ALGEBRA QUAL PREP: REPRESENTATION THEORY

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These are hints/solution sketches; they are not a model for what to write on the quals.

1. FALL 2010 M2

Omitted.

2. FALL 2010 A2

(i) For each $g \in G$, we have operators T_g on V and T'_g on V' giving the action of g. The condition for a linear transformation $S: V \to V'$ to be G-equivariant is that

$$T'_g S = S T_g \text{ for all } g \in G.$$

In coordinates, this is a system of linear equations for the coordinates of S, indexed by $g \in G$. The result then follows from the fact that a linear system defined over K has a solution in L if and only if it has a solution in K. In fact more is true: the solution space over L is (solution space over K) $\otimes_K L$.

(ii) The statement about polynomials is proved by induction. The case $N = 1$ is clear, and for the inductive step we write such a polynomial f as a polynomial in $x_1, \dots, x_{N-1}$ with coefficients in $K[x_N]$. For any specialization of x_N, we get the 0 polynomial by induction, hence all of these coefficients vanish individually for every value of x_N, and we conclude by induction.

(iii) We pick bases for V and V', and consider the determinant function on $\text{Hom}_{k[G]}(V, V')$. It is not the zero polynomial since it has a non-zero specialization in L^N, therefore it has a non-zero specialization in K^N.

3. SPRING 2011 A5

(a) Omitted.

(b) We omit the first statement. The character of V^* is the complex-conjugate of the character of V, since the trace of ${}^t g^{-1}$ is the complex conjugate of the trace of g (using that the eigenvalues are sums of roots of unity). Therefore, $V \cong V^*$ is equivalent to the character being real-valued.

4. FALL 2013 M1

(a) It is a direct sum of GL_{d_i}, since by Schur's Lemma there are no non-zero maps between V_i and V_j if $i \neq j$< and only the constant map between V_i and V_i.

(b) If we can write $V = V_1 \oplus V_2$ in a non-trivial way, then $V_1 \otimes V_2$ and $V_2 \otimes V_1$ both contribute to $V \otimes V$, showing that it has at least one irreducible constituent with multiplicity greater than 1.

5. Fall 2011 A2

(i) Omitted.
(ii) Recall that A_4 has 4 irreducible representations, namely 3 characters of order 3 coming from $A_4 \twoheadrightarrow Z/3$ (he 3-dimensional permutation representation. These inflate to irreducible representations of $\mathrm{SL}_2(\mathbf{F}_3)$. There are 3 more representations, of dimensions d_1, d_2, d_3, such that $d_1^2 + d_2^2 + d_3^2 = 12$. So we must have all $d_i = 2$.

6. Spring 2013 A3

(i) One of the earliest examples is A_5, but we'll omit the proof of simplicity.
(ii) The character $\det \rho$ has to be trivial, since otherwise it would inject G into $\mathbf{C}^\times$, forcing G to be abelian. An element of order 2 in G must be sent to a matrix of order 2 of positive determinant, which can only be $\pm\mathrm{Id}$. Since ρ is injective it must be $-\mathrm{Id}$, but since this is central it implies G has a nontrivial center.

7. Fall 2011 M4

(1) Consider characters:

$$\langle \chi, \chi \rangle_H = \frac{1}{|H|} \sum_{h \in H} |\chi(h)|^2 \leq \frac{2}{|G|} \sum_{g \in G} |\chi(g)|^2 = 2.$$

Each irreducible summand of $V|_H$ contributes 1, and the summands must be non-isomorphic or else there would be additional contribution from a "cross-term".

(2) Suppose V is an irreducible representation of G such that $V|_H = V_1 \oplus V_2$. Let $g \in G - H$. Note that gV_1 is an H-representation, since $hgV_1 = g(g^{-1}hg)V_1$. So we must have $gV_1 \cong V_1$ or $gV_1 \cong V_2$. In the first case, V_1 would be a G-subrepresentation of V, which contradicts V being irreducible. In the second case, the previous formula shows that the character of V_1 agrees with the character of gV_1, which contradicts V_1 and V_2 not being isomorphic.

8. Spring 2010 A3

Let $W \subset V$ be a G-invariant subspace. We have a section $\pi \colon V \to W$ which is H-equivariant. Then

$$\frac{|H|}{|G|} \sum_{g \in G/H} g \circ \pi \circ g^{-1} \colon V \to W$$

is a G-equivariant projection.

9. Fall 2011 A3

We proceed by induction. Since p-groups have non-trivial centers, it suffices to show that the center has a fixed subspace. (The fixed subspace of the center is preserved by all of G.) Take an element g of the center. Since it has p-power order, it satisfies $0 = g^{p^n} - 1 = (g-1)^{p^n}$. Since g is unipotent, it has a fixed subspace.

10. Spring 2012 M4

(a) Example: $k[\mathbf{Z}/p]$ for $k = \mathbf{F}_p$. This is an extension of trivial representations, as one finds by using the filtration by the augmentation ideal, but it is obviously not the trivial representation.
(b) This follows from a previous problem.

11. Spring 2016 M3

(a) One definition is $\mathbf{C}[H] \otimes_{\mathbf{C}[G]} \rho$.
(b) Use that the trace of a tensor product is the product of the traces, and the criterion for irreducibility in terms of characters.
(c) By Frobenius reciprocity

$$\mathrm{Hom}(V \boxtimes W, \mathrm{Ind}_G^{G\times G} 1_G) = \mathrm{Hom}_G(V \otimes W, 1_G) = \mathrm{Hom}_G(V, W^*).$$

So this Hom space has dimension 0 or 1, and 1 if and only if $V \cong W^*$ as G-representations.

12. Fall 2016

(i) We use Mackey theory to analyze

$$\mathrm{Hom}_G(\mathrm{Ind}_Q^G \psi, \mathrm{Ind}_Q^G \psi) = \mathrm{Hom}_Q(\psi, \mathrm{Res}_Q^G \mathrm{Ind}_Q^G \psi).$$

Now, $\mathrm{Res}_Q^G \mathrm{Ind}_Q^G \psi$ is a direct sum of ψ^g for $g \in G/Q$, where $\psi^g(q) = \psi(gqg^{-1})$. Thus the induced representation is reducible if and only if $\psi^g = \psi$ for some $g \in G - Q$. Since Q has index p, this implies the result for all g.
(ii) Let V be an irreducible representation of G. Restrict to Q and take an irreducible summand, say ψ. We have a non-zero map $\mathrm{Ind}_Q^G \psi \to V$ by Frobenius reciprocity, which is surjective by irreducibility of V. If $\mathrm{Ind}_Q^G \psi$ is irreducible, this is an isomorphism. Otherwise ψ extends to G, so $\mathrm{Ind}_Q^G \psi = \psi \otimes \mathbf{C}[G/Q]$, which is a direct sum of characters.