ON THE BRAVERMAN-KAZHDAN PROGRAM

TALK BY NGÔ BÀO CHÂU,
NOTES BY TONY FENG

The Braverman-Kazhdan program is about automorphic L-functions. The goal of the subject is about constructing automorphic L-functions and proving that they share the nice properties enjoyed by the Riemann zeta function. The way that Langlands conceived to study automorphic L-functions is via the so-called "principle of functoriality". But there is actually another route – a much more direct one – which is taken up by Braverman-Kazhdan.

1. Godement-Jacquet theory

There is one case we know a lot about L-functions, which is that of the "standard L-functions", whose theory was developed by Godement-Jacquet.

Let G be a reductive group over k, and $\rho\colon {}^L G \to \mathrm{GL}_n$ be a representation of its dual group. For $G = \mathrm{GL}_n$, we can take ${}^L G = \mathrm{GL}_n(\mathbf{C})$. The general principle is that, given an automorphic representation π of G and ρ as above, we can construct an automorphic L-function $L(s, \rho, \pi)$. A priori this comes as a function on some half-plane, but one should be able to prove a meromorphic continuation and functional equation

$$L(s, \rho, \pi) \leftrightarrow L(1-s, \rho, \pi^\vee).$$

The setting of the standard L-functions have $G = \mathrm{GL}_n$ and $\rho = \mathrm{std}$.

We will now explain the ansatz that implies this story, in the standard (Godement-Jacquet) case. It comes from Fourier and Mellin analysis. We will write out the ingredients. Take k to be a global field.

1.1. Local theory.

For $v \in |K|$, we have a completion k_v, which we also denote by F.

(1) (Schwartz space) We define a space of Schwartz functions $\mathscr{S}^{\mathrm{std}}(G(F))$, which is $\mathscr{S}(M_n(F))$, restricted to $G(F)$.

(2) (Basic function) We have a basic function $\beta^{\mathrm{std}} \in \mathscr{S}^{\mathrm{std}}(G(F))$. When F is non-archimedean, this is taken to be $\beta := \mathbf{I}_{M_n(\mathcal{O})}$. When F is archimedean, we take $\beta := e^{-\pi x^2}$.

(3) (Fourier transform) Fix an additive character $\psi\colon F \to \mathbf{C}^\times$. There is a Fourier transform

$$\varphi \mapsto \widehat{\varphi}(x) = \int_{M_n(F)} \varphi(y)\psi(\mathrm{Tr}(xy))\, dy.$$

It is desirable to write this as $\varphi^\vee * J^{\mathrm{std}}$ where $*$ is convolution with respect to the *multiplicative Haar measure* on $G(F)$, where $J^{\mathrm{std}}(x) = \psi(\mathrm{Tr}(x))|x|^n$.

Now we'll take the Mellin transform.

(1) For π an irreducible representation of $G(F)$ and $f \in \pi \otimes \pi^\vee$, $\varphi \in \mathscr{S}(G(F))$, we consider the family of zeta integrals.

$$Z(\varphi, f, s) := \int_{G(F)} |x|^s \varphi(x) f(x)\, d^*x.$$

These will have a GCD, which is $L(s, \pi, \rho)$.

(2) Recall that in the Riemann zeta function, one needs to complete with Γ-factors to get the functional equation. These are Mellin transforms of the basic functions of the form $e^{-\pi x^2}$. At the non-archimedean places,

$$\mathrm{Tr}(\beta, \pi) = \begin{cases} 0 & \pi \text{ ramified,} \\ L(\frac{1-n}{2}, \pi, \rho) & \pi \text{ ramified.} \end{cases}$$

(3) Since Mellin transform takes convolution to product, we get a functional equation

$$Z(1-s, \pi^\vee, \rho) = \Gamma(\pi, \rho) Z(s, \pi, \rho).$$

1.2. **Global theory.** That was all local. What about the global theory? Let $\mathbf{A} = \mathbf{A}_k$ be the ring of adeles of k.

We define the global Schwartz space to be the restricted product

$$\mathscr{S}(G(\mathbf{A})) := \bigotimes' \mathscr{S}^{\mathrm{std}}(G(k_v))$$

with respect to the basic functions β_v.

