MULTIPLICITIES AND PLANCHEREL FORMULA FOR THE SYMMETRIC SPACE $\mathrm{U}(n)\backslash\ \mathrm{GL}_n(E)$

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1. Goal

We consider a particular case of what has become called the "relative local Langlands correspondence".

Let E/F be a quadratic extension of p-adic fields. Denote $\mathrm{Gal}(E/F) = \{1, \sigma\}$. Let V be a hermitian space over E of dimension n.

We have an embedding $H = U(V) \hookrightarrow G = \mathrm{GL}_n(E)$.

The goal is to study the "spectrum" of $H\backslash G$. This can mean two things.

- Irreducible representations π of G which admit an embedding $\pi \hookrightarrow C^\infty(H\backslash G)$. These are called "distinguished representations".
- A spectral decomposition of $L^2(H\backslash G)$.

2. Multiplicities

Conjecture 2.1 (Jacquet)**.** *If $\pi \hookrightarrow C^\infty(H\backslash G)$, then $\pi \cong \pi^\sigma$. The converse holds if H is quasi-split.*

This has essentially be proved by Feigon-Lapid-Offen [FLO]. They prove the first statement in full generality, and the second part for generic or unitary π.

We want to consider the multiplicities

$$m(\pi) := \dim \mathrm{Hom}_H(\pi, \mathbf{C}).$$

Remark 2.2. We do not have multiplicity one in this situation; in fact, $m(\pi)$ can be as high as 2^{n-1}.

There is a base-change transfer from $\mathrm{Irr}(G')$ to $\mathrm{Irr}(G)$, by work of Arthur-Clozel. We will see that this gives almost everything you want to know about multiplicities.

Using the Langlands classification, we can equip $\mathrm{Irr}(G)$ with the structure of algebraic varieties. It then turns out that this map is a finite morphism. Then one can define a function $(\deg \mathrm{BC})\colon \mathrm{Irr}(G) \to \mathbf{N}$, which has the propery that

$$(\deg \mathrm{BC})(\pi) = \sum_{\sigma \in \mathrm{BC}^{-1}(\pi)} (\deg \mathrm{BC})(\sigma).$$

This defines a locally function on $\mathrm{Im}\,(\mathrm{BC})$, and for general $\pi \in \mathrm{Im}(\mathrm{BC})$ we have $(\deg \mathrm{BC})(\pi) = |\mathrm{BC}^{-1}(\pi)|$.

Remark 2.3. The work of Arthur-Clozel shows that $\pi \in \mathrm{Im}(\mathrm{BC})$ if and only if $\pi \cong \pi^\sigma$.

Example 2.4. Let $\pi \in \mathrm{Im}(\mathrm{BC})$ be generic. Then by work of Bernstein-Zelevinsky, π can be written as $\delta_1 \times \ldots \times \delta_t$ with δ_i essentially square-integrable, and then $(\deg \mathrm{BC})(\pi) = 2^{\#\{i | \delta_i \cong \delta_i^\sigma\}}$. Furthermore, $\deg \mathrm{BC}(\pi) = |\mathrm{BC}^{-1}(\pi)|$ if and only if the Galois stable δ_i are distinct.

Theorem 2.5 (FLO). *For* $\pi \in \mathrm{Irr}(G)$ *generic, we have*

$$m(\pi) \geq \begin{cases} \lceil \frac{\deg \mathrm{BC}(\pi)}{2} \rceil & H \text{ quasisplit,} \\ \lfloor \frac{\deg \mathrm{BC}(\pi)}{2} \rfloor & H \text{ not quasisplit} \end{cases}$$

Moreover, equality holds whenever $(\deg \mathrm{BC})(\pi) = |\mathrm{BC}^{-1}(\pi)|$.

[FLO] and (independently) Prasad have conjectured that equality always holds.

