NEARBY CYCLES OF PARAHORIC SHTUKAS, AND A FUNDAMENTAL LEMMA FOR BASE CHANGE

TONY FENG

ABSTRACT. Using the Langlands-Kottwitz paradigm, we compute the trace of Frobenius composed with Hecke operators on the cohomology of nearby cycles, at places of parahoric reduction, of perverse sheaves on certain moduli stacks of shtukas. Following an argument of Ngô, we then use this to give a geometric proof of a base change fundamental lemma for parahoric Hecke algebras for GL_n over local function fields. This generalizes a theorem of Ngô, who proved the base change fundamental lemma for spherical Hecke algebras for GL_n over local function fields, and extends to positive characteristic (for GL_n) a fundamental lemma originally introduced and proved by Haines for p-adic local fields.

CONTENTS

1. INTRODUCTION

1.1. **Motivation.** There are two main goals of this paper:

   (1) To compute the trace of Frobenius composed with Hecke operators on the cohomology of nearby cycles at places of *parahoric* reduction for certain moduli stacks of shtukas, and
   (2) To parlay the resulting formulas into a *geometric* proof of a fundamental lemma for base change for central elements in parahoric Hecke algebras over local function fields.

The first goal is accomplished by using the Grothendieck-Lefschetz trace formula to break up the computation of the trace into two pieces: (1) counting points on certain moduli spaces, and (2) understanding the stalks of the nearby cycles sheaves. These pieces are then each resolved by a sequence of technical steps whose overall strategy is rather well-known, and which would require a considerable amount of

notation to describe. Therefore, in this introduction we will focus on informally explaining the idea of the second goal.

The fundamental lemma of interest was proposed and proved by Haines [Hai09] for p-adic (i.e. mixed characteristic) local fields, and generalizes the fundamental lemma for base change for spherical Hecke algebras proved (independently) in the p-adic case by Clozel [Clo90] and Labesse [Lab90], building on work of Kottwitz [Kot86a], and in the function field case (for GL_n) by Ngô [Ngo06].

The original motivation for this fundamental lemma was to study the cohomology of a Shimura variety with parahoric level structure, and in particular to determine the semisimple zeta factor at a place of parahoric reduction. The fundamental lemma enters in comparing the trace of Frobenius and Hecke operators on this cohomology with the geometric side of the Arthur-Selberg trace formula. We refer the interested reader to [Hai09], especially p. 573, for more details.

The same applications are available in the function field setting, with Shimura varieties replaced by the *moduli stacks of shtukas*, which have been utilized by Drinfeld ([Dri87], for GL_2), L. Lafforgue ([Laf02], for GL_n), and V. Lafforgue ([Laf18], for general reductive groups) to spectacular success towards the global Langlands correspondence over function fields.

However, in this paper we have chosen to emphasize the *geometric* aspect of the fundamental lemma, rather than its applications to the Langlands program. In contrast to the proof of [Hai09] for the p-adic case, which following in the tradition of [Clo90] and [Lab90] is via p-adic harmonic analysis, our proof works by exploiting additional geometry and structure which is available in the function field setting. Our strategy is very much based on that of [Ngo06], and indeed specializes to it in the case of spherical Hecke algebras.

Broadly speaking, the base change fundamental lemma compares an orbital integral with a twisted orbital integral. To elaborate, let F be a local field, G a reductive group over F, $\gamma \in G(F)$, and f a function on $G(F)$. The *orbital integral* corresponding to this data is

$$\mathrm{O}_\gamma(f) := \int_{G(F)/G_\gamma(F)} f(g^{-1}\gamma g) \, dg \tag{1.1}$$

where $G_\gamma(F)$ is the centralizer of γ in $G(F)$. We will take f to be in an appropriate Hecke algebra $\mathcal{H}_G(F)$. (Of course we also need to discuss the normalization of Haar measures, but let us leave that for later, in §1.3.)

Let E/F be an unramified extension of degree r, $\delta \in G(E)$, and f_E a function on $G(E)$. The *twisted orbital integral* corresponding to this data is

$$\mathrm{TO}_{\delta\sigma}(f_E) := \int_{G(E)/G_{\delta\sigma}(F)} f_E(g^{-1}\delta\sigma(g)) \, dg \tag{1.2}$$

where $\sigma \in \mathrm{Gal}(E/F)$ is the lift of (arithmetic) Frobenius, and

$$G_{\delta\sigma}(F) = \{g \in G(E) : g^{-1}\delta\sigma(g) = \delta\}$$

is the twisted centralizer of γ in $G(E)$. Again, we will take f_E to be in an appropriate Hecke algebra $\mathcal{H}_G(E)$.

If $\mathcal{H}_{G(E),J}$ and $\mathcal{H}_{G(F),J}$ are corresponding *parahoric* Hecke algebras, then there is a base change homomorphism for their centers

$$b \colon Z(\mathcal{H}_{G(E),J}) \to Z(\mathcal{H}_{G(F),J}).$$

There is also a norm map N from stable twisted conjugacy classes in $G(E)$ to stable conjugacy classes in $G(F)$.

In the special case $G = \mathrm{GL}_n$, the base change fundamental lemma for the center of parahoric Hecke algebras predicts that for σ-regular, σ-semisimple $\delta \in G(E)$ and $f_E \in Z(\mathcal{H}_{G(E),J})$, we have

$$\mathrm{TO}_{\delta\sigma}(f_E) = \mathrm{O}_{N(\delta)}(b(f_E)). \tag{1.3}$$

This is almost what we will prove. (For more general G, the formulation is more complicated; see [Hai09], Theorem 1.0.3 and §5.)

1.2. **The idea of the proof.** Now we can describe our strategy of proof of (1.3). The starting point is the seminal work of Kottwitz on counting points of Shimura varieties over finite fields. In [Kot92] Kottwitz proves a formula expressing the trace of Frobenius composed with a Hecke operator on the cohomology of certain PEL Shimura varieties as a sum of a product of (twisted) orbital integrals:

$$\mathrm{Tr}(h \circ \mathrm{Frob}_p, H^*(\mathrm{Sh}_K, \overline{\mathbf{Q}}_\ell)) = \sum (\dots) \mathrm{O}_\gamma(h^p) \mathrm{TO}_{\delta\sigma}(h_p) \tag{1.4}$$

where Sh_K is an appropriate Shimura variety and h is a Hecke operator. In fact the purpose of the fundamental lemma is to re-express the twisted orbital integrals in (1.4), so as to be able to compare the expression with the geometric side of the Arthur-Selberg trace formula. But in this paper we adopt an opposite perspective, instead viewing (1.4) as *giving a geometric interpretation of (twisted) orbital integrals* (in the p-adic case) in terms of the cohomology of Shimura varieties.

In the function field setting, which is the one of interest to this paper, one can prove an analogous formula of the form

$$\mathrm{Tr}(h_A \circ \mathrm{Frob}_{x_0} \circ \tau, H^*(\mathrm{Sht}_A, \mathcal{A})) = \sum (\dots) \mathrm{O}_\gamma(h_A^{x_0}) \mathrm{TO}_{\delta\sigma}(h_{A,x_0}) \tag{1.5}$$

for an appropriate moduli stack Sht_A, an appropriate sheaf $\mathcal{A}$, an appropriate Hecke operator h_A, and an additional symmetry τ. (Roughly, τ is a "rotation" symmetry that arises from the moduli problem.)

However, it turns out that we can *also* construct a moduli stack Sht_B such that

$$\mathrm{Tr}(h_B \circ \mathrm{Frob}_{x_0} \circ \tau, H^*(\mathrm{Sht}_B, \mathcal{B})) = \sum (\dots) \mathrm{O}_\gamma(h_B^{x_0}) \mathrm{O}_{\mathrm{Nm}(\sigma)}(b(h_{B,x_0})) \tag{1.6}$$

for an appropriate sheaf $\mathcal{B}$, an appropriate Hecke operator h_B, and an additional symmetry τ similar to that from (1.5). The crucial point is that in (1.6) the twisted orbital integral is replaced with the orbital integral of a base changed function.

We remark that the computations (1.5) and (1.6) were obtained in [Ngo06] for places of good (hyperspecial) reduction, in which case one finds a spherical Hecke operator. In the present work, which concerns places of parahoric bad reduction, the analogous computations (1.5) and (1.6) are of independent interest, and actually form the main content of this paper. They require several nontrivial inputs, including, for the parahoric setting that we study here, a version of the Kottwitz Conjecture for shtukas, as well as a geometric interpretation of the base change homomorphism for Hecke algebras. Nevertheless, let us elide these points for now.

The upshot is that (1.5) and (1.6) translate the problem of comparing orbital integrals and twisted orbital integrals into comparing (the cohomology of) two different moduli spaces Sht_A and Sht_B. (We remark that the relationship we seek turns out to be subtler than equality, but again we elide this issue for now.) To do this, we realize Sht_A and Sht_B as specializations of "bigger" moduli spaces $\widetilde{\mathrm{Sht}}_A$

and $\widetilde{\mathrm{Sht}}_B$. We then apply the Langlands-Kottwitz method to the spaces $\widetilde{\mathrm{Sht}}_A$ and $\widetilde{\mathrm{Sht}}_B$, obtaining formulas analogous to (1.5) and (1.6), but the crucial point is that over "many" points of these larger moduli spaces (necessarily away from Sht_A and Sht_B, the original spaces of interest), the output of the Langlands-Kottwitz method has *no* twisted orbital integrals, hence does not require any fundamental lemma to compare. We then deduce the desired equality over the specializations to Sht_A and Sht_B by a continuation principle. The key to making this strategy work is a strategic design of the moduli spaces Sht_A and Sht_B, which we take from [Ngo06].

1.3. **Statement of the base change fundamental lemma.** We now give a precise formulation of the fundamental lemma of interest. It is an exact analogue for local function fields of the fundamental lemma studied in [Hai09]. We will impose several assumptions that simplify the formulation, referring the general case to [Hai09]. In particular we let G be a reductive group over a local field F, and assume that G is unramified and G_{der} is simply connected.

1.3.1. *Normalization of Haar measures.* Recall the notation of §1. To give a well-defined meaning to the orbital integral (1.1) and twisted orbital integral (1.2), we need to specify Haar measures on G, G_γ and $G_{\delta\sigma}$. We assume that γ is regular semisimple.

   We fix a hyperspecial vertex and an alcove containing it in the Bruhat-Tits building for G over F. By Bruhat-Tits theory this induces maximal compact subgroups $K_F \subset G(F)$ and $K_E \subset G(E)$.

- We pick the left-invariant Haar measures dg on $G(F)$ and $G(E)$ such that $dg(K_F) = 1$ and $dg(K_E) = 1$.
- We pick the left-invariant Haar measures dh on $G_\gamma(F)$ and $G_{\delta\sigma}(F)$ such that $dg(K_F \cap G_\gamma(F)) = 1$ and dh on $G_{\delta\sigma}(E)$ is the canonical transfer of Haar measure from G_γ to its inner form $G_{\delta\sigma}$. ♠♠♠ TONY: [put a reference to Kottwitz]

   Taking the quotient measure $\frac{dg}{dh}$ on $G(F)/G_\gamma(F)$ and $G(E)/G_{\delta\sigma}(E)$, now (1.1) and (1.2) have been fully defined.

1.3.2. *Parahoric Hecke algebras.* We now fix a facet in the given alcove whose closure contains the fixed hyperspecial point, which induces corresponding (compact open) parahoric groups $J_F \subset G(F)$ and $J_E \subset G(E)$. Let $\mathcal{H}_{G(F),J} = \mathrm{Fun}_c(J_F \backslash G(F)/J_F, \overline{\mathbf{Q}}_\ell)$ and $\mathcal{H}_{G(E),J} = \mathrm{Fun}_c(J_E \backslash G(E)/J_E, \overline{\mathbf{Q}}_\ell)$ be the corresponding parahoric Hecke algebras. (Parahoric Hecke algebras are discussed in more detail in §4.1.)

