

p -adic Hodge Theory

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1 Classical Hodge Theory

Let X be a compact complex manifold. We discuss three properties of classical Hodge theory.

Hodge decomposition. Hodge's theorem says that if X is Kähler, then there is a natural "Hodge decomposition"

$$H^i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{j=0}^i H^{i-j}(X, \Omega_X^j).$$

This is proved by real analysis, and the main step is to represent de Rham cohomology classes by harmonic forms.

de Rham cohomology. A more general (and much easier) statement is that

$$H^i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H_{\text{dR}}^i(X).$$

This is a consequence of the fact that the holomorphic de Rham complex is exact on a polydisc, hence forms a resolution of the constant sheaf. This uses crucially the "locally trivial" nature of complex manifolds. We bring this up because it will fail in the p -adic setting that I'm about to discuss.

Hodge-de Rham degeneration. These two facts together imply that the Hodge-de Rham spectral sequence

$$E_1^{i,j} = H^j(X, \Omega_X^i) \implies H_{\text{dR}}^{i+j}(X).$$

degenerates at E_1 . This just follows from dimension considerations: any non-trivial differential would force the cohomology to be smaller than we know it to be.

Example 1.1. All projective smooth algebraic varieties over \mathbb{C} have rise to Kähler complex manifolds, and hence have Hodge decompositions. Even if you have only properness, you can say something, e.g. that the Hodge-de Rham sequence degenerates.

We now discuss some examples of non-Kähler complex manifolds.

Example 1.2. (The Hopf surface) Let $q \in \mathbb{C}^*$, $|q| < 1$. Then you do something similar to the uniformization of a complex elliptic curve, but in one dimension higher: define the *Hopf surface*

$$X := (\mathbb{C}^2 \setminus \{(0, 0)\})/q^{\mathbb{Z}}.$$

Here $q^{\mathbb{Z}}$ acts diagonally. This is a totally discontinuous action, so you can take the quotient and obtain a complex manifold. This isn't Kähler; for instance one can compute $H^1(X, \mathcal{O}_X) = \mathbb{C}$ while $H^0(X, \Omega_X^1) = 0$, so Hodge symmetry fails.

However, the Hodge-de Rham spectral sequence still degenerates at E_1 .

Example 1.3. (Iwasawa threefold) This is the "first" example where you don't have degeneration. Let N be the unipotent subgroup of GL_3 .

$$N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}.$$

Then $X := N(\mathbb{C})/N(\mathbb{Z}[i])$. You can project to a smaller object by forgetting the upper right hand corner, and you get two copies of \mathbb{G}_a modulo the lattice $\mathbb{Z}[i] \times \mathbb{Z}[i]$. So the projection is onto an abelian surface, and each fiber is the same (twisted) CM elliptic curve. The twist introduces a non-trivial differential in the Hodge-de Rham spectral sequence.

In particular, X is non-Kähler.

2 p-adic Hodge Theory

2.1 Hodge-Tate decomposition

p -adic Hodge theory is about an analogue of these results for manifolds over p -adic fields (discretely valued, complete non-archimedean extension of \mathbb{Q}_p with perfect residue field k) instead of \mathbb{C} (e.g. a finite extension of \mathbb{Q}_p ; the important thing is that the residue field be perfect).

Let $C = \widehat{\overline{K}}$ be the completed algebraic closure of K . (\overline{K}/K is always infinite, because you can extract roots of the uniformizer, and thus it won't be complete. Krasner's Lemma shows that \mathbb{C}_p is algebraically closed.)

The first result in p -adic Hodge theory is:

Theorem 2.1 (Tate, 1967). *Let A/\mathcal{O}_K be an abelian variety. Then there is a natural $\mathrm{Gal}(\overline{K}/K)$ -equivariant isomorphism*

$$H_{\acute{e}t}^1(A_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \cong (H^1(A, \mathcal{O}_A) \otimes_K C) \oplus (H^0(A, \Omega_A^1)(-1) \otimes_K C).$$

Remark 2.2. This also works a proper smooth curve over \mathcal{O} by considering the Jacobian.

This gives non-trivial results about the action of Galois on étale cohomology. After proving this, Tate asked:

One can ask whether a similar Hodge-like decomposition exists for the étale cohomology with values in C in all dimensions, for a scheme X_C coming from a scheme X projective and smooth over O_K , or perhaps even over K , or suitable rigid analytic spaces.

We now call such a decomposition a *Hodge-Tate decomposition*.

This seems natural in light of the fact that the complex-analytic fact holds for Kähler manifolds more generally than smooth projective varieties, and seems to come from analysis.