Theorem 2.6. *This is true: for* $\pi \in \mathrm{Irr}(G)$ *generic,*

$$m(\pi) = \begin{cases} \lceil \frac{\deg \mathrm{BC}(\pi)}{2} \rceil & H \text{ quasisplit,} \\ \lfloor \frac{\deg \mathrm{BC}(\pi)}{2} \rfloor & H \text{ not quasisplit.} \end{cases}$$

3. Plancherel decomposition

Sakellaridis-Venkatesh have made very general conjectures for spherical varieties. It turns out to be convenient to consider also $H' = U(V')$, where $\dim V' = n$ and $V' \not\cong V$ (there are only 2 classes of hermitian spaces). Let $G' = \mathrm{GL}_n(F)$.

Theorem 3.1. *There exists an isomorphism of unitary* G*-representations*

$$L^2(H \backslash G) \oplus L^2(H' \backslash G) \cong \int \mathrm{BC}(\sigma) d\mu_{G'}(\sigma)$$

where $d\mu_{G'}$ *is the Plancherel measure for* G'.

Remark 3.2. This is not a Plancherel decomposition, because BC is not injective. Hence this decomposition is not unique. But we'll later see a refinement that uniquely characterizes the decomposition.

4. FLO functionals

The work of [FLO] constructs invariant functionals.

4.1. **Jacquet-Ye transfer.** Let $N' = N_n(F) \subset G' = \mathrm{GL}_n(F)$ be the standard maximal unipotent. Similarly we let $N = N_n(E) \subset G = \mathrm{GL}_n(E)$. Fix generic characters $\psi_n' \colon N' \to \mathbf{C}^\times$ and $\psi_n \colon N \to \mathbf{C}^\times$.

We will apply the relative trace formula, so we need to introduce relative orbital integrals.

For $f \in C_c^\infty(G)$, $\gamma \in G_{\mathrm{reg}}$ we define

$$O(\gamma, f) = \int_{H \times N} f(h\gamma u)\psi_n(u)\, dh du.$$

For $f' \in C_c^\infty(G')$ and $\delta \in G'_{\mathrm{reg}}$, we define

$$O(\delta, f') = \int_{N' \times N'} f'(u_1 \delta u_2)\psi_n'(u_1 u_2) du_1 du_2.$$

To make a comparison, we need a matching of orbits:

$$H\backslash G_{\mathrm{reg}}/N \hookrightarrow N'\backslash G'_{\mathrm{reg}}/N'.$$

Given this, we introduce the notion of matching functions: we say $f \in C_c^\infty(G)$ matches $f' \in C_c^\infty(G')$ if

$$\Omega(\gamma)O(\gamma, f) = \Omega(\delta)O(\delta, f') \text{ for } \gamma \leftrightarrow \delta.$$

Theorem 4.1 (Jacquet)**.** *Every $f \in C_c^\infty(G)$ matches some $f' \in C_c^\infty(G')$.*

Remark 4.2. There is a converse if you consider H, H' at the same time.

4.2. **The functionals.** By comparison of global relative trace formulas (the Kuznetsove trace formula for G' and the Jacquet-Ye trace formula for G), [FLO] have defined for every $\sigma \in \mathrm{Temp}(G')$ an H-invariant form

$$\alpha_\sigma^H \colon \mathcal{W}(\mathrm{BC}(\sigma)) \to \mathbf{C}$$

where $\mathcal{W}(\mathrm{BC}(\sigma))$ is the Whittaker model of $\mathrm{BC}(\sigma)$. This is characterized by the condition that

$$J^H_{\mathrm{BC}(\sigma)}(f) = I_\sigma(f')$$

for matching f, f', where

$$J^H_{\mathrm{BC}(\sigma)}(f) = \sum_{W \in \mathrm{ON}(\mathcal{W}(\mathrm{BC}(\sigma)))} \alpha_\sigma^H(f \cdot W)\overline{W(1)}$$

with “ON” standing for orthonormal basis (this is called a “relative Bessel distribution”) and

$$I_\sigma(f') = \sum_{W' \in \mathrm{ON}(\mathcal{W}(\sigma))} (f' \cdot W')(1)\overline{W'(1)}$$

(this is called a “Bessel distribution”).

