Counting curves using the Fukaya category

Nick Sheridan
Notes by Tony Feng

1 Introduction

I'm going to talk about getting Gromov-Witten invariants (i.e. curve counts) out of the Fukaya category. This is joint work in progress with S. Ganatra and T. Perutz, based on ideas of Barannikov and Kontsevich from the 90s. Although the new content is purely about symplectic invariants, I'm going to talk a lot about *mirror symmetry*.

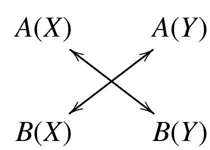
1.1 Mirror symmetry

Mirror symmetry predicts the existence of many "mirror pairs" (X, Y) of Calabi-Yau Kähler manifolds. For a Calabi-Yau manifold X we have two types of invariants:

(a) "A-model." $A(X)$ are *symplectic* invariants of the symplectic manifold (X, ω),

(b) "B-model." $B(X)$ are *algebraic* invariants of the complex manifold (X, J).

Mirror symmetry predicts a correspondence between the A-model of X and the B-model of its mirror pair, and vice versa:



There are several flavors of mirror symmetry.

Enumerative Mirror Symmetry. Here the invariants are numbers. $A(X)$ are the Gromov-Witten invariants, and $B(X)$ are the periods.

Holomorphic Mirror Symmetry. The invariants are homological, namely A_∞-categories. $A(X)$ is the Fukaya category of X and $B(X)$ is $D^b\mathrm{Coh}(X)$.

I'm interested in to what extend the second kind implies the first kind. The bridge is via a third flavor of mirror symmetry, namely "Hodge Mirror Symmetry". Here the invariants are variations of Hodge structure. You assemble the Gromov-Witten invariants into $A(X)$. The passage from Hodge mirror symmetry to enumerative mirror symmetry is classical, and the passage from homological mirror symmetry to Hodge mirror symmetry is our result. ♠♠♠ TONY: [ehh?]

1.2 A running example

Let $X = V(\sum_i z_i^5) \subset \mathbb{P}^4$, a smooth quintic threefold. We will also need a divisor, which we take to be $D = V(\prod_i z_i = 0)$ (the union of the coordinate axes). The mirror predicted by physicists is $Y^5 = \widetilde{Y}^5/G$ where

$$\widetilde{Y}^5 := \{-z_1 \dots z_5 + q \sum_i z_i^5 = 0\} \subset \mathbb{P}^4_{K_B}$$

You can think of this as a family of threefolds parametrized by q, which we think of as living in $K_B := \mathbb{C}((q))$. What is G? We have a $(\mathbb{Z}/5)^5$ action by scaling each variable, but we should quotient by one factor since simultaneous scaling does nothing. Finally, we take the $(\mathbb{Z}/5)^3$ subgroup preserving the monomial $z_1 \dots z_5$.

The invariant n_d is the number of degree d rational curves on X^5. We have $n_1 = 2875$, etc. This can be packaged into an element $\mathrm{Yuk}_A(Q) \in \mathrm{Sym}^3(\Omega^1 M_A)$ where $M_A = \mathrm{Spec}\ K_A$ and $K_A = \mathbb{C}((Q))$. The formula is

$$\mathrm{Yuk}_A(Q\partial_Q, Q\partial_Q, Q\partial_Q) := 4 + \sum_{d \geq 1} n_d d^3 \frac{Q^d}{1 - Q^d}.$$

We can also define $\mathrm{Yuk}_B(q) \in \mathrm{Sym}^3(\Omega^1 M_B)$, where $M_B = \mathrm{Spec}\ K_B = \mathrm{Spec}\ \mathbb{C}((q))$, as before.

Mirror symmetry predicts an isomorphism $\psi\colon M_a \xrightarrow{\sim} M_B$ intertwining Yuk_A with Yuk_B, and it gives an explicit formula for $\psi^*\colon K_B \to K_A$,

$$q \mapsto Q + a_2 Q^2 + a_3 Q^3 + \dots$$

This in turn gives an explicit formula for Yuk_A in terms of Yuk_B, which is defined in terms of Hodge theory. This was proved by Givental.

2 Homological Mirror Symmetry

In 1994, Kontsevich conjectured a form of homological mirror symmetry. Let X be a smooth Calabi-Yau mnaifold and $D \subset X$ a smooth normal crossing divisor, with a technical condition meaning that D "supports enough ample divisors" (the D from our example works).

Definition 2.1. We define the *Fukaya category* $\mathrm{Fuk}(X, D)$ to have

- objects being closed exact Lagrangians $L \subset X \setminus D$ (i.e. the symplectic form restricted to $X \setminus D$ is exact, say $d\alpha$, and we require $\alpha|_L$ to be exact)

- morphisms between L_0 and L_1 are the K_A vector space spanned by $L_0 \cap L_1$ (perhaps perturbed to make this transverse).
- A_∞ structure maps count $u(\mathbb{D}, \partial) \to (X, L_i)$, weighted by $Q^{\mathbb{D} \cdot D} \in K_A$. ♠♠♠ TONY: [???]

