get the Hilbert-Schmidt inner product of two operators, which is the same as the trace of their convolution.

The RTF decomposes as an integral of automorphic representations, and those that appear are the "automorphic spectrum of X".

As with the usual trace formula, we would like to do the following: given a map ${}^L X_1 \to {}^L X_2$, compare the geometric sides of RTF_{X_1} and RTF_{X_2}. What does it mean to "compare" them? This is a very subtle issue, as one wants to "extract" the correct part of the spectrum. (This is already very hard in the group case, and is the subject of Beyond Endoscopy.) So this is very hard, but there are more cases to play with.

This is highly non-trivial already when ${}^L X_1 \cong {}^L X_2$. (In the group case, this would only happen for inner forms.)

Example 1.3. The following varieties have L-group SL_2.

- $\mathrm{PGL}_{n+1} / \mathrm{GL}_n$
- $\mathrm{SO}_{2n+1} / \mathrm{SO}_{2n}$
- $\mathrm{Sp}_{2n} / \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2$
- F_4/Spin_9
- G_2/SL_3

Example 1.4. The following varieties have L-group PGL_2.

- $\mathrm{SO}_{2n} / \mathrm{SO}_{2n-1}$
- $\mathrm{Spin}_8/\mathrm{Spin}_7$
- Spin_7/G_2

Moreover, there are all the affine homogeneous examples, up to finite automorphisms.

These cases were treated locally by Gan-Gomez, who analyzed $L^2(X)$ in terms of $L^2((N,\psi)\backslash \mathrm{PGL}_2)$, using the theta correspondence and generalizations (on a case-by-case basis).

2. Schwartz functions

I would like to study these spaces using the trace formula, and in a *uniform* way – as uniform as the formulation of the conjecture.

The input to the relative trace formula is: $\Phi_1 \otimes \Phi_2 \in \mathcal{S}(X \times X)$. But really it's expressed in terms of orbital integrals, so I prefer to work with measures. (This is equivalent to a Schwartz function times a Haar measure.) Instead of orbital integrals, we then consider "orbital measures".

So we have
$$\mathcal{S}(X \times X) \to \mathrm{Meas}(X \times X//G) \tag{2.1}$$
where $X \times X//G = \mathrm{Spec}\ k[X \times X]^G$.

Example 2.1. For $X = H$, $G = H \times H$,
$$X \times X//G = H//H - \mathrm{conj}$$
is the space of characteristic polynomials.

Example 2.2. For $X = H\backslash G$, $X \times X//G = H\backslash G//H$. This is the usual formulation of the relative trace formula.

The image of (2.1) will be denoted by $\mathcal{S}(X \times X/G)$. (Everything here is over a local field F, and $X = X(F)$.)

Example 2.3. What does $\mathcal{S}(X \times X/G)$ look like? Baby case: $\mathbf{A}^2/\mathbf{G}_m$, with $a \cdot (x, y) = (ax, a^{-1}y)$. Then $\mathbf{A}^2//\mathbf{G}_m = \mathbf{A}^1$ via $(x, y) \mapsto \xi := xy$.

Exercise: functions that are smooth away from $\xi = 0$, but at 0 there will be a singularity of the form $c_1(\xi) \log |\xi| + c_2(\xi)$ where c_1, c_2 are smooth.

In general, you get measures with singularities. (In the group case, you get measures with singularities along the discriminant divisor.)

Conjecture 2.4. *Suppose* ${}^L X_1 = {}^L X_2$ *and denote by* $\mathcal{X}_i := X_i \times X_i/G_i$. *There should be a "functoriality transfer operator"* $\mathcal{T}$*:*

$$\begin{array}{ccc} \mathcal{S}(\mathcal{X}_1(\mathbf{A})) & \xrightarrow{\quad\mathcal{T}\quad} & \mathcal{S}(X_2(\mathbf{A})) \\ & {\scriptstyle\mathrm{RTF}}\searrow \quad \swarrow{\scriptstyle\mathrm{RTF}} & \\ & \mathbf{C} & \end{array}$$

This is a bit imprecise, because on the spectral side of RTF_X the automorphic representation π is weighted by a factor
$$\frac{L_X(\pi)}{L(\pi, \mathrm{Ad}, 1)}$$
where L_X is some L-value *depending on* X.