1.3.3. *The base change homomorphism.* Let $Z(\mathcal{H}_{G(F),J})$ be the center of $\mathcal{H}_{G(F),J}$, and define $Z(\mathcal{H}_{G(E),J})$ similarly. There is a base change homomorphism

$$b \colon Z(\mathcal{H}_{G(E),J}) \to Z(\mathcal{H}_{G(F),J}),$$

which is defined in §6.1. To give a brief characterization of the base change homomorphism: under the Bernstein isomorphism

$$- *_J \mathbb{I}_K \colon Z(\mathcal{H}_{G(F),J}) \xrightarrow{\sim} \mathcal{H}_{G(F),K}$$

obtained by convolving with the indicator function $\mathbb{I}_K$, it corresponds to the usual base change homomorphism for spherical Hecke algebras.

$$
\begin{array}{ccc}
Z(\mathcal{H}_{G(E),J}) & \xrightarrow{\ b\ } & Z(\mathcal{H}_{G(F),J}) \\
{\scriptstyle\sim}\big\downarrow{\scriptstyle -*_J \mathbb{I}_K} & & {\scriptstyle\sim}\big\downarrow{\scriptstyle -*_J \mathbb{I}_K} \\
\mathcal{H}_{G(E),K} & \xrightarrow{\ b\ } & \mathcal{H}_{G(F),K}
\end{array}
$$

1.3.4. *The norm map.* Let $\sigma \in \mathrm{Gal}(E/F)$ be a lift of (arithmetic) Frobenius. The "concrete norm"

$$\mathrm{Nm}_{E/F} \colon G(E) \to G(F)$$

defined by

$$\mathrm{Nm}_{E/F}(\delta) := \delta \cdot \sigma(\delta) \cdot \ldots \sigma^{r-1}(\delta)$$

descends to a norm map

$$N \colon G(E)/\text{stable } \sigma\text{-conjugacy} \to G(F)/\text{stable conjugacy}.$$

1.3.5. *Formulation of the fundamental lemma.* The following fundamental lemma was proved by Haines in the p-adic setting [Hai09, Theorem 1.0.3].

Theorem 1.1 (Haines)**.** *Let E/F be an unramified extension of p-adic local fields of degree r and residue characteristic p. Let $\psi \in Z(\mathcal{H}_{G(E),J})$ and $\delta \in G(E)$ such that $N(\delta)$ is semisimple. Then we have*

$$\mathrm{SO}_{\delta\sigma}^{G(E)}(\psi) = \mathrm{SO}_{N(\delta)}^{G}(b(\psi)).$$

Here SO are *stable (twisted) orbital integrals*, for whose definition we refer to [Hai09, §5.1]. Since our eventual result will be for $G = \mathrm{GL}_n$, where stable conjugacy coincides with conjugacy, we can ignore the issue of stabilization.

Remark 1.2. Haines has informed us that his proof, which is based on the global simple trace formula and Kottwitz's stabilization of the twisted trace formula, does not carry over (at least, not without nontrivial additional work) to the positive characteristic setting. [1]

1.4. **Statement of results.** We now formulate our main result, which is an extension (in a special case) of Theorem 1.1 to positive characteristic.

By the Bernstein isomorphism, a basis for $Z(\mathcal{H}_{G(E),J})$ is given by the functions ψ_μ for μ a dominant coweight of G, which correspond under $-*_J \mathbb{I}_K$ to the indicator functions of the double coset in $K_E \backslash G(E)/K_E$ indexed by μ.

Example/Definition 1.3. If $G = \mathrm{GL}_n$, and $T \subset \mathrm{GL}_n$ is the usual (diagonal) maximal torus, then we may identify $X_*(T) \cong \mathbf{Z}^n$ in the standard way. The dominant weights coweights $X_*(T)_+$ are those $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. We define

$$|\mu| := \mu_1 + \ldots + \mu_n.$$

In this paper we prove:

---

[1] However, we note that W. Ray Dulany proved the base change fundamental lemma for GL_2 by hand in the function field case [RD]. We thank Tom Haines for informing us about Ray Dulany's work.

Theorem 1.4. *Let E/F be an unramified degree r extension of characteristic p local fields. If $\delta \in \mathrm{GL}_n(E)$ is such that $N(\delta)$ is regular semisimple and separable, and $\psi \in Z(\mathcal{H}_{\mathrm{GL}_n(E),J})$ is a linear combination of ψ_μ with $|\mu| = 0$, then we have*

$$\mathrm{TO}_{\delta\sigma}(\psi) = \mathrm{O}_{N(\delta)}(b(\psi)).$$

Remark 1.5. Let us make some remarks on the hypotheses. The hypothesis $|\mu| = 0$ arises geometrically as a condition for the non-emptiness of moduli stacks of shtukas. It can be interpreted as saying that ψ comes from the Hecke algebra of SL_n.

The restriction to GL_n comes from a need to obtain a proper moduli stack, in order to have enough control over the cohomology of the relevant moduli stacks of shtukas. In general the moduli stacks of shtukas are of infinite type, and their cohomology not constructible. However, for GL_n we can use the trick of globalizing to a division algebra in order to create a proper global space with the right local behavior.

Remark 1.6. As T. Haines pointed out to us, another key aspect of the fundamental lemma is the assertion that $\mathrm{O}_\gamma(b(\psi)) = 0$ if γ is not a norm. Our strategy does not seem to naturally give access to this statement. On the other hand, since Labesse gave a *purely local* proof of this statement for the spherical case in mixed characteristic, which was extended to the center of parahoric Hecke algebras in [Hai09] §5.2, it should generalize to positive characteristic.

1.4.1. *Related work.* The fundamental lemma for base change for *spherical* Hecke algebras, which arises from Theorem 1.1 in the special case where $J = K$ is a hyperspecial maximal compact subgroup, was proved in the p-adic case by Clozel [Clo90] and Labesse [Lab90], using key input from Kottwitz [Kot86a] who checked it for the unit element. These arguments were generalized by Haines to proved the base change fundamental lemma for centers of parahoric Hecke algebras, as has been discussed.[2]

For local function fields (i.e. positive characteristic), the spherical case $J = K$ of Theorem 1.4 was established by Ngô [Ngo06], also for GL_n and also for $|\mu| = 0$ (with the same reasons for the restrictions). Indeed, our strategy as described in §1 is the one pioneered by [Ngo06]. Similar results were obtained independently and simultaneously by Lau [Lau04].

1.5. **Summary of the paper.** Although our current argument does not work beyond GL_n, we hope that after future technical improvements in the theory of shtukas it can be generalized to a much wider class of reductive groups. For this reason, for the individual steps we have tried to work with more general groups when possible. It seems worthwhile to give a brief outline of the organization of the paper, pointing out exactly where we can be more general.

In §3 we review the theory of shtukas for nonconstant reductive group schemes, summarizing the essential background facts. Also, a key point is to define an "integral model" for the moduli stack of shtukas, extending over points of parahoric bad reduction.

---

[2]See the last paragraph of the introduction to [Hai09] for a discussion of to what extent Theorem 1.1 follows from the fundamental lemma for twisted endoscopy. Although base change is a special case of twisted endoscopy, the fundamental lemma for twisted endoscopy should imply that there is a matching function for ψ, but does not identify it in terms of the base change homomorphism.

In §4 we establish an analogue of the Kottwitz Conjecture for moduli of shtukas. This works even for fairly general (not necessarily constant) reductive groups $\mathcal{G} \to X$: Gaitsgory originally proved it for constant groups, and his argument was generalized by Zhu in [Zhu14, Theorem 7.3] and Pappas-Zhu in [PZ13].

In §5 we establish some counting formulas for points of shtukas over finite fields. This is a minor variant of the work of Ngô B.C. and Ngô Dac T., which was previously only formulated at places of hyperspecial level structure. Our contribution is to write it out for the case of parahoric bad reduction that we require.

In §6 we provide a geometric interpretation of the base change homomorphism for spherical Hecke algebras and the center of parahoric Hecke algebras for general split reductive groups G over a local field. For GL_n this was proved by Ngô, in a formulation that was rather specific to GL_n. We generalize the argument to spherical Hecke algebras for arbitrary split reductive groups, and then use that to deduce a result for (central elements in) parahoric Hecke algebras.

In §7 we introduce the two moduli problems Sht_A and Sht_B which are to be compared, and recall Ngô's theorem stating the precise comparison. Using this we deduce an equality of traces on the nearby cycles sheaves at the point of parahoric reduction. Here we also crucially use that the moduli of shtukas associated to a sufficiently ramified division algebra is proper, which implies that the cohomology is a local system.

In §8 we compute these traces in terms of (twisted) orbital integrals, giving formulas in the paradigm of Kottwitz, and then use them in §8.3 to deduce the cases of the base change fundamental lemma claimed in Theorem 1.4.

1.6. **Acknowledgments.** I am indebted to Zhiwei Yun for teaching me basically everything that I know about shtukas, and in particular for directing me to [Ngo06]. I thank Zhiwei, Laurent Clozel, Gurbir Dhillon, Tom Haines, Jochen Heinloth, Urs Hartl, Bao Le Hung, and Rong Zhou for helpful conversations related to this work, and Brian Conrad and Timo Richarz for comments and suggestions on a draft. I am particularly grateful to Tom Haines and the very thorough referee for crucial corrections and explanations.

This project was conceived at the 2017 Arbeitsgemeinschaft in Oberwolfach, and completed while I was a guest at the Institute for Advanced Study, and under the support of an NSF Graduate Fellowship. I am grateful to these institutions for their support.

2. Notation

We collect some notation that will be used frequently throughout the paper.

- Let X be a smooth projective curve over a finite field $k = \mathbf{F}_q$, and – changing notation from the introduction – let $F = k(X)$ be its global function field for the rest of the paper. We assume that X has a rational point, and fix such a point $x_0 \in X(\mathbf{F}_q)$.
- We will let X° be an open subset of X, usually the complement of some points for ramification and possibly also x_0.
- We denote by $|X|$ the set of closed points of X, and for $x \in X$ we write $k(x)$ for the residue field of x.
- For $x \in X$, we let $\mathcal{O}_x$ be the completion of $\mathcal{O}_{X,x}$ at its maximal, and F_x be the fraction field of $\mathcal{O}_x$. We set $D_x := \operatorname{Spec} \mathcal{O}_x$.

- We let G be a connected reductive group over F, whose derived group is simply connected. We extend G to a parahoric group scheme $\mathcal{G} \to X$ (which is also possible – cf. [Laf18] §12.1), and that $\mathcal{G} \otimes F_x$ is split at all points $x \in X$ where $\mathcal{G}(\mathcal{O}_x)$ is not hyperspecial.
- We let $U \subset X$ be the dense open subscheme where $\mathcal{G}$ is reductive.
- We denote by $\mathcal{E}^0$ the trivial (fppf) $\mathcal{G}$-torsor over X.

3. Moduli of shtukas

In this section we recall material concerning shtukas and their perverse sheaves. This is mostly background, but we emphasize that it is important for us to work at the generality of nonconstant groups. This allows us to define an "integral model" for parahoric shtukas, which is *much* easier than the corresponding problem for Shimura varieties. References for this section are [Zhu14, §3], [Laf18, §12], and [HaRa].

3.1. $\mathcal{G}$-bundles.
Let $\mathcal{G} \to X$ be a smooth affine group scheme with (connected) reductive generic fiber G, such that $\mathcal{G}|_{\mathcal{O}_x}$ is a parahoric group scheme for each $x \in X$. We assume that $\mathcal{G}|_{F_x}$ is split at all points $x \in X$ where $\mathcal{G}(\mathcal{O}_x)$ is not hyperspecial.

3.1.1. We recall the notion of $\mathcal{G}$-bundles and affine Grassmannians, the study of which seems to have been initiated by [PR10].

Definition 3.1. A $\mathcal{G}$-*bundle* $\mathcal{E}$ is a $\mathcal{G}$-torsor for the fppf topology. We define $\mathrm{Bun}_{\mathcal{G}}$ to be the (Artin) stack[3] representing the functor

$$\mathrm{Bun}_{\mathcal{G}} \colon S \mapsto \{\mathcal{G}\text{-bundles on } X \times S\}.$$

Definition 3.2. We define the *global affine Grassmannian* $\mathrm{Gr}_{\mathcal{G}}$ to be the ind-scheme representing the functor

$$\mathrm{Gr}_{\mathcal{G}} \colon S \mapsto \left\{ (x, \mathcal{E}, \beta) \colon \begin{array}{c} x \in X(S) \\ \mathcal{E} \in \mathrm{Bun}_{\mathcal{G}}(S) \\ \beta \colon \mathcal{E}|_{X_S - \Gamma_x} \cong \mathcal{E}^0|_{X_S - \Gamma_x} \end{array} \right\}$$

where here and throughout $\mathcal{E}^0$ denotes the trivial $\mathcal{G}$-torsor, and Γ_x is the graph of x, viewed as a divisor in $S \times X$.

We have a map

$$\pi \colon \mathrm{Gr}_{\mathcal{G}} \to X$$

sending $(x, \mathcal{E}, \beta) \mapsto x$.

Example 3.3. For any closed point x, the fiber $\mathrm{Gr}_G|_x$ is the partial affine flag variety associated with $G|_{D_x}$, as was studied for example in [PR08]. If $G|_{D_x}$ happens to be *reductive*, then $G|_{D_x} \cong (G|_x) \otimes_{k(x)} \mathcal{O}_x$, where $G|_x := G \times_X x$ is a constant group scheme, hence $\mathrm{Gr}_G|_x$ is the usual affine Grassmannian attached to $G|_x$ over $k(x)$. If on the other hand $G|_{D_x}$ happens to be an *Iwahori* group scheme, then $\mathrm{Gr}_G|_x$ is an affine flag variety.

