Tropical methods in Brill-Noether theory

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1 Introduction

Notation.

- K is a complete discretely valued field,
- X/K is a smooth projective cure of genus g.
- $R \subset K$ is the valuation ring,
- $\mathcal{X}/R$ is a regular *semi*stable model of X.
- G is the dual graph of $\mathcal{X}_s$ (The special fiber).

This talk is about "tropical Brill-Noether theory." The idea is to study algebraic geometry of linear series on X (the smooth general fiber) using combinatorial/piecewise-linear geometry on G.

1.1 Applications.

We want to prove some result which is open on the moduli space of curves. In principle this can be established by exhibiting a single smooth curve with the desired property, but in practice this is impossible. The classical approach is through degeneration, and we perform a different kind of degeneration.

We can (re)prove the existence of curves X (over any algebraically closed fields or complete rationally valued fields) such that

1. (*Brill-Noether Theorem*) $\dim W^r_d(X) = \rho(g, r, d)$ (empty if $\rho < 0$) where

$$W^r_d(X) = \{[L] \in \operatorname{Pic}^d(X) \colon h^0(L) > r\}$$

and

$$\rho(g, r, d) = g - (r + 1)(g - d + r).$$

It is very hard to write down such a curve explicitly.

2. (*Gieseker-Petri Theorem*) The map

$$\Gamma(X, L) \otimes \Gamma(X, L^{-1} \otimes K) \to \Gamma(X, K_X)$$

   is injective for all L. This implies that the variety $G^r_d(X)$, which lives over $W^r_d(X)$ and parametrizes choices of a linear system of dimension r as well, is smooth (at least on the locus away from $G^{r+1}_d(X)$).

3. When $\rho \geq 0$, there exists $L \in W^r_d(X)$ such that $\mathrm{Sym}^2 \Gamma(X, L) \to \Gamma(X, L^{\otimes 2})$ is injective or surjective.

   This is saying that the image of the embedding is contained in the expected number of quadrics.

   More generally, the space of embeddings is irreducible, so there is a "general Hilbert function." This is predicted by the *maximal rank conjecture*, and the statement here is the special case for quadrics.

As mentioned, the existence of a single curve with any of these properties immediately implies it for the general curve in the moduli space.

Tropical proofs were given by

1. Cools-Darisma-Robeva (2012)
2. Jensen (2014)
3. Jensen (2015)

This is *not* a reframing of old arguments. Eisenbud-Harris proved 1 and 2 using the theory of limit linear series and degenerations to curves of compact type. These lie on K3 surfaces, while the maximal rank conjecture fails for such curves. So the degenerations we use are necessarily different.

For example, an open problem is to give an algebro-geometric proof of *tropical* Riemann-Roch.

1.2 Ingredients of the proofs

The input involves:

- Ideas from limit linear series (Harris and Eisenbud)
- Non-archimedean potential theory (Berkovich and Thuillier)
- Metrized complex (Amini, Baker)
- Tropical Abel-Jacobi theory

2 Tropical geometry

Let G be a metric graph, where all edges have length 1. We have $G \subset X^{\mathrm{an}}$ as a skeleton (in the sene of Berkovich). So G is a strong deformation retract of X^{an}.

$$\begin{array}{ccc} & \overset{\pi}{\curvearrowleft} & \\ G & \longrightarrow & X^{\mathrm{an}} \\ & & \Big\downarrow p \\ & & X \end{array}$$

The way these maps play together is important.

- We have $p(G) = \eta_X$.
- G is a subset of the valuations on $K(X)^*$ that extend a valuation on K. Concretely this means that any rational function on X is a regular function on G.
- One way to think okf the analytification is as $\{x \in X(K') \mid K' \mid K \text{ a valued extension}\}$.

Think of the linear series as points moving on the curve. They can even be interpreted as moving on X^{an}, the shadow of which movement is reflected in G. The key is to understand how the movement on G is reflected on X.

Let D be a divisor on X, say $D = \sum a_i x_i$. The tropicalization is $\pi_*(D)$, a divisor on G:

$$\pi_*(D) = \sum a_i \pi(x_i).$$

Topologically the map is proper, so pushforward is nice. Suppose $D \sim D'$. Then we have $D' = D + \mathrm{Div}(f)$ for some $f \in K(X)^*$. We get a map $\mathrm{Trop}(f)\colon G \to \mathbb{R}$ sending $v \mapsto \mathrm{val}_v(f)$. This is piece-wise linear with integer slopes.

Proposition 2.1 (Non-archimedean Poincaré formula)**.** *If* $\mathrm{Div}(f) = \sum b_j x_j$, *then*

$$\pi_* \mathrm{Div}(f) = \sum_{v \in G} \Big(\sum \textit{incoming slopes at } v\Big) v.$$

So if D moves a lot, then there are many rational functions with poles boudned by D, hence many piece-wise linear functions on G that bend in precise ways.

Definition 2.2. If $\psi\colon G \to \mathbb{R}$ is a piece-wise linear function with integer slopes, then

$$\mathrm{Div}(\psi) = \sum_v (\text{incoming slope at } v) \cdot v.$$

Definition 2.3. A *divisor* on G is a formal sum $\sum a_i v_i$ with $a_i \in \mathbb{Z}$, $v_i \in G$. We say that

- $\deg D = \sum a_i$
- D is *effective* if $a_i \geq 0$,

- We have an equivalence relation $D \sim D'$ if and only if $D - D' = \mathrm{Div}(\psi)$.

We define $\mathrm{Pic}(G) = \mathrm{Div}(G)/\sim$, whose degree 0 piece is $\mathrm{Alb}(G)$. This is a compact torus of dimension $h^1(G)$.

The theory is quite similar to that of linear series in algebraic geometry. What we need is a theory of the rank or dimension of a linear series of a divisor. I've been trying to hint that this should have to do with how the divisor moves.

Definition 2.4. The *rank* $r(D)$ is the largest integer r such that $D - E$ is effective for all effective E of degree r.

This is completely analogous to the dimension of linear series in algebraic geometry.

Finally, we need a notion of canonical divisor.

Definition 2.5 (SW Zhang, 1993). We define

$$K_G := \sum_v (\deg v - 2) \cdot v.$$

The inner degree is valence. You can check that $\deg K_G = 2h^1(G) - 2$.

Proposition 2.6 (Baker)**.** *We have $\pi_* K_X \sim K_G$.*

Theorem 2.7 (Tropical Riemann-Roch)**.** *We have*

$$r(D) - r(K - D) = \deg D + 1 - h^1(G).$$

Specialization. The pushforward $\pi_* \colon \mathrm{Div}(X) \to \mathrm{Div}(a)$ respects equivalence, hence descends to the Picard group: $\pi_* \colon \mathrm{Pic}(X) \to \mathrm{Pic}(G)$. The specialization theorem says that

$$\boxed{r(\pi_* D) \geq r(D).}$$

We hinted at this before, and if you try to make this precise you can probably prove it.

3 Brill-Noether Theorem

Definition 3.1. Define $w^r_d(G)$ is the largest integer w such that for every effective E of degree $r + w$, there exists D such that $r(D) \geq r$, $\deg D = d$, and $D - E$ is effective.

Proposition 3.2. *We have $w^r_d(G) \geq \dim W^r_d(X)$.*

To prove Brill-Noether, we just have exhibit a single curve with $w^r_d(G) \geq \rho(X)$.

Theorem 3.3 (CDPR 2012)**.** *Let G be a chain of loops with generic edge lengths. Then $w^r_d(G) = \rho(g, r, d)$.*