We have a *Poisson summation formula*

$$\sum_{\gamma \in G(k)} \varphi(\gamma) = \sum_{\gamma \in G(k)} \widehat{\varphi}(\gamma)$$

under some local conditions on $\varphi, \widehat{\varphi}$ (whose role is to annihilate boundary terms).

Under Mellin transform, this then gives a global functional equation.

2. Braverman-Kazhdan program

How can we generalize this?

2.1. **Reductive monoids.** First question: what is the Schwartz space? In the case of GL_n, we defined a bigger ambient object M_n, then took Schwartz functions on that and restricted them to GL_n. So the first problem is to find a space $M^\rho \supset G$ which generalizes this. We want the $G \times G$-action to extend to M_ρ. We ask for a monoid structure on M_ρ, generalizing the multiplication on M_n.

How can we construct such M^ρ? It just so happens that a theory of "reductive monoids" has been developed by Putcha, Renner, and Vinberg, which are suitable for this purpose. Take a maximal torus $T \subset G$. Let M_T^ρ be the closure; this will be a normal affine algebraic (toric) variety. We have a W-action on T, which extends to an action on M_T^ρ. Conversely, we can construct M^ρ out of the data of

T, W, M_T^ρ. Namely, start with a W-equivariant, strictly convex cone $\sigma \subset \Lambda_{\mathbf{R}} = \mathrm{Hom}(\mathbf{G}_m, T) \otimes \mathbf{R}$.

It's not completely obvious that such a thing exists. Indeed, if G is semisimple then such a thing *cannot* exist. You need to have a center which will allow you to "shift" the translates of the cone to a half.

Let $\rho \colon {}^L G \to \mathrm{GL}(V_\rho)$. Then we can take the cone generated by the weight of ρ. This will give rise to a satisfactory M^ρ.

Remark 2.1. Note that M^ρ doesn't capture all the information of ρ. The cone forgets the multiplicities. For example, $\rho \oplus \rho$ gives the same monoid.

2.2. **The basic function.** The Godement-Jacquet case is basically the only one where the monoid is smooth. When it is not smooth, it isn't right to take smooth functions.

> **Problem:** Define a sheaf $\widetilde{\mathscr{S}^\rho}$ (in the p-adic topology) on $M^\rho(F)$ such that $\mathscr{S}^\rho(G) = \Gamma_c(M^\rho(F), \widetilde{\mathscr{S}^\rho})$.

When v is non-archimedean, β_v should be the trace of Frobenius on the intersection complex of $\mathscr{L}M^\rho$ (when ρ is irreducible). This is a joint result of Bouthier-Sakellaridis-N. The philosophy is that the basic function should "only depend on the singularities".

2.3. **The Fourier transform.** A more difficult problem seems to be to develop a theory of "ρ-Fourier transform". This should stabilize the Schwartz space, preserve the basic function, and have the form $\varphi \mapsto \varphi^\vee * J^\rho$. Experience suggests that we want J^ρ to be a stably invariant smooth function on $G_{\mathrm{rss}}(F)$.

For all irreducible representations π, we believe $J^\rho * f = \gamma^\rho(\pi) f$ for $f \in \pi$. Since γ^ρ is a function *packets*, this is consistent with the property that J_ρ should be stably invariant.

Example 2.1. For GL_n, we saw that $J^\rho(g) = \psi(\mathrm{Tr}(g))|\det g|^n$.

2.4. **The finite field case.** Let G be a reductive group over a *finite* field K. Let $\rho \colon {}^L G \to \mathrm{GL}(V)$. For every irreducible representation π of $G(K)$, we can define $\gamma^\rho(\pi) \in \mathbf{C}$. (Think of this as something like a Gauss sum.) We then get an invariant function $J^\rho \colon G(K) \to \mathbf{C}$ determined by: for $v \in \pi$, $J^\rho * v = \gamma^\rho(\pi) v$.