---

[3]For the fact that this is really an Artin stack in the generality required here, see [Bro13]. We thank Brian Conrad for bringing this reference to our attention.

3.1.2. *Adding level structure.* Let $\mathrm{Bun}_{\mathcal{G},n\Gamma}$ be the moduli stack of $\mathcal{G}$-bundles with "n-th order level structure", i.e.

$$\mathrm{Bun}_{\mathcal{G},n\Gamma} \colon S \mapsto \left\{ (x,\mathcal{E},\psi) \colon \begin{array}{c} x \in X(S) \\ \mathcal{E} \in \mathrm{Bun}_{\mathcal{G}}(S) \\ \psi \colon \mathcal{E}|_{n\Gamma_x} \xrightarrow{\sim} \mathcal{E}^0|_{n\Gamma_x} \end{array} \right\}$$

where $n\Gamma_x$ is viewed as a divisor in $S \times X$.

Let $\mathrm{Bun}_{\mathcal{G},\infty\Gamma}$ be the moduli stack of $\mathcal{G}$-bundles with "infinite level structure", i.e.

$$\mathrm{Bun}_{\mathcal{G},\infty\Gamma} \colon S \mapsto \left\{ (x,\mathcal{E},\psi) \colon \begin{array}{c} x \in X(S) \\ \mathcal{E} \in \mathrm{Bun}_{\mathcal{G}}(S) \\ \psi \colon \mathcal{E}|_{\widehat{\Gamma}_x} \xrightarrow{\sim} \mathcal{E}^0|_{\widehat{\Gamma}_x} \end{array} \right\}$$

where $\widehat{\Gamma}_x$ is the completion of $X \times S$ along Γ_x. One can also think of ψ as a compatible family of level structures over $n\Gamma_x$ as $n \to \infty$. We will use the notation $\widehat{\Gamma}_x^\circ " := \widehat{\Gamma}_x - \Gamma_x"$ with the meaning as in [Laf18, Notation 1.7].

Let $\mathcal{L}^+\mathcal{G}$ be the global "arc group", defined by

$$\mathcal{L}^+\mathcal{G} \colon S \mapsto \left\{ (x,\beta) \colon \begin{array}{c} x \in X(S) \\ \beta \in \mathcal{G}(\widehat{\Gamma}_x) \end{array} \right\}.$$

We clarify that $\mathcal{L}^+\mathcal{G}$ is a pro-algebraic group scheme over X, as it is the restriction of scalars of smooth affine group schemes over X, and $\mathrm{Bun}_{\mathcal{G},\infty\Gamma}$ is an Artin stack of infinite type, as it is and a $\mathcal{L}^+\mathcal{G}$-torsor over $\mathrm{Bun}_{\mathcal{G}}$.

Remark 3.4. There is an action of $\mathcal{L}^+\mathcal{G}$ on $\mathrm{Gr}_{\mathcal{G}}$ by changing the level structure ψ.

3.1.3. *Global Schubert varieties.* Let $T \subset G$ be a maximal torus. In [Ric16, §2] (generalizing work in the tamely ramified case of [Zhu14, §3.3]) it is shown how to associate to $\mu \in X_*(T_{\overline{F}})$ a global Schubert variety $\mathrm{Gr}_{\mathcal{G}}^{\leq\mu}$. Of course, this is well-known in the split case.

3.1.4. *Geometric Satake.* We fix some notation pertaining to the Geometric Satake correspondence [MV07]. For a space Y over X, we denote by $Y|_U$ the fibered product of Y with $U \hookrightarrow X$ (recall that §2 that $U \subset X$ is the locus where $\mathcal{G}$ is reductive, so G is the generic fiber of $\mathcal{G}$.)

Since we want to work over $\mathbf{F}_q$, we need a slightly modified version of the Geometric Satake equivalence – [RZ15] for an explanation of the theory over general fields. Fix ℓ and a choice of $\sqrt{q} \in \overline{\mathbf{Q}}_\ell$, so we get a half Tate twist as in [RZ15, Definition A.1], so that [RZ15, Theorem A.12] applies.

Definition 3.5. Given a finite-dimensional representation W of Langlands' L-group ${}^L G$, we let $\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(W)$ be the associated $\mathcal{L}^+\mathcal{G}$-equivariant (for the action of Remark 3.4) perverse sheaf on $\mathrm{Gr}_{\mathcal{G}}|_U$ furnished by Geometric Satake, in the sense of [Zhu, Proposition 5.5.16]. Note that $\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(W)$ is automatically $\mathcal{L}^+\mathcal{G}|_U$-equivariant. (Strictly speaking, [RZ15] concerns the local affine Grassmannian. For a statement of Geometric Satake for non-constant groups phrased in terms of the Beilinson-Drinfeld Grassmannian, see [Laf18, Theorem 12.16].)

If G is split then irreducible finite-dimensional representations W of ${}^L G = \widehat{G}$ are indexed by dominant coweights $\mu \in X_*(T)_+$ for a maximal split torus $T \subset G$, and we denote by $\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu) := \mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(W_\mu)$ the corresponding perverse sheaf.

This is the primal source for constructing perverse sheaves on a plethora of objects, which will all be denoted $\mathrm{Sat}_{\cdots}(W)$ or $\mathrm{Sat}_{\cdots}(\mu)$.

3.2. **Hecke stacks.**

3.2.1.   We now define objects that geometrize the Hecke operators.

Definition 3.6. We define the *Hecke stack* $\mathrm{Hecke}_{\mathcal{G}}$ by the functor of points

$$\mathrm{Hecke}_{\mathcal{G}} \colon S \mapsto \left\{ (x, \mathcal{E}, \mathcal{E}', \varphi) \colon \begin{array}{c} x \in X(S) \\ \mathcal{E}, \mathcal{E}' \in \mathrm{Bun}_{\mathcal{G}}(S) \\ \varphi \colon \mathcal{E}'|_{X \times S - \Gamma_x} \xrightarrow{\sim} \mathcal{E}|_{X \times S - \Gamma_x} \end{array} \right\}.$$

We have structure maps

$$
\begin{array}{ccc}
 & \mathrm{Hecke}_{\mathcal{G}} & \\
 {}^{h^{\leftarrow}}\swarrow & \downarrow{\scriptstyle\pi} & \searrow^{h^{\rightarrow}} \\
\mathrm{Bun}_{\mathcal{G}} & X & \mathrm{Bun}_{\mathcal{G}}
\end{array}
$$

where the map $h^{\leftarrow}$ takes $(\mathcal{E}, \mathcal{E}') \mapsto \mathcal{E}$, and the map $h^{\rightarrow}$ takes $(\mathcal{E}, \mathcal{E}') \mapsto \mathcal{E}'$.

One can think of the $\mathrm{Hecke}_{\mathcal{G}}$ as looking locally, in the smooth topology, like $\mathrm{Bun}_{\mathcal{G}} \times_X \mathrm{Gr}_{\mathcal{G}}$. To make this precise, recall that there is an action of $L^+\mathcal{G}$ on $\mathrm{Bun}_{\mathcal{G},\infty\Gamma}$, by changing the level structure.

Proposition 3.7. *There is an isomorphism*

$$\xi \colon \mathrm{Hecke}_{\mathcal{G}} \xrightarrow{\sim} (\mathrm{Gr}_{\mathcal{G}} \times_X \mathrm{Bun}_{\mathcal{G},\infty\Gamma})/\mathcal{L}^+\mathcal{G}$$

where the quotient is for the diagonal action of $\mathcal{L}^+\mathcal{G}$.

This is actually taken as the *definition* of the Hecke stack in [Laf18, §12.3.1]. Although it is well-known we have not found the statement formulated in quite this way, so we give a proof for completeness.

Proof. Giving an isomorphism $\mathrm{Hecke}_{\mathcal{G}} \xrightarrow{\sim} (\mathrm{Gr}_{\mathcal{G}} \times_X \mathrm{Bun}_{\mathcal{G},\infty\Gamma})/\mathcal{L}^+\mathcal{G}$ is equivalent to giving an $\mathcal{L}^+\mathcal{G}$-equivariant isomorphism from an $\mathcal{L}^+\mathcal{G}$-torsor over $\mathrm{Hecke}_{\mathcal{G}}$ to $\mathrm{Gr}_{\mathcal{G}} \times_X \mathrm{Bun}_{\mathcal{G},\infty\Gamma}$, so we will construct the latter.

Let $\widetilde{\mathrm{Hecke}_{\mathcal{G}}} \to \mathrm{Hecke}_{\mathcal{G}}$ be the $\mathcal{L}^+\mathcal{G}$-torsor representing $(x, \varphi \colon \mathcal{E}' \dashrightarrow \mathcal{E}) \in \mathrm{Hecke}_{\mathcal{G}}$ *plus* a choice of trivialization $\psi \colon \mathcal{E}|_{\widehat{\Gamma}_x} \cong \mathcal{E}^0|_{\widehat{\Gamma}_x}$.

There is a map

$$\widetilde{\mathrm{Hecke}_{\mathcal{G}}} \to \mathrm{Gr}_{\mathcal{G}} \times_{X^I} \mathrm{Bun}_{\mathcal{G},\infty\Gamma}$$

sending

$$(x, \varphi \colon \mathcal{E}' \dashrightarrow \mathcal{E}, \psi) \mapsto (x, \mathcal{E}', \psi \circ \varphi), (x, \mathcal{E}, \psi)$$

where we have implicitly used the Beauville-Laszlo theorem [Zhu14, Lemma 3.1] to extend $\psi \circ \varphi$, which is a priori only defined on $\widehat{\Gamma}_x^{\circ}$, to $X - \Gamma_x$. It is easily checked that this is an isomorphism, by defining an inverse directly, and that it is $\mathcal{L}^+\mathcal{G}$-equivariant.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 3.8. In practice, we can always translate these statements into ones about locally finite type Artin stacks by considering substacks obtained by bounding the type of the modification, and noting that the action of $\mathcal{L}^+\mathcal{G}$ on such a substack factors through a finite type quotient.

3.2.2. *Geometric Satake for Hecke stacks.*

Definition 3.9. Denoting by $D_c^b(-)$ the bounded derived category, we define a functor

$$\mathrm{Sat}_{\mathrm{Hecke}_{\mathcal{G}}} \colon \mathrm{Rep}_{{}^L G} \to D_c^b(\mathrm{Hecke}_{\mathcal{G}}|_U)$$

as follows (by definition, $D_c^b(\mathrm{Hecke}_{\mathcal{G}})$ is the $\mathcal{L}^+\mathcal{G}$-equivariant constructible derived category of $\mathrm{Gr}_{\mathcal{G}}$ with coefficients in $\mathbf{Q}_\ell$). If $W \in \mathrm{Rep}_{{}^L G}$, then

$$\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(W) \boxtimes \overline{\mathbf{Q}}_{\ell, \mathrm{Bun}_{\mathcal{G}, \infty\Gamma}} \in D_c^b(\mathrm{Gr}_{\mathcal{G}} \times_X \mathrm{Bun}_{\mathcal{G}, \infty\Gamma}|_U)$$

descends to the quotient by $\mathcal{L}^+\mathcal{G}$ by the fact that $\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(W)$ is $\mathcal{L}^+\mathcal{G}$-equivariant. We set $\mathrm{Sat}_{\mathrm{Hecke}_{\mathcal{G}}}(W)$ to be the pullback of this descent via the isomorphism ξ^* from Proposition 3.7.

3.2.3. *Hecke stacks with bounded modification.* For $\mu \in X_*(T_{\overline{F}})$ we define $\mathrm{Hecke}_{\mathcal{G}}^{\leq\mu}$ as follows. First, we have the Schubert variety $\mathrm{Gr}_{\mathcal{G}}^{\leq\mu} \to \mathrm{Gr}_{\mathcal{G}}$, which has an $\mathcal{L}^+\mathcal{G}$-action. This induces a substack of $(\mathrm{Gr}_{\mathcal{G}} \times_X \mathrm{Bun}_{\mathcal{G}, \infty\Gamma})/\mathcal{L}^+\mathcal{G}$, and we define $\mathrm{Hecke}_{\mathcal{G}}^{\leq\mu}$ to be the pullback via ξ^* of Proposition 3.7.

If $\mathcal{G} = G \times X$ is constant and split over X, then up to taking reduced substacks, $\mathrm{Hecke}_{\mathcal{G}}^{\leq\mu}$ admits a very concrete definition as "modifications of G-bundles with invariant bounded by μ". In §3.5 we explicate this for GL_n-bundles, which may be an enlightening example.

3.3. **Shtukas.**

3.3.1. We now define the moduli stack of $\mathcal{G}$-shtukas. At places $x \in X$ where $\mathcal{G}|_{D_x}$ is a parahoric group scheme, this should be thought of as an "integral model" of the usual moduli stacks in which the legs are demanded to be disjoint from the level structure.