[FLO] prove the inequality by counting the number of independent such functionals.

5. Proof of Theorem 2.6

We reduce to the statement for $\pi \in \mathrm{Temp}(G)$. It is convenient in the proof to consider H and H' together, so define

$$m'(\pi) := \dim \mathrm{Hom}_{H'}(\pi, \mathbf{C}).$$

Since we know a lower bound for each of $m(\pi)$ and $m'(\pi)$, we just need to show an upper bound for the sum: $m(\pi) + m'(\pi) \le (\deg \mathrm{BC})(\pi)$.

Let $X = H\backslash G \coprod H'\backslash G$. Then

$$m(\pi) + m'(\pi) = \dim \mathrm{Hom}_N(C_c^\infty(X)_\pi, \psi_n)$$

where the subscript π refers to the maximal π-isotypic quotient. This is easy, using just Frobenius reciprocity and uniqueness of Whittaker functions.

For $\sigma \in \mathrm{Temp}(G')$, we have $J_\sigma^H, J_\sigma^{H'} \in \mathrm{Hom}_N(C_c^\infty(X), \psi_n)$. Let $J_\sigma = J_\sigma^H + J_\sigma^{H'}$.

The key lemma is:

Lemma 5.1. *We have*

$$\mathrm{Hom}_N(C_c^\infty(X)_\pi, \psi_n) \subset \overline{\langle J_\sigma \colon \sigma \in \mathrm{BC}^{-1}(\Omega_\pi^t)\rangle}$$

where $\Omega_\pi^t \subset \mathrm{Temp}(G)$ *is the connected component of* π*, and the closure is for the weak topology.*

In other words, any $f \in C_c^\infty(X)_\pi$ which is called by all the J_σ is killed by all Whittaker functionals.

Proof. Use the local Kuznetsov trace formula:

$$\int_{N'\backslash G'/N'} O(\delta, f_1')\overline{O(\delta_1, f_2')} = \int_{\mathrm{Temp}(G')} I_\sigma(f_1')\overline{I_\sigma(f_2')} d\mu_{G'}(\sigma).$$

If f_1' match, we can write both sides in terms of G.

$$\int_{X/N} O(x, \varphi_1)\overline{O(x, \varphi_2)} = \int_{\mathrm{Temp}(G')} J_\sigma(\varphi_1)\overline{J_\sigma(\varphi_2)} d\mu_{G'}(\sigma).$$

A localization principle implies that $\langle J_\sigma \mid \sigma \in \mathrm{Temp}(G')\rangle$ is dense in $\mathrm{Hom}_N(C_c^\infty(X), \psi_n)$. □

Let $\mathcal{B}(G') = \mathrm{Spec}\, \mathscr{Z}(G')$ and $\mathcal{B}(G) = \mathrm{Spec}\, \mathscr{Z}(G)$ (where $\mathscr{Z}$ denotes the Bernstein center). We have

$$\begin{array}{ccc} \mathrm{Irr}(G') & \xrightarrow{\mathrm{BC}} & \mathrm{Irr}(G) \\ \downarrow \lambda & & \downarrow \lambda \\ \mathcal{B}(G') & \xrightarrow{\mathrm{BC}} & \mathcal{B}(G) \end{array}$$

By Jacquet's theory of Whittaker functionals, we have

$$\{\sigma \mapsto I_\sigma(f') \mid f' \in C_c^\infty(G')\} = \mathbf{C}[\mathcal{B}(G')] = \{\sigma \in J_\sigma(f) \mid f \in C_c^\infty(X)\}.$$

We denote $\Omega_\pi \subset \mathrm{Irr}(G)$ the connected component of π, and $\Omega_\pi' \subset \mathrm{Irr}(G')$ the pre-image under BC. Let $(\Omega_\pi')^\lambda$ be the image under λ.