Take Y to be a smooth proper Calabi-Yau variety over K_B. Then the category mirror to the Fukaya category is $D^b\mathrm{Coh}(Y)$. This is a K_B-linear DG category (in particular, it's an A_∞ category). Homological mirror symmetry predicts that there exists an identification $\widetilde{\psi}^* \colon K_B \xrightarrow{\sim} K_A$ and a quasi-equivalence of A_∞-categories (which implies an equivalence of ordinary categories)

$$D^b\,\mathrm{Coh}(Y) \cong \widetilde{\psi}^* D^\pi\,\mathrm{Fuk}(X, D).$$

This $D^\pi Fuk(X, D)$ means the "split closure" meaning that we add in all summands, cones, and shifts. The $\widetilde{\psi}^*$ matches up the coefficient fields. Another way to think of it is that the left hand side is a family of varieties parametrized by q and the right hand side is a family of varieties parametrized by Q, and you need to match parameters in some way to identify these notions.

Theorem 2.2 (Sheridan, 2011)**.** *This statement holds for* (X^5, D) *and* Y^5.

Questions.

1. Is $\widetilde{\psi}$ equal to the ψ from Givental's work? (The $\widetilde{\psi}$ is produced by deformation theory, so it is not so clear what it is.)
2. Does this imply enumerative mirror symmetry?

The answer to both is yes, and the proof is via Hodge-theoretic mirror symmetry.

3 Hodge Mirror Symmetry

Since I'm running out of time, I'll stress the ideas over accuracy.

Definition 3.1 (Barannikov). A *variation of semi-infinite Hodge structures* (HSMS) over M is

- A finite-dimensional vector bundle $\mathcal{E}/M$,
- A filtration $\ldots \supset F^p\mathcal{E} \supset F^{p+1}\mathcal{E} \supset \ldots$
- A flat connection on $\mathcal{E}$,
$$\nabla \colon TM \otimes F^p\mathcal{E} \to F^{p-1}\mathcal{E}$$
- A covariant nondegenerate pairing $(\cdot, \cdot)$, with some "sesquilinearity property."

Example 3.2. If $Y \to M_B$ is a family of compact smooth Kähler manifolds, then you get a Gauss-Manin local system $V^B(Y)$ whose fiber over a point is the cohomology of the fiber of Y. This is the B-structure.

Example 3.3. Let X be a Calabi-Yau Kähler manifold. Define $V^A(X) := VSHS/M_A$. Let $\mathcal{E} := H^*(X, K_A)$. Define a connection

$$\nabla_{Q\partial_Q}(\alpha) := Q\partial_Q(\alpha) - [D] \star_Q \alpha.$$

Here $[D] = [\omega]$ and $\star_Q$ is the quantum product.

The filtration is $F^p\mathcal{E} := H^{*\leq -2p}(X; K_A)$ and pairing

$$(\alpha, \beta) = \int_X \alpha \smile \beta.$$

Hodge MS says that there exists an isomorphism

$$\begin{array}{ccc} V^A(X) & \xleftrightarrow{\sim} & V^B(Y) \\ \downarrow & & \downarrow \\ M_A & \xleftrightarrow[\psi]{} & M_B \end{array}$$

respecting all the structures.

Example 3.4. Let C be an A_∞ category over K. Then there is a VSHS over Spec K, the "negative cyclic homology" $HC_\bullet^-(C)$ ♠♠♠ TONY: [ehhh?] if C is proper, homologically smooth, and satisfies the Hodge-de Rham degeneration conjecture.

Theorem 3.5. *There exists a natural map of VSHS*

$$OC : HC_\bullet^-(Fuk(X, D)) \to V^A(X).$$

Remark 3.6. OC stands for "open-closed."

Theorem 3.7. *If (X, D) and Y satisfy "core homological mirror symmetry", meaning that there are full subcategories $A \cong B$, and B split generates in $Fuk(X, D)$ and $D^bCoh(Y)$, where Y is smooth CY with some technical condition that is satisfied in all known cases, then*

1. *the partial identification extends to a fulll identification $D^\pi Fuk(X, D) \cong D^bCoh(Y)$*
2. *the OC from the previous theorem is an isomorphism.*

The hypotheses are known for the quintic. There are also sketch-proofs in other cases, for example everything in the Gross-Siebert program.

The point is that we get a match

$$\begin{array}{ccc} HC_\bullet^-(\text{Fuk}) & \overset{\text{HMS}}{=\!=\!=\!=} & HC_\bullet^-(D^b\,\text{Coh}(Y)) \\ \downarrow{\scriptstyle OC} & & {\scriptstyle \text{conj?}}\downarrow \\ V^A(X) & & V^B(Y) \end{array}$$

The identification ? is known at least partially, but perhaps not completely.