Definition 3.10. We define the moduli stack of $\mathcal{G}$-shtukas by the following cartesian diagram

$$
\begin{array}{ccc}
\mathrm{Sht}_{\mathcal{G}} & \longrightarrow & \mathrm{Bun}_{\mathcal{G}} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{Id} \times \mathrm{Frob}} \\
\mathrm{Hecke}_{\mathcal{G}} & \xrightarrow{h^{\leftarrow} \times h^{\rightarrow}} & \mathrm{Bun}_{\mathcal{G}} \times \mathrm{Bun}_{\mathcal{G}}
\end{array}
$$

More explicitly, $\mathrm{Sht}_{\mathcal{G}}$ represents the following moduli problem:

$$\mathrm{Sht}_{\mathcal{G}} \colon S \mapsto \left\{ (x, \mathcal{E}, \varphi) \colon \begin{array}{c} x \in X(S) \\ \mathcal{E} \in \mathrm{Bun}_{\mathcal{G}}(S) \\ \varphi \colon {}^\sigma\mathcal{E}|_{X \times S - \Gamma_x} \xrightarrow{\sim} \mathcal{E}|_{X \times S - \Gamma_x} \end{array} \right\}$$

where σ is the Frobenius on the S factor in $X \times S$, and ${}^\sigma\mathcal{E}$ is the pullback of $\mathcal{E}$ under the map $1 \times \sigma \colon X \times S \to X \times S$.

We have an evident map

$$\pi \colon \mathrm{Sht}_{\mathcal{G}} \to X \quad \text{sending} \quad (x, \mathcal{E}, \mathcal{E}', \varphi) \mapsto x.$$

3.3.2. *Perverse sheaves on shtukas.* From Definition 3.10 we have a tautological map

$$\iota \colon \mathrm{Sht}_{\mathcal{G}} \to \mathrm{Hecke}_{\mathcal{G}} \,.$$

Definition 3.11. For $W \in \mathrm{Rep}(^L G)$, we define $\mathrm{Sat}_{\mathrm{Sht}_{\mathcal{G}}}(W) := \iota^*(\mathrm{Sat}_{\mathrm{Hecke}_{\mathcal{G}}}(W))$. This is a perverse sheaf up to shift on $\mathrm{Sht}_{\mathcal{G}}|_U$, since the affine Grassmannian is étale-locally equivalent to $\mathrm{Sht}_{\mathcal{G}}$ (see §4, or [Laf18, §1.1]).

3.3.3. *Schubert varieties of shtukas.* For $\mu \in X_*(\widehat{T_{\overline{F}}})$, we define $\mathrm{Sht}_{\mathcal{G}}^{\leq \mu} = \iota^* \mathrm{Hecke}_{\mathcal{G}}^{\leq \mu}$. This is a closed substack of $\mathrm{Sht}_{\mathcal{G}}$ which is the support of $\mathrm{Sat}_{\mathrm{Sht}_{\mathcal{G}}}(\mu)$. We call these "Schubert varieties of shtukas" even though they are, of course, not varieties but (Deligne-Mumford) stacks.

3.3.4. *Hecke operators on shtukas.* There are Hecke correspondences of shtukas that induce Hecke operators on $\mathrm{Sht}_{\mathcal{G}}$, hence on their cohomology.

Definition 3.12. We define $\mathrm{Hecke}(\mathrm{Sht}_{\mathcal{G}})$ to be the moduli stack parametrizing $x, y \in X(S)$ along with a diagram

$$
\begin{array}{ccc}
{}^{\sigma}\mathcal{E} & \dashrightarrow{\ x\ } & \mathcal{E} \\
{\scriptstyle \sigma(\beta)}\big\downarrow{\scriptstyle \sigma(y)} & & {\scriptstyle \beta}\big\downarrow{\scriptstyle y} \\
{}^{\sigma}\mathcal{E}' & \dashrightarrow{\ x\ } & \mathcal{E}'
\end{array}
$$

Here we note:

- $\mathcal{E}$ and $\mathcal{E}'$ are $\mathcal{G}$-torsors on $X \times S$, and ${}^{\sigma}\mathcal{E}$ and ${}^{\sigma}\mathcal{E}'$ are their twists by $1 \times \sigma$.
- The x above the horizontal arrows mean an isomorphism on the complement of Γ_x.
- The y (resp. $\sigma(y)$) next to the vertical arrows means an isomorphism on the complement of Γ_y (resp. $\Gamma_{\sigma(y)}$).
- The map $\sigma(\beta)$ is the twist of β. We emphasize that it is determined by β, rather than being an additional datum.

We evidently have a diagram

$$
\begin{array}{ccc}
\mathrm{Hecke}(\mathrm{Sht}_{\mathcal{G}}) & \longrightarrow & \mathrm{Hecke}_{\mathcal{G}} \\
& {\scriptstyle \pi_2}\searrow \quad \swarrow {\scriptstyle \pi} & \\
& X &
\end{array}
$$

where the horizontal arrow sends this data to $(y, \mathcal{E}, \mathcal{E}', \beta)$, which allows us to define $\mathrm{Hecke}(\mathrm{Sht}_{\mathcal{G}})^{\leq \mu}$ for $\mu \in X_*(T_{\overline{F}})$, and $\mathrm{Sat}_{\mathrm{Hecke}(\mathrm{Sht}_{\mathcal{G}})}(W)$ for $W \in \mathrm{Rep}(^L G)$.

We also evidently have a diagram

$$
\begin{array}{ccc}
& \mathrm{Hecke}(\mathrm{Sht}_{\mathcal{G}}) & \\
{\scriptstyle h^{\leftarrow}}\swarrow & & \searrow {\scriptstyle h^{\rightarrow}} \\
\mathrm{Sht}_{\mathcal{G}} & & \mathrm{Sht}_{\mathcal{G}}
\end{array}
$$

where the arrows $h^{\leftarrow}$ and $h^{\rightarrow}$ send this data to $(x, {}^{\sigma}\mathcal{E} \dashrightarrow \mathcal{E})$ and $(x, {}^{\sigma}\mathcal{E}' \dashrightarrow \mathcal{E}')$ respectively. For $v \in X$, let

$$\mathrm{Hecke}(\mathrm{Sht}_{\mathcal{G}})_v^{\leq \mu} := \pi_2^{-1}(v).$$

A choice of $v \in X$ and $\mu \in X_*(T_{\overline{F}})$ induces a correspondence

$$
\begin{array}{ccc}
& \mathrm{Hecke}(\mathrm{Sht}_{\mathcal{G}})^{\leq \mu}_v & \\
{}^{h^{\leftarrow}}\swarrow & & \searrow^{h^{\rightarrow}} \\
\mathrm{Sht}_{\mathcal{G}} & & \mathrm{Sht}_{\mathcal{G}} \\
{}_{\pi}\searrow & & \swarrow_{\pi} \\
& X &
\end{array}
\qquad (3.1)
$$

which is the analogue of the classical Hecke correspondences.

Definition 3.13. Since $\pi \circ h^{\leftarrow} = \pi \circ h^{\rightarrow}$ and $h^{\leftarrow *}(\mathrm{Sat}_{\mathrm{Sht}_{\mathcal{G}}}(W)) \cong h^{\rightarrow *}(\mathrm{Sat}_{\mathrm{Sht}_{\mathcal{G}}}(W))$, from $\mathrm{Hecke}(\mathrm{Sht}_{\mathcal{G}})^{\leq \mu}_v$ we get a corresponding Hecke operator on $R\pi_! \, \mathrm{Sat}_{\mathrm{Sht}_{\mathcal{G}}}(W) \in D^+(X)$.

3.4. **Iterated shtukas and factorization.** This entire discussion carries through to "iterated" versions of $\mathrm{Gr}_{\mathcal{G}}$, $\mathrm{Hecke}_{\mathcal{G}}$ and $\mathrm{Sht}_{\mathcal{G}}$. We will content ourselves with stating the essentials, leaving the reader to generalize the preceding discussion. (A reference is [Laf18, §1,2].)

3.4.1. *Iterated affine Grassmannian.* The iterated global affine Grassmannian

$$\pi \colon \mathrm{Gr}_{\mathcal{G}} \widetilde{\times} \mathrm{Gr}_{\mathcal{G}} \to X^2$$

is defined by the functor of points

$$
\mathrm{Gr}_{\mathcal{G}} \widetilde{\times} \mathrm{Gr}_{\mathcal{G}} \colon S \mapsto \left\{ (x_1, x_2, \mathcal{E}_1, \mathcal{E}_2, \varphi, \beta) \colon
\begin{array}{c}
x_1, x_2 \in X(S) \\
\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Bun}_{\mathcal{G}}(S) \\
\varphi \colon \mathcal{E}_1|_{X_S - \Gamma_{x_1}} \xrightarrow{\sim} \mathcal{E}_2|_{X_S - \Gamma_{x_1}} \\
\beta \colon \mathcal{E}_2|_{X_S - \Gamma_{x_2}} \xrightarrow{\sim} \mathcal{E}^0|_{X_S - \Gamma_{x_2}}
\end{array}
\right\}
$$

We may denote $\mathrm{Gr}_{\mathcal{G}, X^r} = \mathrm{Gr}_{\mathcal{G}} \widetilde{\times} \ldots \widetilde{\times} \mathrm{Gr}_{\mathcal{G}}$ (r times), although the reader should be warned that this notation is sometimes used elsewhere in the literature to denote a different object. We also have Schubert cells: given $\mu_1, \ldots, \mu_r \in X_*(T_{\overline{F}})$, we can define $\mathrm{Gr}^{\leq (\mu_1, \ldots, \mu_r)}_{\mathcal{G}, X^r}$ in a way that is by now obvious.

3.4.2. *Iterated shtukas.* We now define the iterated shtukas.

Definition 3.14. We define the moduli stack $\mathrm{Sht}_{\mathcal{G}, X^r}$ by the functor of points:

$$
\mathrm{Sht}_{\mathcal{G}, X^r} \colon S \mapsto \left\{
\begin{array}{c}
x_1, \ldots, x_r \in X(S) \\
\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_r \xrightarrow{\sim} {}^{\sigma}\mathcal{E}_0 \in \mathrm{Bun}_{\mathcal{G}}(S) \\
\varphi_i \colon \mathcal{E}_i|_{X \times S - \Gamma_{x_{i+1}}} \xrightarrow{\sim} \mathcal{E}_{i+1}|_{X \times S - \Gamma_{x_{i+1}}}
\end{array}
\right\}
$$

Remark 3.15. The stack of iterated shtukas $\mathrm{Sht}_{\mathcal{G}, X^r}$ can also be defined as a repeated fibered product of $\mathrm{Sht}_{\mathcal{G}}$ over $\mathrm{Bun}_{\mathcal{G}}$, which is more analogous to how we defined $\mathrm{Sht}_{\mathcal{G}}$.

We have an evident map

$$\pi \colon \mathrm{Sht}_{\mathcal{G}, X^r} \to X^r$$

projecting to the datum of $(x_1, \ldots, x_r)$.

We can similarly define $\mathrm{Hecke}_{\mathcal{G}, X^r}$ and $\mathrm{Hecke}(\mathrm{Sht}_{\mathcal{G}, X^r})$. For a tuple $W_1, \ldots, W_r \in \mathrm{Rep}({}^L G)$ we can define a shifted perverse sheaf $\mathrm{Sat}_{\mathrm{Sht}_{\mathcal{G}, X^r}}(W_1, \ldots, W_r)$ using Geometric Satake.

We also have Schubert cells for $\mathrm{Sht}_{\mathcal{G},X^r}$: given $\mu_1, \ldots, \mu_r \in X_*(\widehat{T_{\overline{F}}})$, we can define $\mathrm{Sht}_{\mathcal{G},X^r}^{\leq(\mu_1,\ldots,\mu_r)}$ in a way that is by now obvious.

3.5. $\mathcal{D}$-**shtukas.** One of the main difficulties with $\mathrm{Sht}_{\mathcal{G}}$ is that it is of infinite type in general. To study the fundamental lemma for GL_n, we can globalize to a division algebra instead of the constant group GL_n, which gives us a *proper* moduli problem. We now explain the salient facts about this special case. Since the literature already contains several excellent expositions of the theory of $\mathcal{D}$-shtukas, we will content ourselves with a brief summary. A reference for everything here is [Ngo06, §1]; see [Laf97] or [Lau04] for more extensive treatments.

Let D be a division algebra F of dimension n^2, ramified over a (necessarily finite) set of points $Z \subset X$. We assume that our fixed (rational) point $x_0 \notin Z$, so $D_{x_0} := D \otimes_F F_{x_0} \cong \mathfrak{gl}_n(F_{x_0})$. Later we will need to assume that $\#Z$ is sufficiently large.

We extend $\mathcal{D}$ to an $\mathcal{O}_X$-algebra $\mathcal{D}$ such that $\mathcal{D}_x$ is a maximal order in D_x for all x, and we let $G \to X$ be the associated group scheme of units. Let $\mathcal{G} \to X$ be a a parahoric group scheme which is hyperspecial away from x_0 but perhaps not hyperspecial at x_0; we will be most interested in the case where $\mathcal{G}$ is not hyperspecial at x_0.