Here is Braverman-Kazhdan's proposal for constructing J^ρ. Consider restricting $\rho \colon \widehat{G} \to \mathrm{GL}(V_\rho)$ to $\widehat{T}$. This will break up as

$$\rho|_{\widehat{T}} = \chi_1 \oplus \ldots \oplus \chi_n$$

which corresponds (by local Langlands for tori) to the character $\rho_T \colon \mathbf{G}_m^n \to T$ given by

$$\rho_T(x_1, \ldots, x_n) = \prod_{i=1}^n \chi_i(x_i).$$

We have a diagram

$$\begin{array}{ccc} \mathbf{G}_m^n & \xrightarrow{\Sigma} & \mathbf{A}^1 \\ \downarrow{\scriptstyle \rho_T} & & \\ T & & \end{array}$$

Let $\mathscr{L}_\psi$ be the Artin-Schreier sheaf on $\mathbf{A}^1$, and form $J_T^\rho := \rho_{T!}\Sigma^*\mathscr{L}_\psi$.

Example 2.1. For $T = \mathbf{G}_m, \rho = \mathrm{std} \oplus \mathrm{std}$ then J_T^ρ corresponds to a Kloosterman sum.

We have now defined a sheaf on T, and we want to define a sheaf on G. A process called *Lusztig induction* takes in W-invariant perverse sheaves on T and constructs perverse sheaves on G.

Remark 2.2. In Lusztig's case, he uses Kummer sheaves. Here we use the Artin-Schreier sheaf, which is more complicated, but his formalism still goes through.

But the W-equivariant structure is not obvious. There is an S_n-action on $\mathbf{G}_m^n$, but it's not clear how this plays with the W-action on T.

What happens is that there is W' mapping to both S_n and W, and you have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod_{i=1}^n S_{n_i} & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & S_n & & & & \end{array}$$

You then find that ρ_T is W'-equivariant, but $\prod_{i=1}^n S_{n_i}$ doesn't act trivially (so it doesn't descend to a W-action). It turns out that the kernel acts by a sign character. So you twist the action by the sign character of S_n, at which point it descends, and then you untwist by the sign character of W.

Braverman-Kazhdan conjecture that this gives the correct J^ρ. This is basically known now, even in a more geometric version. It was proved by Braverman-Kazhdan for G semisimple of rank 1. Chen-Ngô extended the result to $G = \mathrm{GL}_n$ and all ρ. Tsao-Hsien Chen proved it in general in the $\mathbf{C}$-setting, reformulating in terms of D-modules. Laumon-Lettelian have recently proved it in general.

2.5. **The p-adic case.** The twisting and untwisting by the sign character is puzzling – what would correspond to the sign character in the p-adic case? But I've realized that you don't really need it.

Let $F = k_v$ and T be a torus, not split, over F. Let ${}^LT = \widehat{T} \rtimes \Gamma \to \mathrm{GL}(V_\rho)$. It is an elementary exercise to show that this is equivalent to $\rho\colon D_\rho \to T$ where D_ρ is the induced torus. (Namely, $\Gamma \to S_n$ corresponds to an extension E/F, and $D_\rho := E^\times$.) We compose with $E^\times \to \psi(\mathrm{Tr})F$. Integrating along the fibers, with appropriate regularization, gives J_ρ^T.

Now consider $\rho\colon {}^LG \to \mathrm{GL}(V_\rho)$. We want to make a function J_G^ρ on G. We can consider $T \subset G$ a maximal torus. There is no canonical ${}^LT \to {}^LG$, but it turns out that one can make a canonical composition $\rho_T\colon {}^LT \to {}^LG \to \mathrm{GL}(V_\rho)$. You then get a function J_T^ρ on T. This is compatible, and "glues" to a (stably invariant) function on $G(F)$. However, it is not the correct distribution: it stops working after the standard representation.

Lastly let me explain the work of Lafforgue. He studied GL_2. Start with $J_G^{\rho,\text{naïve}}(c,a)$ where c is the trace and a is the determinant. Then do Fourier transform on the first variable and multiply by $|\xi|$, getting $|\xi|\mathcal{F}_1 J^\rho(\xi,a)$, and then Fourier transform on the second variable, getting $\mathcal{F}_G^\rho$. He proved that this satisfies compatibility with constant terms, so it's right on the induced representations. It seems probable that it is correct in general.