SUPERCUSPIDAL L-PACKETS

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The goal for today is to describe an explicit construction of the Local Langlands Correspondence, in the following setting.

We consider discrete Langlands parameters $\varphi \colon W_F \to {}^L G$, where G is connected reductive over a non-archimedean field F split over a tame extension, and $p \nmid |W|$.

There will be two situations:

(1) The regular case,
(2) The singular case.

Remark 0.1. Conjecturally, the L-packet Π_φ corresponding to such a φ consists entirely of supercuspidal representations. However, this doesn't mean that all supercuspidals appear in such L-packets.

1. Real groups

Since the construction is motivated by what happens for real groups, we'll start by reviewing that. According to Harish-Chandra, $G(\mathbf{R})$ has discrete series if and only if there exists $S \subset G$ an eliptic maximal torus, which is unique up to $G(\mathbf{R})$-conjugacy, and in that case there is a bijection

$$\{(\theta, B)\}/\text{conj} \leftrightarrow \{\text{irreducible discrete series } \pi\}.$$

Here $S \subset B/\mathbf{C} \subset G$ be a Borel subgroup, θ is a character $S(\mathbf{R}) \to \mathbf{C}^\times$, such that $d\theta$ is B-dominant.

If $s \in S(\mathbf{R})_{\text{reg}}$, Harish-Chandra established a character formula

$$\theta_\pi(s) = (-1)^{q(G)} \sum_w \frac{\theta(s^w)}{\prod_{\alpha>0}(1-\alpha(s^w)^{-1})}.$$

This even uniquely characterizes the representation.

If we restrict to $d\theta$ which are regular, we get $\{(S,\theta)\}/G(\mathbf{R}) - \text{conj}$ is in bijection with regular π.

2. Regular supercuspidals

2.1. **Yu's construction.** Let F be non-archimedean and G/F connected reductive, split over a tame extension. There is a construction due to Yu, which produces supercuspidals from the following input datum:

(1) A tower of reductive subgroups $G^0 \subset \ldots \subset G^d = G$, where G_i is a "tame twisted Levi"
(2) $\phi_0, \ldots, \phi_d$, where ϕ_i is a character of G_i,

(3) π_{-1} is a depth 0 supercuspidal representation of G^0.

What is depth? By work of Moy-Prasad, any supercuspidal π has associated $d(\pi) \in \mathbf{Q}_{\geq 0}$ such that: if $d(\pi) = 0$, then π is obtained by compact-induction of ρ from a compact-mod-center $K \subset G(F)$, and ρ is an irreducible representation of K/P^+ such that $\rho|_{P/P+}$, which is a finite group of Lie type, contains a cuspidal irrep.

What is known about Yu's construction (2001)?

- 2007 – Julee Kim showed that the construction is surjective when $\mathrm{char}(F) > 0$ and $p \gg 0$.
- 2018 – Fintzen improved this to all F and $p \nmid |W|$.
- 2008 – Hakim-Murnaghan studied the fibers.

In some sense these results give a classification of supercuspidals. But we want something simpler, as in the real case.

Definition 2.1. We say π is *regular* if

- it comes from Yu's construction,
- $\pi_{-1} = c - \mathrm{Ind}_K^G \rho$ where ρ is a *regular* Deligne-Lusztig character.

Theorem 2.2 (K). *Assume p is not a bad prime (which is implied by $p \nmid |W|$). There is a bijection between*

$$\left\{ \substack{\textit{regular supercuspidal} \\ \textit{representations}} \right\} \leftrightarrow \{(S, \theta)\}/G(F) - \mathrm{conj}$$

where $S \subset G$ is an elliptic tame maximal torus, and $\theta \colon S(F) \to \mathbf{C}^\times$ is a regular character.

2.2. **Character formula.** Work of Adler-DeBacker-Spice gives a character formula for any supercuspidal. We give a re-interpretation of the interesting roots of unity which occur.

Notation: $R(S, G)$ is an absolute root system, and $\Gamma = \mathrm{Gal}(F^s/F)$ acting.

We have $\Gamma_\alpha = \mathrm{Stab}(\alpha, \Gamma) \supset \Gamma_{\pm\alpha} = \mathrm{Stab}(\{\pm\alpha\}, \Gamma)$ corresponding to fields $F_\alpha / F_{\pm\alpha}$.

We choose a-data $a_\alpha \in F_\alpha^\times$ with $a_{-\alpha} = -a_\alpha$ and $a_{\sigma\alpha} = \sigma(a_\alpha)$, and also χ-data $\chi_\alpha \colon F_\alpha^\times \to \mathbf{C}^\times$ satisfying similar conditions, and that $\chi_d|_{F_{\pm\alpha}^\times}$ corresponds to κ_α corresponding to F_α.