3.5.1. *Modification types.* Let $T \subset \mathrm{GL}_n$ be the standard maximal torus. The dominant coweights are
$$X_*(T)_+ \cong \mathbf{Z}_+^n := \{\mu = (\mu^{(1)}, \ldots, \mu^{(n)}) \colon \mu^{(1)} \geq \mu^{(2)} \geq \ldots \mu^{(n)}\}.$$

The *relative position* of lattices in $k[[t]]^n$ is a $\mu \in X_*(T)_+$ determined by the Cartan decomposition
$$\mathrm{GL}_n(k((t))) = \bigcup_{\mu \in X_*(T)_+} \mathrm{GL}_n(k[[t]])t^\mu \, \mathrm{GL}_n(k[[t]]).$$

Let $X^\circ := X - Z - \{x_0\}$. For $x \in X^\circ$, a modification of $\mathcal{G}$-bundles outside x is an isomorphism $\varphi \colon \mathcal{E}'|_{X^\circ - x} \xrightarrow{\sim} \mathcal{E}|_{X^\circ - x}$. Let F_x be the completion of F at x, and $\overline{F}_x = \overline{\mathbf{F}}_q \widehat{\otimes} F_x$, i.e. the completion of the maximal unramified extension F_x. Then $\mathcal{E}'|_{\mathrm{Spec}\,\mathcal{O}_x}$ and $\mathcal{E}|_{\mathrm{Spec}\,\mathcal{O}_x}$ induce two lattices in $\overline{F}_x^{\oplus n}$ by using $\varphi \otimes F_x$ to identify their generic fibers. Since a choice of uniformizer at x induces an isomorphism $\overline{F}_x \cong \overline{\mathbf{F}}_q((t))$, the previous discussion applies so that we can speak of the "relative position" $\mu \in X_*(T)_+$ of these two lattices (it is easily checked to be well-defined, independently of the choice of uniformizer). We will call this the *modification type* of φ.

3.5.2. *The global affine Grassmannian.* Up to taking the reduced substack, we can interpret $\mathrm{Gr}_{\mathcal{G}}^{\leq\mu}|_{X^\circ}$ as the subscheme of $\mathrm{Gr}_{\mathcal{G}}|_{X^\circ}$ parametrizing
$$\beta \colon \mathcal{E}|_{X_S - \Gamma_x} \cong \mathcal{E}^0|_{X_S - \Gamma_x}$$

such that for each geometric point of S, the modification type of β is $\leq \mu$. The sheaf $\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu)$ is the IC sheaf of $\mathrm{Gr}_{\mathcal{G}}^{\leq\mu}$, i.e. the middle extension of the constant weight-zero sheaf on the open cell.

Proposition 3.16. *The sheaf* $\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu)$ *is universally locally acyclic with respect to the morphism* $\mathrm{Gr}_{\mathcal{G}}|_{X^\circ} \to X^\circ$.

Proof. This is [Ngo06, §1.1 Corollaire 6]. Note that Ngô's formulation is slightly different, but is actually deduced from the version that we state, which is the usual formulation in Geometric Satake. □

3.5.3. *The Hecke stack.* The Hecke stack $\mathrm{Hecke}_{\mathcal{G}}^{\leq \mu}|_{X^\circ}$ parametrizes modifications of G-torsors over X°

$$(x, \varphi \colon \mathcal{E}'|_{X \times S - \Gamma_x} \xrightarrow{\sim} \mathcal{E}|_{X \times S - \Gamma_x})$$

with modification type $\leq \mu$ at all geometric points of S.

3.5.4. *Shtukas and iterated shtukas.* The moduli stack $\mathrm{Sht}_{\mathcal{G},(X-Z)^r}$ parametrizes

$$\left\{ \begin{array}{c} x_1, \ldots, x_r \in (X - Z)(S) \\ \mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_r \cong {}^\sigma \mathcal{E}_0 \in \mathrm{Bun}_{\mathcal{G}}(S) \\ \varphi_i \colon \mathcal{E}_i|_{X \times S - \Gamma_{x_{i+1}}} \xrightarrow{\sim} \mathcal{E}_{i+1}|_{X \times S - \Gamma_{x_{i+1}}} \end{array} \right\}$$

The Schubert "variety" $\mathrm{Sht}_{\mathcal{G},(X-Z)^r}^{\leq(\mu_1,\ldots,\mu_r)}$ associated to $(\mu_1, \ldots, \mu_r) \in (X_*(T)_+)^r$ can be interpreted, up to reduced structure, as the substack where the modification type of φ_i is bounded by μ_i at all geometric points of S. For such a tuple we can also form $\mathrm{Sat}_{\mathrm{Sht}_{\mathcal{G},(X-Z)^r}}(\mu_1, \ldots, \mu_r)$ on $\mathrm{Sht}_{\mathcal{G},(X-Z)^r}$, which is perverse up to shift and supported on $\mathrm{Sht}_{\mathcal{G},(X-Z)^r}^{\leq(\mu_1,\ldots,\mu_r)}$.

Proposition 3.17. *The (shifted) perverse sheaf* $\mathrm{Sat}_{\mathrm{Sht}_{\mathcal{G},(X-Z)^r}}(\mu_1, \ldots, \mu_r)|_{(X^\circ)^r}$ *is locally acylic with respect to the morphism* $\pi \colon \mathrm{Sat}_{\mathrm{Sht}_{\mathcal{G},(X-Z)^r}}|_{(X^\circ)^r} \to (X^\circ)^r$.

Proof. This [Ngo06, §1.4 Corollaire 2], but with the same remark as in the proof of Proposition 3.16. □

3.5.5. *Global geometry.* The stack $\mathrm{Sht}_{\mathcal{G}}$ has infinitely many connected components owing to the positive-dimensional center of $\mathcal{G}$. We wish to and can rectify this in the usual way: let $a \in \mathbf{A}_F^\times$ be a non-trivial idele of degree 1, and let $\mathrm{Sht}_{\mathcal{G}}/a^{\mathbf{Z}}$ be the quotient obtained by formally adjoining an isomorphism $\mathcal{E} \cong \mathcal{E} \otimes \mathcal{O}(a)$. Similarly define $\mathrm{Sht}_{\mathcal{G},(X-Z)^r}/a^{\mathbf{Z}}$. We still have the map

$$\pi \colon \mathrm{Sht}_{\mathcal{G},(X-Z)^r}/a^{\mathbf{Z}} \to (X - Z)^r$$

and the Geometric Satake sheaves descend to $\mathrm{Sht}_{\mathcal{G},(X-Z)^r}/a^{\mathbf{Z}}$, which in an effort to curtail increasingly monstrous notation we continue to denote by $\mathrm{Sat}_{\mathrm{Sht}_{\mathcal{G},(X-Z)^r}}(\mu_1, \ldots, \mu_r)$. Furthermore, we still have:

Proposition 3.18 ([Ngo06, §1.4 Corollaire 2]). *The shifted perverse sheaf* $\mathrm{Sat}_{\mathrm{Sht}_{\mathcal{G},(X-Z)^r}}(\mu_1, \ldots, \mu_r)$ *restricted to* $\mathrm{Sht}_{\mathcal{G},(X^\circ)^r}/a^{\mathbf{Z}}$ *is locally acyclic with respect to the map*

$$\pi \colon \mathrm{Sht}_{\mathcal{G},(X^\circ)^r}/a^{\mathbf{Z}} \to (X^\circ)^r.$$

We now prepare to state the properness result for the morphism $\mathrm{Sht}_{\mathcal{G},(X^\circ)^r}/a^{\mathbf{Z}} \to (X^\circ)^r$. Any $\mu \in X_*(T)_+$ can be uniquely written as

$$\mu = \mu^+ + \mu^-$$

where

$$\mu^+ := (\mu_1^+ \geq \ldots \geq \mu_r^+ \geq 0)$$
$$\mu^- := (0 \geq \mu_1^- \geq \ldots \geq \mu_r^-)$$

and for all $1 \leq i \leq r$, we have either $\mu_i = \mu_i^+$ or $\mu_i = \mu_i^-$. We define

$$||\mu|| := \max(|\mu^+|, |\mu^-|).$$

Proposition 3.19 ([Ngo06, §1.6 Proposition 2]). *Let* $(\mu_1, \ldots, \mu_r) \in (X_*(T)_+)^r$. *Suppose that the locus Z of ramification points of $\mathcal{D}$ satisfies*

$$\#Z \geq n^2(||\mu_1|| + \ldots + ||\mu_r||).$$

Then the morphism

$$\pi^\circ \colon \operatorname{Sht}_{\mathcal{G},(X^\circ)^r}^{\leq(\mu_1,\ldots,\mu_r)} / a^{\mathbf{Z}} \to (X^\circ)^r$$

is proper.

We need to extend this result to our integral model $X - Z = X^\circ \cup \{x_0\}$.

Proposition 3.20. *Let* $(\mu_1, \ldots, \mu_r) \in (X_*(T)_+)^r$. *Suppose that the locus Z of ramification points of $\mathcal{D}$ satisfies*

$$\#Z \geq n^2(||\mu_1|| + \ldots + ||\mu_r||).$$

Then the morphism

$$\pi \colon \operatorname{Sht}_{\mathcal{G},(X-Z)^r}^{\leq(\mu_1,\ldots,\mu_r)} / a^{\mathbf{Z}} \to (X - Z)^r$$

is proper.

Proof. Let G be the group scheme isomorphic to $\mathcal{G}$ at places away from x_0 but hyperspecial at x_0. Then Proposition 3.19 applied to Sht_G shows that

$$\pi' \colon \operatorname{Sht}_{G,(X-Z)^r}^{\leq(\mu_1,\ldots,\mu_r)} / a^{\mathbf{Z}} \to (X - Z)^r$$

is proper. The map $\mathcal{G} \to G$ induces a projection

$$\operatorname{pr} \colon \operatorname{Sht}_{\mathcal{G},(X-Z)^r}^{\leq(\mu_1,\ldots,\mu_r)} / a^{\mathbf{Z}} \to \operatorname{Sht}_{G,(X-Z)^r}^{\leq(\mu_1,\ldots,\mu_r)} / a^{\mathbf{Z}}$$

which we claim is proper. It obviously suffices to prove the claim. For that we consider the commutative diagram below, omitting some subscripts and superscripts, etc. for clarity of presentation.

In this diagram the squares with solid arrows are cartesian. Let $\operatorname{Hecke}_{\mathcal{G}}'$ and $\operatorname{Sht}_{\mathcal{G}}'$ denote the fibered products

and

$$
\begin{array}{ccc}
\mathrm{Sht}'_{\mathcal{G}} & \longrightarrow & \mathrm{Bun}_{\mathcal{G}} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{Id}\times\mathrm{Frob}\,\cdot} \\
\mathrm{Hecke}'_{\mathcal{G}} & \xrightarrow{h^{\leftarrow}\times h^{\rightarrow}} & \mathrm{Bun}_{\mathcal{G}}\times\mathrm{Bun}_{\mathcal{G}}
\end{array}
$$

The map $\mathrm{Bun}_{\mathcal{G}}\to\mathrm{Bun}_G$ is proper, hence so is pullback $\mathrm{Hecke}'_{\mathcal{G}}\to\mathrm{Hecke}_G$. Since the map $\mathrm{Hecke}_{\mathcal{G}}^{\leq(\mu_1,\ldots,\mu_r)}\to(\mathrm{Hecke}'_{\mathcal{G}})^{\leq(\mu_1,\ldots,\mu_r)}$ is proper, the map $\mathrm{Sht}_{\mathcal{G}}^{\leq(\mu_1,\ldots,\mu_r)}\to$ $(\mathrm{Sht}'_{\mathcal{G}})^{\leq(\mu_1,\ldots,\mu_r)}$ is also proper. As the map $\to(\mathrm{Sht}'_{\mathcal{G}})^{\leq(\mu_1,\ldots,\mu_r)}\to\mathrm{Sht}_G^{\leq(\mu_1,\ldots,\mu_r)}$ is also proper, so is the composition

$$
\mathrm{Sht}_{\mathcal{G}}^{\leq(\mu_1,\ldots,\mu_r)}\to(\mathrm{Sht}'_{\mathcal{G}})^{\leq(\mu_1,\ldots,\mu_r)}\to\mathrm{Sht}_G^{\leq(\mu_1,\ldots,\mu_r)}\ .
$$

$\square$

4. The Kottwitz Conjecture for shtukas

4.1. **Parahoric Hecke algebras.** Let G be a split reductive group over a non-archimedean local field F_t with uniformizer t. (The splitness assumption is not necessary, but is certainly adequate for our eventual purposes and simplifies the notation significantly.)