Example 2.5. In the group case, $L_X = L(\pi, \mathrm{Ad}, 1)$ and it cancels with the denominator.

For $X = N, \psi\backslash G$ we have $L_X = 1$.

For $X = U_n\backslash U_n \times U_{n+1}$ and $\pi = \pi_1 \otimes \pi_2$, we have $L_X = L(\pi_1 \times \pi_2, \frac{1}{2})$.

This means that the conjecture should not work as stated. Really, one should enlarge the space of test functions to introduce extra L-functions on the spectral side.

Theorem 2.6. *Let* $X = H\backslash G$ *be of rank* 1*, as in the list of examples. Let* $G^* = \mathrm{PGL}_2$ *or* SL_2 *if* ${}^L X = \mathrm{SL}_2$ *or* PGL_2. *Then there is an explicit transfer operator*
$$\mathcal{S}(X \times X/G) \xleftarrow{\sim} \mathcal{S}^+_{L_X}((N, \psi)\backslash G^*/(N, \psi)) \colon \mathcal{T}$$

given by explicit Fourier transforms determined by L_X. (Here $L_X = L(\pi, s_1)L(\pi, s_2)$ if ${}^L X = \mathrm{SL}_2$ and $L_X = L(\pi, \mathrm{Ad}, s_0)$ if ${}^L X = \mathrm{PGL}_2$.)

At the heart of the proof is an explicit description of $\mathcal{S}(X \times X/G)$, which goes as follows. For simplicity, let's assume G and H are split.

Theorem 2.7. *For X of rank 1 as above, the quotient $C := X \times X//G \cong \mathbf{A}^1$ and $\mathcal{S}(X \times X/G)$ has singularities ξ_- and ξ_+ where the orbital measures behave like pushforwards under a quadratic form*

$$q_\pm \colon V_\pm \to \mathbf{A}^1$$

where $V_\pm$ is a split quadratic space of dimension $d_\pm$ [which has an explicit formula, omitted].

3. On the proof

How do you analyze $X \times X/G = H\backslash G/H$?

Remark 3.1. For X symmetric, Richardson has studied $H\backslash G/H = A_X//W_X$.

For G acting on X, you get $\mathfrak{g} \to \mathrm{Vect}(X)$ and then dually a moment map $T^*X \to \mathfrak{g}^*$. Knop has shown that although the X's look very different, their cotanget spaces T^*X are very similar.

More precisely, consider the composition

$$T^*X \to \mathfrak{g}^* \to \mathfrak{g}^*//G = \mathfrak{a}^*//W = \mathfrak{a}^*//W.$$

One then forms the cartesian diagram

$$\begin{array}{ccc} T^*X & \longrightarrow & \mathfrak{a}^*//W \\ \uparrow & & \uparrow \\ \widehat{T^*X} & \longrightarrow & \mathfrak{a}^* \end{array}$$

Knop has shown that there is an irreducible component $\widehat{T^*X}^\circ \to \mathfrak{a}_X^* \hookrightarrow \mathfrak{a}^*$ where the action of B on the open B-orbit X° is through a quotient A_X of A. (The "rank of X" is the rank of A_X.)

The map $\widehat{T^*X} \to T^*X$ is W_X, the Weyl group of X (which is the Weyl group of ${}^L X$).

Fact 3.2. Over $\mathfrak{a}_X^{*,\mathrm{reg}}$ (the locus with no W_X-stabilizer) the bottom horizontal map is the moment map for an A_X-action.

We give an example of how the geometry of T^*X is related to the modified space of Schwartz functions. Consider $\Delta(X) \subset X \times X$. The conormal bundle can be identified with $T^*X \to \Delta(X)$. You have the commutative group A_X acting on T^*X. Hence you get an action map $A_X \times T^*X \to T^*X \times T^*X$. So you consider flows on the cotangent spaces instead of on X. It turns out that in rank 1, the composition $A_X \times T^*X \to T^*X \times T^*X \to X \times X$ identifies $X \times X/G \cong A_X//W_X = \mathbf{A}^1$, and shows that there are only 2 singularities.