Progress.

- Fontaine-Messing (1985) proved a Hodge-Tate decomposition for X a smooth projective scheme over $O_K = W(k)$ (i.e. unramified), and $p > \dim X$. In fact, they proved something much stronger, namely that the étale cohomology is *crystalline*.
- Faltings (1990) proved the result for X a proper smooth scheme over K , completely resolving the question for algebraic varieties.
- Other proofs were given by Tsuji, Niziol, and Beilinson, for algebraic varieties.
- The rigid-analytic case remained open. One reason that this was hopeless was that we not even the finiteness of $H_{\text{ét}}^i(X_C, \mathbb{Z}_p)$ was known!

3 Rigid analytic varieties

3.1 Examples

Let's see how the examples I discussed in the complex case translate here. Let X be a proper smooth rigid-analytic variety over K .

Example 3.1. The analytification of a proper smooth scheme over K .

Example 3.2. You can start with an abelian variety or K3 surface. It is known that there are non-algebraizable deformations, and the generic fiber is a proper smooth rigid-analytic variety.

Example 3.3. There is a rigid analytic Hopf surface; in fact, it's exactly the same construction. Let $q \in K^*$ with $|q| < 1$ and define

$$X := (\mathbb{A}^2 \setminus \{(0, 0)\})/q^{\mathbb{Z}}.$$

Dividing by $q^{\mathbb{Z}}$ is bad in algebraic geometry because this is not a proper discontinuous action, but it is in rigid analytic geometry.

We don't have a definition of Kähler rigid-analytic varieties, but this examples would *not* be Kähler. One reason is that you can compute the cohomology, and it will fail to satisfy Hodge symmetry.

Example 3.4. However, there is no p -adic analogue of the Iwasawa manifold, because \mathbb{C}_p has no cocompact discrete subgroups like $\mathbb{Z}[i] \subset \mathbb{C}$. Consider the subgroup generated by 1: it contains all the powers of p , which are not discrete. So we don't have something that we can "divide out by" to mimic the Iwasawa manifold.

3.2 Rigid-analytic Hodge-Tate decomposition

Theorem 3.5 (Scholze, 2012). *Let X be a smooth rigid-analytic variety.*

1. For all $i \geq 0$, $H_{\text{ét}}^i(X_C, \mathbb{Z}_p)$ is a finitely generated \mathbb{Z}_p -module, which vanishes for $i > 2 \dim X$.
2. There is a natural $\text{Gal}(\overline{K}/K)$ -equivariant Hodge-Tate decomposition

$$H_{\text{ét}}^i(X_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{C} \cong \bigoplus_{j=0}^i H^{i-j}(X, \Omega_X^j)(-j) \otimes_K \mathbb{C}.$$

3. There is a natural $\text{Gal}(\overline{K}/K)$ -equivariant de Rham comparison isomorphism

$$H_{\text{ét}}^i(X_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{dR}} \cong H_{\text{dR}}^i(X) \otimes_K B_{\text{dR}}$$

preserving filtrations; in particular, $H_{\text{ét}}^i(X_C, \mathbb{Q}_p)$ is de Rham in the sense of Fontaine.

4. The Hodge-de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Omega_X^i) \implies H_{\text{dR}}^{i+j}(X).$$

degenerates on E_1 .

Very roughly, one can think of "de Rham" as meaning that there are "enough" Galois invariants.

3.3 Sketch of proof

3.3.1 Finiteness of cohomology

The finiteness for \mathbb{Z}_ℓ -cohomology for $\ell \neq p$ is known. The idea is that any rigid analytic variety admits a formal model, i.e. a formal scheme whose generic fiber is that rigid analytic variety. One way to compute cohomology is to look at the "nearby cycles" and take their cohomology. In the classical case one knows that if ℓ is different from all the relevant characteristics, then forming nearby cycles preserves finiteness properties (i.e. constructibility). This works for any quasicompact separated smooth rigid-analytic variety over k , i.e. without properness. In particular, it works for a *disc*. Then, in analogy to the complex case, you cover your space with discs and patch.

However, this doesn't work for $\ell = p$. In fact, if X is the closed unit disk over K , then $H_{\text{ét}}^1(X_C, \mathbb{F}_p)$ is infinite-dimensional. This is related to non-finiteness of $H_{\text{ét}}^1(\mathbb{A}_k^1, \mathbb{F}_p)$, i.e. the

existence of Artin-Schreier curves. In addition, the étale cohomology groups are critically dependent on the base field C . You embed this into the other group. Roughly, the formal model has special fiber the affine line, and an étale cover of such can be deformed, hence extended to the generic fiber. Said differently, nearby cycles of \mathbb{F}_p contains \mathbb{A}^1 on the special fiber.