4.1.1. *Spherical Hecke algebra.* Let K be a hyperspecial maximal compact subgroup of $G(F_t)$. By Bruhat-Tits theory we may extend G to an integral model over the valuation subring $\mathcal{O}_t\subset F_t$ such that $G(\mathcal{O}_t)=K$.

Let $\mathcal{H}_{G,K}=\mathrm{Fun}_c(K\backslash G(F_t)/K,\overline{\mathbf{Q}}_\ell)$ be the corresponding spherical Hecke algebra. This has several canonical bases, so we fix notation for them. Let $T\subset G$ be a maximal split torus. As is well known, we have a Cartan decomposition

$$
G(F_t)=\bigcup_{\mu\in X_*(T)_+}Kt^\mu K \tag{4.1}
$$

indexed by the dominant coweights $X_*(T)_+\cong\mathbf{Z}^n$, where t^μ is the character such that for a character $\chi\in X^*(T)$, we have $\chi(t^\mu)=t^{\langle\chi,\mu\rangle}$.

Definition 4.1. For $\mu\in X_*(T)_+$, we denote by $f_\mu\in\mathcal{H}_{G,K}$ the indicator function of $Kt^\mu K$.

4.1.2. *Geometrization of spherical Hecke algebra.* A second basis is obtained by interpreting categorifying the Hecke algebra in terms of L^+G-equivariant perverse sheaves on the affine Grassmannian Gr_G. Since we want to work over $\mathbf{F}_q$, we need a slightly modified version of the Geometric Satake equivalence; see [Zhu] for an explanation of the theory over general fields. We summarize the essential points. Recall that $\mathrm{Gr}_G(k_t)=G(F_t)/G(\mathcal{O}_t)$ where k_t is the residue field of F_t. Fixing ℓ and a choice of $\sqrt{q}\in\overline{\mathbf{Q}}_\ell$, so we get a half Tate twist as in [RZ15, Definition A.1], Geometric Satake furnishes a fully faithful symmetric monoidal functor

$$
\mathrm{Rep}(\widehat{G})\to\mathrm{Perv}_{G(\mathcal{O}_t)}(\mathrm{Gr}_G).
$$

The simple objects in $\mathrm{Rep}(\widehat{G})$ are indexed by $\mu\in X_*(T)_+$, and we denote by $\mathrm{Sat}_{\mathrm{Gr}_G}(\mu)$ the corresponding perverse sheaf on Gr_G, which is the IC sheaf of the Schubert variety $\mathrm{Gr}_G^{\leq\mu}$, Tate twisted to be pure of weight zero.

For any $\mathcal{F} \in \mathrm{Perv}_{G(\mathcal{O}_t)}(\mathrm{Gr}_G)$, we have under the function-sheaf dictionary [KW01, §III.12] a trace function $f_{\mathcal{F}}$ on $G(\mathcal{O}_t)\backslash G(F_t)/G(\mathcal{O}_t)$ given by

$$f_{\mathcal{F}}(x) = \mathrm{Tr}(\mathrm{Frob}, \mathcal{F}_{\overline{x}}). \tag{4.2}$$

Definition 4.2. We define ψ_μ to be the trace function associated to $\mathrm{Sat}_{\mathrm{Gr}_G}(\mu)$. (By [RZ15], especially Lemma A.13, this corresponds under the Satake equivalence to the character of the highest weight representation V_μ.)

Definition 4.3. Since $\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu)$ is a $\mathcal{G}(\mathcal{O}_{x_0})$-equivariant sheaf on $\mathrm{Gr}_{\mathcal{G},x_0}$, its stalks are the same on any open Schubert cell $\mathrm{Gr}^{=\nu}$ corresponding to $Kt^\nu K$ in the Cartan decomposition (4.1). We denote this common stalk by $\mathrm{Sat}_{\mathrm{Gr}_G}(\mu)_\nu$.

Lemma 4.4. *We have*

$$\psi_\mu = \sum_{\nu \leq u} \mathrm{Tr}(\mathrm{Frob}, \mathrm{Sat}_{\mathrm{Gr}_G}(\mu)_\nu) f_\nu$$

Proof. This is immediate from the fact that $\mathrm{Sat}_{\mathrm{Gr}}(\mu)_\nu$ is supported on $\mathrm{Gr}^{\leq \mu}$, which is the union of the $\mathrm{Gr}^{=\nu}$ for $\nu \leq \mu$, and the definition of f_ν as the characteristic function on $Kt^\nu K$. $\qquad\square$

We have a Satake isomorphism

$$\mathcal{H}_G(K) \xrightarrow{\sim} R(\widehat{G}) \cong \overline{\mathbf{Q}}_\ell[X_*(T)]^W$$

where W is the Weyl group of T. (The Satake isomorphism is reviewed in §6.2.) Here $R(\widehat{G})$ is the representation ring of $\widehat{G}$, which is generated by the classes of the highest weight representations V_μ.

4.1.3. *Parahoric Hecke algebras.* Let J be a parahoric subgroup of G stabilizing a facet whose closure contains the vertex corresponding to K in the Bruhat-Tits building of $G(F_t)$. Let $\mathcal{H}_{G,J} = \mathrm{Fun}_c(J\backslash G(F_t)/J, \overline{\mathbf{Q}}_\ell)$ be the corresponding parahoric Hecke algebra.

Theorem 4.5 (Bernstein, [Hai09, Theorem 3.1.1]). *Convolution with* $f_0 = \mathbb{I}_K$ *(the identity of* $\mathcal{H}_{G,K}$*) induces an isomorphism*

$$- *_J \mathbb{I}_K : Z(\mathcal{H}_{G,J}) \xrightarrow{\sim} \mathcal{H}_{G,K}.$$

Definition 4.6. For $\mu \in X_*(T)_+$,
  - We denote by f'_μ the unique element of $Z(\mathcal{H}_{G,J})$ such that $f'_\mu * \mathbb{I}_K = f_\mu$.
  - We denote by $\psi'_\mu \in Z(\mathcal{H}_{G,J})$ the unique element such that $\psi'_\mu * \mathbb{I}_K = \psi_\mu \in \mathcal{H}_{G,K}$.

4.1.4. *Geometrization of parahoric Hecke algebras.* There is a geometrization of the parahoric Hecke algebra analogous to §4.1.2, which goes as follows (it will be elaborated on later in §6). Briefly, let $\mathcal{G}$ be the parahoric group scheme corresponding to J by Bruhat-Tits theory. Then the Hecke algebra $\mathcal{H}_{G,J}$ the Grothendieck ring (i.e. the group completion of equivalence classes of objects, with ring structure induced by the convolution) of $\mathrm{Perv}_{\mathcal{G}(\mathcal{O})}(\mathrm{Gr}_{\mathcal{G}})$.

Note that if $J = I$ is an Iwahori subgroup (the stabilizer of full alcove), then $\mathrm{Gr}_{\mathcal{G}}$ is the affine flag variety Fl_G. In general $\mathrm{Gr}_{\mathcal{G}}$ is a partial affine flag variety, which one can think of as a mix between the affine Grassmannian and affine flag variety.

One might ask which sheaves the functions ψ'_μ correspond to. The answer is that they can be realized as nearby cycles of certain global degenerations, and it is the key point underlying this section.

4.2. **Nearby cycles.** We recall the definition and essential (for us) properties of the nearby cycles functor. For a reference, see [Del73, Exposé XIII].

A *Henselian trait* is a triple (S, s, η) where S is a the spectrum of a discrete valuation ring, s is the special point of S and η is the generic point of S. Choose geometric points $\overline{s}$ and $\overline{\eta}$ lying over s and η, respectively. Let $\overline{S}$ be the normalization of S in $\overline{\eta}$. We denote by

$$\overline{i} \colon \overline{s} \to \overline{S}$$
$$\overline{j} \colon \overline{\eta} \to \overline{S}$$

the obvious maps.

Let $f \colon Y \to S$ be a finite type scheme over S. One defines a topos $Y \times_s \eta$ as in [Del73, Exposé XIII §1.2], so that the category $D_c^b(Y \times_s \eta, \overline{\mathbf{Q}}_\ell)$ is the category of $\mathcal{F} \in D_c^b(Y \times_s \overline{s}, \overline{\mathbf{Q}}_\ell)$ together with a continuous $\mathrm{Gal}(\overline{\eta}/\eta)$-action compatible with the $\mathrm{Gal}(\overline{s}/s)$-action on $Y_{\overline{s}}$ via the natural map $\mathrm{Gal}(\overline{\eta}/\eta) \to \mathrm{Gal}(\overline{s}/s)$.

Definition 4.7. Given $\mathcal{F} \in D_c^b(Y_\eta, \overline{\mathbf{Q}}_\ell)$ we define the *nearby cycles* $R\Psi(\mathcal{F}) \in D_c^b(Y \times_s \overline{s}, \overline{\mathbf{Q}}_\ell)$ by

$$R\Psi(\mathcal{F}) := \overline{i}^* R\overline{j}_*(\mathcal{F}_{\overline{\eta}})$$

with the $\mathrm{Gal}(\overline{\eta}/\eta)$-action obtained by transport of structure from that on $\mathcal{F}_{\overline{\eta}}$.

Remark 4.8. When the nearby cycles construction is performed with $S = \mathrm{Spec}\, \mathbf{F}_q[\![t]\!]$, the sheaf $R\Psi(\mathcal{F})$ is a priori only defined over $Y_{\overline{\mathbf{F}}_q}$, but can be descended to $Y_{\mathbf{F}_q}$ by choosing a splitting $\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q) \to \mathrm{Gal}(\overline{\mathbf{F}_q((t))}/\mathbf{F}_q((t)))$. When dealing with nearby cycles on affine Grassmannians (or related objects) this is often what we mean (see [Gai01, Footnote 4 on page 8]). Only after such a descent one can associate a trace function to $R\Psi(\mathcal{F})$. We shall point out when this descent is being used, but as a blanket rule it is necessary every time we wish to talk about a trace function.

Lemma 4.9. *If $f \colon Y \to S$ is proper, then the base change homomorphism*

$$R\Psi f_* \to f_* R\Psi$$

is an isomorphism.

Proof. This is [Del73, Exposé XIII (2.1.7.1)]. $\qquad\square$

Corollary 4.10. *If $f \colon Y \to S$ is proper, then the natural map*

$$H_c^i(Y_{\overline{\eta}}, \overline{\mathbf{Q}}_\ell) \to H_c^i(Y_{\overline{s}}, R\Psi(\overline{\mathbf{Q}}_\ell)).$$

is an isomorphism.

Lemma 4.11. *If $f \colon Y \to S$ is lisse, then the base change homomorphism*

$$f^* R\Psi \to R\Psi f^*$$

is an isomorphism.

Proof. This is [Del73, Exposé XIII (2.1.7.2)]. $\qquad\square$

4.3. **Degeneration to affine flag varieties.** Let X be a smooth curve (not necessarily projective) over $\mathbf{F}_q$ and let $\mathcal{G} \to X$ be a parahoric group scheme, with parahoric level structure at x_0. We consider the global affine Grassmannian for $\mathcal{G}$ as in §3.1:

$$\pi \colon \mathrm{Gr}_\mathcal{G} \to X.$$

Consider the restriction $\mathrm{Gr}_\mathcal{G}|_{D_{x_0}}$, where $D_{x_0} := \mathrm{Spec}\, \mathcal{O}_{x_0}$ is the spectrum of the completed local ring at x_0. We apply the nearby cycles construction of §4.2 and in the form of Remark 4.8, to

- $S = D_{x_0}$, $Y = \mathrm{Gr}_\mathcal{G}|_{D_{x_0}}$, and
- $\mathcal{F} = \mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu)|_{D^*_{x_0}}$, where $D^*_{x_0} = \mathrm{Spec}\, F_{x_0}$ is thought of as a local "punctured disk" around x_0.

This produces a $\mathcal{L}^+\mathcal{G}$-equivariant perverse sheaf $R\Psi(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu)|_{D^\circ_{x_0}})$ on $\mathrm{Gr}_\mathcal{G}|_{x_0}$, which we will abbreviate by $R\Psi(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu))$.

Theorem 4.12 (Gaitsgory [Gai01], Zhu [Zhu14]). *The sheaf $R\Psi(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu)))$ is central.*

Remark 4.13. In the present formulation and level of generality, this theorem is actually due to X. Zhu in [Zhu14, Theorem 7.3]. Gaitsgory proved the first prototype of Theorem 4.12, but working with constant group schemes $\mathcal{G}$, and a slightly different degeneration.

Corollary 4.14. *Assume that $G := \mathcal{G}|_{F_{x_0}}$ is split. Then the trace function (in the sense of (4.2)) associated to $R\Psi(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu)))$ is ψ'_μ (Definition 4.6).*

Remark 4.15. Note that we need to use Remark 4.8 to descend $R\Psi(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu)))$ to $\mathrm{Gr}_\mathcal{G}|_{x_0}$, so that it makes sense to speak of the trace function.