By Lemma 5.1, every element of $\mathrm{Hom}_N(C_c^\infty(X)_\pi, \psi_n)$ factors through $C_c^\infty(X) \twoheadrightarrow \mathbf{C}[(\Omega_\pi')^\lambda]$, and then further through the quotient by the maximal ideal $\mathfrak{m}_{\lambda(\pi)}$, whose dimension is $(\deg \mathrm{BC})(\pi)$.

6. Explicit Plancherel

Let $X = H\backslash G \coprod H'\backslash G$. Let $\sigma \in \mathrm{Temp}(G')$. We have a G-invariant semi-definite scalar product

$$(,)_{X,\sigma} \colon C_c^\infty(X) \times C_c^\infty(X) \to \mathbf{C}.$$

This is defined by

$$(\varphi_1, \varphi_2)_{X,\sigma} = \sum_{W \in \mathscr{W}(\mathrm{BC}(\sigma)))} \alpha_\sigma(\varphi_1 \cdot W)\overline{\alpha_\sigma(\varphi_2 \cdot W)}$$

If $\varphi = \varphi^H + \varphi^{H'} \in C_c^\infty(X)$, then

$$\alpha_\sigma(\varphi \cdot W) = \alpha_\sigma^H(\varphi^H \cdot W) + \alpha_\sigma^{H'}(\varphi^{H'} \cdot W).$$

Remark 6.1. We have $(C_c^\infty(X), (,)_{X,\sigma})^\wedge \cong \widehat{\mathrm{BC}(\sigma)}$.

Theorem 6.2. *For every $\varphi_1, \varphi_2 \in C_c^\infty(X)$ we have*

$$(\varphi_1, \varphi_2)_{L^2(G)} = \int_{\mathrm{Temp}(G')} (\varphi_1, \varphi_2)_{X,\sigma} d\mu_{G'}(\sigma).$$

Unitary periods: let k'/k be a quadratic extension of number fields. Let

$$H := U(V) \hookrightarrow G := \mathrm{GL}_{n,k'}.$$

For $\pi \hookrightarrow \mathcal{A}_{\mathrm{cusp}}$, we define

$$P_H \colon \phi \in \pi \mapsto \int_{[H]} \phi.$$

Dually, we have $C_c^\infty(H(\mathbf{A})\backslash G(\mathbf{A})) \to \pi$ given by the composition

$$\varphi \mapsto \Sigma\varphi(-) := \sum_{x \in H(k)\backslash G(k)} \varphi(x-),$$

and then projecting $(\Sigma\varphi) \mapsto (\Sigma\varphi)_\pi$.

It's better to replace $H(\mathbf{A})\backslash G(\mathbf{A})$ by $(H\backslash G)(\mathbf{A}) = X(\mathbf{A})$. It's the (infinite) disjoint union of $H'(\mathbf{A})\backslash G(\mathbf{A})$ where H' runs over unitary groups.

Theorem 6.3 (FLO, Jacquet (n=3))**.** *Let $\varphi = \prod_v \varphi_v \in C_c^\infty(X(\mathbf{A}))$. Then $(\sum \varphi)_\pi = 0$ unless $\pi = \mathrm{BC}(\sigma)$ for $\sigma \hookrightarrow \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{n,k})$, in which case*

$$\langle (\sum \varphi)_\pi, (\sum \varphi)_\pi \rangle_{L^2} = \frac{2}{n} \frac{L^S(1, \sigma, \mathrm{Ad} \otimes \eta)}{L^S(1, \sigma, \mathrm{Ad})} \left(\prod_{v \in S} (\varphi_v, \varphi_v)_{X_v, \sigma_v} + \prod_{v \in S} (\varphi_v, \varphi_v)_{X_v, \sigma_v \otimes \eta_v} \right).$$

where η is the character corresponding to k'/k.