Theorem 2.1. *Let $s \in S(F)_{\mathrm{reg}}$ be shallow (i.e. not in Iwahori subgroup). Then*

$$\theta_\pi(s) = e(G)\epsilon(\frac{1}{2}, X^*(T_0)_{\mathbf{C}} - X^*(S)_{\mathbf{C}}, 1) \sum_{w \in N(S,G)(F)/S(F)} \Delta_{II}^{\mathrm{abs}}(s^w)\theta(s^w)$$

where T_0 is the torus of the minimal Levi in the quasi-split inner form, $e(G)$ is the Kottwitz sign, and $\Delta_{II}^{\mathrm{abs}}$ is some explicit character.

This also makes sense when $F = \mathbf{R}$, and it becomes Harish-Chandra's formula.

2.3. **Local Langlands correspondence for regular supercuspidals.** Let $\varphi \colon W_F \to {}^L G$ be a Langlands parameter, and assume that $Z_{\widehat{G}}(\varphi(I_F))$ is abelian. We'll construct a corresponding supercuspidal L-packet.

Fix a Borel pair $(\widehat{T}, \widehat{B})$ which is Γ-invariant. Up to equivalence we can factor

$$\varphi\colon W_F \to N(\widehat{T}, \widehat{G}) \rtimes \Gamma.$$

Let $\widehat{S} := \widehat{T}$ with the new Γ-action, which gives a torus S/F coming with $j\colon S \hookrightarrow G$.

From the choice of χ-data (χ_α), we get $j_\chi\colon {}^L S \hookrightarrow {}^L G$ through which φ factors, giving φ_χ. By LLC for tori one gets $\theta_\chi\colon S(F) \to \mathbf{C}^\times$, which is regular. For each $j\colon S \hookrightarrow G$, write down

$$e(G)\epsilon \sum_w \Delta_{II}[a,\chi](s^w)\theta_\chi(s^w)$$

Take the representation with this character (see §2.2). This seems to depend on our choices, but the choices appear twice and cancel, so this character depends only on φ, j. This gives a regular supercuspidal $\pi_{\varphi,j}$.

We define $\Pi_\varphi := \{\pi_{\varphi,j} \mid j\colon S \hookrightarrow G\}$.

We also want to parametrize the packet. Fact: $S_\varphi = \widehat{S}^\Gamma$. Hence $\pi_0(S_\varphi)^* = \pi_0(\widehat{S}^\Gamma)^* = H^1(\Gamma, S)$ acts simply transitively on the embedding $j\colon S \to G$ (or its pure inner forms). Trivialize the torsor by picking a Whittaker datum.

3. Singular case

3.1. **Summary.** It's not too far from being regular, and we can carry through many of the same steps.

Fact: $Z_{\widehat{G}}(\varphi(I_F))$ is not abelian (which would mean φ was regular), but its connected component is a torus. This is enough to repeat the recipe: for each $j\colon S \hookrightarrow G$ we get a supercuspidal $\pi_{\varphi,j}$.

But there is a big difference: this is usually reducible. So we just try defining Π_φ as the irreducible constituents of the $\pi_{\varphi,j}$.

But where is the difference balanced on the Galois side? It's because $\pi_0(S_\varphi)$ is usually non-abelian. Now the challenge is to match the representations with the constituents.

We have a short exact sequence

$$1 \to \widehat{S}^\Gamma \to S_\varphi \to \Omega(S,G)(F)_{\theta_\chi} \to 1.$$

Reduction 1: it's enough to obtain a bijection $[\pi_{\varphi,j}] \leftrightarrow \{\rho \in \mathrm{Irr}(\pi_0 S_\varphi) \mid \varphi|_{\widehat{S}^\Gamma} \ni j\}$.

Reduction 2: reduce to $d(\pi) = 0$. This is still in progress, but it's very difficult and technical, and we won't comment on it.

How do you handle $d(\pi) = 0$?

3.2. **Geometric intertwining operators.** Let G be connected reductive over a finite field k, $S \subset G$ an elliptic maximal torus, and $\theta\colon S(k) \to \mathbf{C}^\times$ non-singular. Fix $S \subset B/\overline{k}$. We have a Deligne-Lusztig variety Y_B. Then $H^d_c(Y_B, \overline{\mathbf{Q}}_\ell)_\theta$ is the representation of $G(k)$. This is reducible, which leads to the reducibility of the supercuspidal representation $\pi_{\varphi,j}$ mentioned previously.

Problem: parametrize the irreducible constituents. Idea: think of DL-theory as a generalization of parabolic induction. Classically, decompose parabolic induction using self-intertwining and the theory of the Weyl group. This is composed of:

shifting by Weyl goup, and an integral operator changing parabolics. What would correspond to the integral operator in this setting?

You have $Y_B \to Y_{nBn^{-1}}$ as usual, but recently [Bonnafe-Dat-Rouquier, 2017] defined the analog of the integral. This goes through a new object, the "linking variety".

Theorem: there exists *some* a normalization of the self-intertwining operators so define an action of $N(S, G)(k)_\theta$. This is analogous to a similar fact proved by Arthur in the p-adic case. Langlands had conjectured a more precise form that specified the normalization constants.