Proof. Since $R\Psi(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu))$ is a $\mathcal{L}^+\mathcal{G}$-equivariant perverse sheaf on $\mathrm{Gr}_\mathcal{G}|_{x_0}$, which is central by Theorem 4.12, we have a priori that its trace function

$$\mathrm{Tr}(\mathrm{Frob}, R\Psi(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu)))$$

lies in $Z(\mathcal{H}_{G, \mathcal{G}(\mathcal{O}_{x_0})}(F_x))$. Since G is split we can extend it to a constant group scheme over D_{x_0}, which we continue to denote G, such that $G(\mathcal{O}_{x_0}) =: K$ is a hyperspecial maximal compact subgroup of $G(F_x)$. Write also $J := \mathcal{G}(\mathcal{O}_{x_0})$ for the parahoric subgroup. By the Bernstein isomorphism (Theorem 4.5)

$$- * \mathbb{I}_K \colon Z(\mathcal{H}_{G,J}) \xrightarrow{\sim} \mathcal{H}_{G,K}$$

it suffices to check that

$$\mathrm{Tr}(\mathrm{Frob}, R\Psi(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu))) * \mathbb{I}_K = \psi_\mu. \tag{4.3}$$

By [Gai01, Theorem 1 (d)] the map (4.3) is realized sheaf-theoretically by the pushforward via the proper map

$$\mathrm{pr} \colon \mathrm{Gr}_\mathcal{G} \to \mathrm{Gr}_G$$

or in other words,

$$\mathrm{Tr}(\mathrm{Frob}, R\Psi(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu))) * \mathbb{I}_K = \mathrm{Tr}(\mathrm{Frob}, \mathrm{pr}_! R\Psi(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu))).$$

Now, by Lemma 4.9 and the fact that pr is an isomorphism over $D^*_{x_0}$ (since $G|_{D^*_{x_0}} \cong \mathcal{G}|_{D^*_{x_0}}$) we have

$$\mathrm{pr}_! R\Psi(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu)) = R\Psi(\mathrm{Sat}_{\mathrm{Gr}_G}(\mu))$$

but since $\mathcal{G}_0|_{D_x} \to D_x$ is constant, we simply have

$$R\Psi(\mathrm{Sat}_{\mathrm{Gr}_G}(\mu)) = \mathrm{Sat}_{\mathrm{Gr}_G}(\mu)|_{x_0},$$

whose trace function is ψ_μ by definition. $\qquad\qquad\qquad\qquad\qquad\square$

Schubert stratification. Let $\mathrm{Gr}_{\mathcal{G},x_0}$ be the fiber of $\mathrm{Gr}_{\mathcal{G}}$ over x_0. We discuss the stratification induced by the $\mathcal{G}(\mathcal{O}_{x_0})$-action on $\mathrm{Gr}_{\mathcal{G},x_0}$.

The analogue of the Cartan decomposition (4.1) is

$$\mathcal{G}(\mathcal{O}_{x_0})\backslash\mathcal{G}(F_{x_0})/\mathcal{G}(\mathcal{O}_{x_0}) \cong \widetilde{W}_J\backslash\widetilde{W}/\widetilde{W}_J$$

where $\widetilde{W}$ is the extended affine Weyl group, and $\widetilde{W}_J$ is the subgroup corresponding to the parahoric subgroup $J := \mathcal{G}(\mathcal{O}_{x_0})$. We refer to [Hai09, §2.6] for the notation and definitions; all that we require are the following abstract facts:

- The $\mathcal{G}(\mathcal{O}_{x_0})$-orbits on $\mathrm{Gr}_{\mathcal{G},x_0}$ are indexed by $\nu \in \widetilde{W}_J\backslash\widetilde{W}/\widetilde{W}_J$. We denote the orbit corresponding to $\nu \in \widetilde{W}_J\backslash\widetilde{W}/\widetilde{W}_J$ by $\mathrm{Gr}_{\mathcal{G},x_0}^{=\nu}$ and its closure by $\mathrm{Gr}_{\mathcal{G},x_0}^{\leq\nu}$.
- There is a partial order on $\widetilde{W}_J\backslash\widetilde{W}/\widetilde{W}_J$, which can be characterized by the property that $\mu \geq \nu$ if and only if $\mathrm{Gr}_{\mathcal{G},x_0}^{\leq\mu} \supset \mathrm{Gr}_{\mathcal{G},x_0}^{=\nu}$.

Definition 4.16. Since $R\Psi(\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu))$ is a perverse sheaf on $\mathrm{Gr}_{\mathcal{G},x_0}$, equivariant for the proalgebraic group underlying J [Gai01, Theorem 1], its stalks are the same on any open Schubert cell $\mathrm{Gr}_{\mathcal{G}}^{=\nu}$. We denote this common stalk by $R\Psi(\mathrm{Sat}_{\mathrm{Gr}}(\mu))_\nu$.

Lemma 4.17. *We have*

$$\psi'_\mu = \sum_{\nu \leq \mu} \mathrm{Tr}(\mathrm{Frob}, R\Psi(\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu))_\nu)f_\nu$$

Proof. The argument is the same as for Lemma 4.4. $\qquad\qquad\qquad\qquad\square$

4.4. Local models for shtukas.

Definition 4.18. Let $\mathcal{X}$ and $\mathcal{Y}$ be Artin stacks. We say that $\mathcal{Y}$ is a *smooth local model* for $\mathcal{X}$ if there exists an Artin stack $\mathcal{W}$ and a diagram

A diagram as above is called a (smooth) *local model diagram.*

Theorem 4.19. *The stack* $\mathrm{Gr}_{\mathcal{G},X^r}^{\leq(\mu_1,\ldots,\mu_r)}$ *is a smooth local model for* $\mathrm{Sht}_{\mathcal{G}}^{\leq(\mu_1,\ldots,\mu_r)}$.

Remark 4.20. We learned from the referee that Theorem 4.19 – in fact, the stronger statement that one gets an *étale* local model – has already been proved for smooth affine group schemes in [RH, §3]. (Our proof also only uses that $\mathcal{G}$ is smooth and affine.)

Proof. For ease of presentation, we assume that $r = 1$ in the proof; the argument for the general case is a completely straightforward generalization.

As in [Laf18, Proposition 2.11] (beginning of the proof) we can add a level structure at a closed subscheme $N \subset X$ to rigidify all spaces under consideration from stacks to schemes. Since this addition of level structure induces smooth covers

of all objects, we will suppress it from the notation; the upshot is that we can reason with all objects as if they were schemes.

We let $\text{Bun}_{G,n\Gamma}$ be the smooth torsor over $\text{Bun}_{\mathcal{G}}$ as in §3.1.2, and $\text{Hecke}_{\mathcal{G},n\Gamma}$ the pullback torsor over $\text{Hecke}_{\mathcal{G}}$. Informally, $\text{Hecke}_{\mathcal{G},n\Gamma}$ parametrizes modifications $(x, \varphi\colon \mathcal{E} \dashrightarrow \mathcal{E}')$ together with the additional datum of a trivialization of $\mathcal{E}'$ on Γ_{nx}.

Since for any given μ the $\mathcal{L}^+\mathcal{G}$-action on $\text{Hecke}_{\mathcal{G}}^{\leq\mu}$ and $\text{Gr}_{\mathcal{G}}^{\leq\mu}$ factors through a quotient group scheme of finite type, Proposition 3.7 implies that for n sufficiently large relative to fixed μ, there is an isomorphism $\text{Hecke}_{\mathcal{G},n\Gamma}^{\leq\mu} \to \text{Gr}_{\mathcal{G}}^{\leq\mu} \times \text{Bun}_{\mathcal{G},n\Gamma}$.

Consider the diagram below, where all squares are cartesian:

$$
\begin{array}{ccc}
\text{Sht}_{\mathcal{G},n\Gamma}^{\leq\mu} \xrightarrow{\;\;\text{smooth}\;\;} & \text{Sht}_{\mathcal{G}}^{\leq\mu} \xrightarrow{\hspace{2cm}} & \text{Bun}_{\mathcal{G}} \\
\downarrow & \downarrow & \downarrow{\scriptstyle \text{Id} \times \text{Frob}} \\
\text{Hecke}_{\mathcal{G},n\Gamma} \xrightarrow{\;\;\text{smooth}\;\;} & \text{Hecke}_{\mathcal{G}}^{\leq\mu} \xrightarrow{h^{\leftarrow} \times h^{\rightarrow}} & \text{Bun}_{\mathcal{G}} \times \text{Bun}_{\mathcal{G}} \\
\downarrow{\scriptstyle \sim} & & \\
\text{Bun}_{\mathcal{G},n\Gamma} \times \text{Gr}_{G}^{\leq\mu} & &
\end{array}
\qquad (4.4)
$$

It suffices to show that map $\text{Sht}_{\mathcal{G},n\Gamma}^{\leq\mu} \to \text{Gr}_{G}^{\leq\mu}$, induced by the composition of the leftmost vertical arrows, is smooth. This follows by the same transversality calculation in [Var04, Lemma 4.3] applied to the outer diagram in (4.4)

$$
\begin{array}{ccc}
\text{Sht}_{\mathcal{G},n\Gamma}^{\leq\mu} \xrightarrow{\hspace{2.5cm}} & \text{Bun}_{\mathcal{G}} \\
\downarrow & \downarrow{\scriptstyle \text{Id} \times \text{Frob}} \\
\text{Hecke}_{\mathcal{G},n\Gamma} \xrightarrow{\hspace{1.5cm}} & \text{Bun}_{\mathcal{G}} \times \text{Bun}_{\mathcal{G}} \\
\downarrow{\scriptstyle \sim} & \\
\text{Bun}_{\mathcal{G},n\Gamma} \times \text{Gr}_{G}^{\leq\mu} &
\end{array}
\qquad (4.5)
$$

using that $\text{Bun}_{\mathcal{G},n\Gamma} \to \text{Bun}_{\mathcal{G}}$ is smooth. $\qquad\square$

Corollary 4.21. *Let $\underline{\mu} \in X_*(T)^r$. There is a local model diagram*

$$
\begin{array}{ccc}
 & \mathcal{W}^{\leq\underline{\mu}} & \\
{\scriptstyle smooth}\;\swarrow{\scriptstyle f} & & {\scriptstyle g}\searrow\;{\scriptstyle smooth} \\
\text{Sht}_{\mathcal{G}}^{\leq\underline{\mu}} & & \text{Gr}_{\mathcal{G}}^{\leq\underline{\mu}} \\
& \searrow \qquad \swarrow & \\
& X^r &
\end{array}
$$

with

$$
f^* \text{Sat}_{\text{Sht}_{\mathcal{G}}}(\underline{\mu}) = g^* \text{Sat}_{\text{Gr}_{\mathcal{G}}}(\underline{\mu}).
$$

Proof. This follows from the diagram (4.4) and the definition of the Satake sheaves, taking $\mathcal{W}^{\leq\underline{\mu}} = \text{Sht}_{\mathcal{G},N}^{\leq\underline{\mu}}$. $\qquad\square$

4.5. **The trace function of nearby cycles.** Recall that the nearby cycles $R\Psi(\mathcal{F})$ has an action of the inertia group. We have a decomposition

$$R\Psi(\mathcal{F}) \cong R\Psi(\mathcal{F})^{\mathrm{un}} \oplus R\Psi(\mathcal{F})^{\mathrm{non-un}}$$

into unipotent and non-unipotent parts for this inertial action. The associated trace function is independent of the choice of splitting in Remark 4.8 if $R\Psi(\mathcal{F})$ is unipotent, i.e. if $R\Psi(\mathcal{F})^{\mathrm{non-un}} = 0$. (Otherwise, to get a well-defined trace function we need to project to the unipotent summand – this is the "semisimple trace of Frobenius".)