This means that we need a *global* argument, using finiteness of coherent cohomology on proper guys to control Artin-Schreier type covers. The problem is that the Artin-Schreier sequence lives in characteristic p , whereas we want to prove something in characteristic 0.

3.4 Local structure of rigid-analytic varieties

Complex manifolds are locally contractible. In contrast, rigid-analytic varieties have large étale fundamental group, even locally, as we just saw. But at least they have no higher homotopy groups:

Theorem 3.6 (Scholze, 2012). *Let X be a connected affinoid rigid-analytic variety over C . Then X is a $K(\pi, 1)$ for p -torsion coefficients, i.e. for all p -torsion local systems \mathbb{L} on X ,*

$$H_{\text{ét}}^i(X, \mathbb{L}) \cong H_{\text{cont}}^i(\pi_1(X, \bar{x}), \mathbb{L}_{\bar{x}})$$

where \bar{x} is a geometric basepoint.

In particular, the étale cohomology of X is some group cohomology.

This is also true and *easy* for X over equal characteristic fields $\overline{k((t))}$. It is enough to consider $\mathbb{L} = \mathbb{F}_p$, and the Artin-Schreier sequence reads

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X \rightarrow 0.$$

As X is affinoid, its higher coherent cohomology vanishes. Hence $H_{\text{ét}}^i(X, \mathbb{F}_p)$ vanishes for $i \geq 2$. Moreover, we get a long exact sequence

$$0 \rightarrow H_{\text{ét}}^0(X, \mathbb{F}_p) \rightarrow R \rightarrow R \rightarrow H_{\text{ét}}^1(X, \mathbb{F}_p) \rightarrow 0$$

where $R = H^0(X, \mathcal{O}_X)$. As the exactness of Artin-Schreier sequence needs only finite étale covers, one gets by the same computation the same result for $H_{\text{cont}}^i(\pi_1(X, \bar{x}), \mathbb{L}_{\bar{x}})$. The point was that the sequence was already exact on the “finite étale site” instead of the full étale site.

To prove the result in general, we reduce the mixed characteristic case to the equal characteristic case using *perfectoid spaces*.

I’ll discuss perfectoid spaces in detail tomorrow, so I won’t go into them too much today. Perfectoid spaces are built from topological algebras, but they are unusual in that you must assume that they have lots of p -power roots. This is encoded in requiring that Frobenius is surjective on the “integral subring,” which will never be the case for finite type stuff.

Definition 3.7. A *perfectoid C-algebra* is a Banach C-algebra R such that the subring of powerbounded elements $R^\circ \subset R$ is bounded and the Frobenius $\Phi: R^\circ/p \rightarrow R^\circ/p$ is surjective.

Theorem 3.8 (Scholze, 2011). *The category of perfectoid C-algebras is canonically equivalent to the category of perfectoid C^b -algebras.*

Again, you can associate “rigid-analytic varieties” (more precisely, adic spaces) to perfectoid C or C^b -algebras, called *affinoid perfectoid spaces* over C , resp. C^b . The crucial result is that this “tilting” induces an equivalence of categories $X \mapsto X^b$, and the étale sites are equivalent: $X_{\text{ét}} \cong X^b_{\text{ét}}$.

The functor from perfectoid C -algebras R to perfectoid C^b -algebras R^b is given by Fontaine’s construction:

$$R^b = \varprojlim_{\phi} (R^\circ/p) \otimes_{\varprojlim_{\phi} \mathcal{O}_C/p} C^b.$$

The slogan is that all topological information passes from characteristic 0 to p , but no coherent information passes through (the linearity structure is obviously incompatible).

Let’s see how this finishes the proof.

- Find affinoid perfectoid space \widetilde{X} over C , so that $\widetilde{X} \rightarrow X$ is an inverse limit of finite étale covers. (You construct \widetilde{X} by iteratively adjoining p -power roots of units.)
- The tilt \widetilde{X}^b is affinoid and lives in equal characteristic, hence is a $K(\pi, 1)$ for p -torsion coefficients.
- As $\widetilde{X}_{\text{ét}} \cong \widetilde{X}^b_{\text{ét}}$, also \widetilde{X} is a $K(\pi, 1)$ for p -torsion coefficients.
- As $\widetilde{X} \rightarrow X$ is pro-finite étale, and taking finite covers doesn’t change the higher homotopy groups, it will also be the case that X is a $K(\pi, 1)$ for p -torsion coefficients.