Lemma 4.22. *The complex $R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Sht}_\mathcal{G}}(\underline{\mu}))$ is unipotent, i.e.*

$$R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Sht}_\mathcal{G}}(\underline{\mu}))^{\mathrm{non-un}} = 0.$$

Proof. By Corollary 4.21 plus the compatibility of the inertia action with the isomorphism of Lemma 4.11, it suffices to know that $R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\underline{\mu}))^{\mathrm{non-un}} = 0$. This is established in [Gai01, §5.1 Proposition 7.]. □

Let $\underline{\mu} \in X_*(T)^r$. By Corollary 4.21, we may set

$$\mathrm{Sat}_\mathcal{W}(\underline{\mu}) := f^* \mathrm{Sat}_{\mathrm{Sht}_\mathcal{G}}(\underline{\mu}) = g^* \mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\underline{\mu}).$$

We write $R\Psi_{x_0}$ to emphasize that we are taking nearby cycles over the point x_0. By Lemma 4.11, and implicitly using Corollary 4.21, we have

$$f^* R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Sht}_\mathcal{G}}(\underline{\mu})) = R\Psi_{x_0}(\mathrm{Sat}_\mathcal{W}(\underline{\mu})) = g^* R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\underline{\mu})).$$

Thus, for $w \in \mathcal{W}^{\leq\underline{\mu}}(k)$ lying over $y \in \mathrm{Sht}_\mathcal{G}^{\leq\underline{\mu}}(k)$ and $z \in \mathrm{Gr}_\mathcal{G}^{\leq\underline{\mu}}(k)$, we have

$$\mathrm{Tr}(\mathrm{Frob}, R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Sht}_\mathcal{G}}(\underline{\mu}))_y) = \mathrm{Tr}(\mathrm{Frob}, R\Psi_{x_0}(\mathrm{Sat}_\mathcal{W}(\underline{\mu}))_w) = \mathrm{Tr}(\mathrm{Frob}, R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\underline{\mu}))_z).$$

Therefore, the stalks of $R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Sht}_\mathcal{G}}(\underline{\mu}))$ are constant along the stratification

$$\mathrm{Sht}_\mathcal{G}^{\leq\underline{\mu}} = \coprod_{\underline{\nu}\leq\underline{\mu}} \mathrm{Sht}_\mathcal{G}^{=\underline{\nu}},$$

and we deduce:

Corollary 4.23. *For $\underline{\nu} \in X_*(T)_+$, we have*

$$\mathrm{Tr}(\mathrm{Frob}, R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Sht}_\mathcal{G}}(\underline{\mu}))_{\underline{\nu}}) = \mathrm{Tr}(\mathrm{Frob}, R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\underline{\mu}))_{\underline{\nu}}).$$

Remark 4.24. We will actually need to work with $\mathrm{Sht}_\mathcal{G}/a^{\mathbf{Z}}$ instead. Since this is obtained from $\mathrm{Sht}_\mathcal{G}$ by gluing isomorphic components, the result is exactly the same.

5. COUNTING PARAHORIC SHTUKAS

Our eventual goal is to establish a formula for the trace of an operator, formed as a composition of Hecke operators and Frobenius, on the cohomology of the nearby cycles sheaf of (a variant of) $\mathrm{Sht}_\mathcal{G}/a^{\mathbf{Z}} \to X$, at a place of parahoric bad reduction. The mold for such calculations was set by Kottwitz in [Kot92], who computed this sort of trace for certain PEL Shimura varieties, at places of good (hyperspecial) reduction. It has since been extended vastly by work of many authors; we note that in particular that Kisin and Pappas constructed integral models for Shimura varieties with parahoric level structure (a problem which itself has a long history, with contributions from many authors – see the references in [KP]) and computed the trace of Frobenius on nearby cycles for unramified groups in [KP], and for

tamely ramified groups in [HaRi]. Our result is a function field analogue of this computation.

In this section we carry out one step of this calculation, which deals with counting the number of fixed points of Frobenius composed with Hecke correspondences. (The precise setup will be explained in §5.1.) In fact most of the work has already been done by B.C. Ngô and T. Ngô Dac, who studied the case of moduli of shtukas with hyperspecial reduction in the series of papers [Ngo06], [NND08], [ND13], and [ND15]. The only new element here is that we are considering parahoric reduction. We note also that our results should follow from work of Hartl and Arasteh Rad proving the analogue of the Langlands-Rapoport Conjecture for shtukas [HaRa].

5.1. **Setup.** Throughout this section X° is an unspecified open subset X, which in the case of $\mathcal{D}$-shtukas will be $X - Z$ where Z is the set of ramification places of D. We let G be a quasi-split, connected reductive group over F with simply-connected derived group, or the group attached to a division algebra D as in §3.5. (This unwieldy hypothesis is in place because the statements of [NND08] and [ND13] use the first general hypothesis, but apply also $\mathcal{D}$-shtukas, cf. [Ngo06, §4], and we are also interested in the latter.) Let $\mathcal{G} \to X$ a parahoric group scheme, with parahoric reduction at x_0.

Let $K_v = \mathcal{G}(\mathcal{O}_v)$. Let $K_v t^{\beta_v} K_v \in K_v \backslash G(F_v)/K_v$ be a choice of double coset for all v, trivial for almost all v. Let $T' \subset X^\circ$ be the set of all v where $\beta_v \neq 0$, i.e. where the corresponding Hecke operator h_{β_v} is not the identity. *We assume that K_v is hyperspecial for all $v \in T'$.*

There is a Hecke correspondence (§3.3.4)

$$
\begin{array}{ccc}
 & \text{Hecke}(\text{Sht}_\mathcal{G})^{\leq \beta_v}_v / a^{\mathbf{Z}} & \\
 & \swarrow^{h^\leftarrow} \qquad \searrow^{h^\rightarrow} & \\
\text{Sht}_\mathcal{G} / a^{\mathbf{Z}} & & \text{Sht}_\mathcal{G} / a^{\mathbf{Z}} \qquad (5.1) \\
 & \searrow^{\pi} \qquad \swarrow^{\pi} & \\
 & X &
\end{array}
$$

for each β_v. This induces a Hecke operator h_{β_v} on the cohomology of $\text{Sht}_\mathcal{G} / a^{\mathbf{Z}}$ (Definition 3.13). See [NND08, §3] for more discussion about the Hecke correspondences.

We abbreviate $\beta := (\beta_v)_{v \in T'}$ and denote the corresponding Hecke operator $\prod h_{\beta_v}$ by $h_{\beta,T'}$. We want to compute (a variant of)

$$
\text{Tr}(h_{\beta,T'} \circ \text{Frob}, \pi_! R\Psi_{x_0}(\text{Sat}_{\text{Sht}_\mathcal{G}}(\mu)))
$$

where x_0 is our fixed place of parahoric reduction. By the Grothendieck-Lefschetz trace formula, we have

$$
\text{Tr}(h_{\beta,T'} \circ \text{Frob}, R\Psi_{x_0}(\text{Sat}_{\text{Sht}_\mathcal{G}}(\mu)))
$$
$$
= \sum_{\xi \in \text{Fix}(h_{\beta,T'} \circ \text{Frob})} \frac{1}{\#\text{Aut}\,\xi} \text{Tr}(h_{\beta,T'} \circ \text{Frob}_\Xi, R\Psi_{x_0}(\text{Sat}_{\text{Sht}_\mathcal{G}}(\mu))_\Xi).
$$

We will compute this by focusing first on counting $\text{Fix}(h_{\beta,T} \circ \text{Frob})$. This was done by [Ngo06] for $\mathcal{D}$-shtukas at points of with no level structure (good reduction), and extended by [NND08] for general reductive groups and [ND13] for more

complicated setups; however, these counts only account for the contribution from the "elliptic part".

In the case where G_F is anisotropic mod center, the elliptic part will obviously compose everything. This is one of the reasons why it is convenient to work with division algebras, and one of the difficulties in carrying out the strategy for general groups. Since $\mathrm{Sht}_{\mathcal{G}}$ is of infinite type in general, it will have infinitely many points even over finite fields.

5.2. **The groupoid of fixed points.** We consider a slightly more general situation. We will define a groupoid $\mathcal{C}(\alpha, \beta; T, T'; d)$ which occurs as the fixed points of a composition of Hecke and Frobenius operators on a certain moduli stack of shtukas. Then we will count its mass in the sense of groupoids.

Definition 5.1. If $\mathcal{C}$ is a finite groupoid with finite automorphism groups then we define
$$\#\mathcal{C} := \sum_{c \in \mathcal{C}} \frac{1}{\# \operatorname{Aut}(c)}.$$
Note that the $\mathbf{F}_q$-points of a finite type Deligne-Mumford stack, which includes any Schubert cell in a moduli stack of shtukas, satisfies this assumption.

Definition 5.2 ([NND08, §4]). Let $T, T' \subset |X| - I$. Let
$$\alpha = (\alpha_v \in K_v \backslash G(F_v)/K_v)_{v \in T}$$
$$\beta = (\beta_v \in K_v \backslash G(F_v)/K_v)_{v \in T'}.$$
(In terms of the notation of §5.1, we are identifying β_v with $K_v t^{\beta_v} K_v$.) We define the groupoid of fixed points $\mathcal{C}(\alpha, \beta; T, T'; d)$ as follows: its objects are triples $(\mathcal{E}, t, t')$ with

(1) $t \colon \mathcal{E}^\sigma|_{\overline{X}-\overline{T}} \xrightarrow{\sim} \mathcal{E}|_{\overline{X}-\overline{T}}$, with modification type α on T, and
(2) $t' \colon \mathcal{E}^{\sigma^d}|_{\overline{X}-\overline{T}'} \xrightarrow{\sim} \mathcal{E}|_{\overline{X}-\overline{T}'}$, with modification type β on T',
(3) satisfying the following compatibility:

$$
\begin{array}{ccc}
\mathcal{E}^{\sigma^{d+1}}|_{\overline{X}-\overline{T}-\overline{T}'} & \xrightarrow{\ \sigma^d(t)\ } & \mathcal{E}^{\sigma^d}|_{\overline{X}-\overline{T}-\overline{T}'} \\
\downarrow{\scriptstyle\sigma(t')} & & \downarrow{\scriptstyle t'} \\
\mathcal{E}^{\sigma}|_{\overline{X}-\overline{T}-\overline{T}'} & \xrightarrow{\quad t \quad} & \mathcal{E}|_{\overline{X}-\overline{T}-\overline{T}'}
\end{array}
$$

The automorphisms of $(\mathcal{E}, t, t')$ are defined to be automorphisms of $\mathcal{E}$ commuting with t and t'.

The relation to our initial problem is given by the following.

Lemma 5.3. *Suppose $x \in |X|$ is a point of degree d. Then we have an isomorphism of groupoids*
$$\operatorname{Fix}(h_{\beta, T'} \circ \operatorname{Frob}, \mathrm{Sht}_{\mathcal{G}}^{=\mu}|_x) \cong \mathcal{C}(\mu, \beta; \{x\}, T'; d).$$

Proof. This is immediate upon writing down the definitions. $\qquad\square$

We actually want to study the truncated space $\mathrm{Sht}_{\mathcal{G}}^{=\mu}/a^{\mathbf{Z}}$, so we modify the discussion accordingly. Let $\Xi \subset Z(G)(\mathbf{A})$ be a cocompact lattice. Then Ξ acts on $\mathrm{Sht}_{\mathcal{G}}^{=\mu}$ via Hecke correspondences, and we define $\mathrm{Sht}_{\mathcal{G}}^{=\mu}/\Xi$ to be the quotient. Similarly we define $\mathcal{C}(\mu, \beta; \{x\}, T'; d)_\Xi$ to be the quotient by the Ξ-action. (See [NND08, near the end of §4], for more details.)

Lemma 5.4. *Suppose $x \in |X|$ is a point of degree d. Then we have an isomorphism of groupoids*

$$\mathrm{Fix}(h_{\beta,T'} \circ \mathrm{Frob}_x, \mathrm{Sht}_{\mathcal{G}}^{=\mu}/\Xi|_x) \cong \mathcal{C}(\mu, \beta; \{x\}, T'; d)_\Xi.$$

Proof. Immediate by taking the quotient of Lemma 5.3 with respect to the Ξ-action. $\qed$

Hence we want to study $\#\mathcal{C}(\mu, \beta; \{x\}, T'; d)_\Xi$. The strategy for these counts goes back to Kottwitz's study of points of Shimura varieties (with hyperspecial level structure) over finite fields [Kot92].

(1) We first show that there is a cohomological invariant, the *Kottwitz invariant*, which controls the possible "generic fibers" of members of $\mathcal{C}(\alpha, \beta; T, T'; d)$.
(2) We then express the *size of an isogeny class* as a product of (twisted) orbital integrals.
(3) We then express the *number of isogeny classes* associated to each Kottwitz invariant in terms of certain cohomology groups.

These steps have been carried out in papers of B.C. Ngô and T. Ngô Dac, as already mentioned, but not quite in the generality required here. In particular, these previous papers avoid the case where T meets a point with non-trivial level structure (because the moduli problem was not defined over such points), which is exactly the situation that we are interested in. So we will describe the modifications needed to extend the argument to our setting, and only briefly summarize the parts that are already covered in the papers of B.C. Ngô and T. Ngô Dac.

5.3. **Kottwitz triples and classification of generic fibers.** Our first step is to define a category that looks like the category of "generic fibers of $\mathcal{C}(\alpha, \beta; T, T'; d)$".

Definition 5.5 ([NND08, §5]). Let $T, T' \subset |X| - I$. We define the groupoid $C(T, T'; d)$ as follows: its objects are triples (V, τ, τ') with

(1) V a G-torsor over $F_{\overline{k}} := F \otimes_k \overline{k}$,
(2) an isomorphism $\tau \colon V^\sigma \xrightarrow{\sim} V$, where $V^\sigma = V \otimes_{F_{\overline{k}}, \sigma} F_{\overline{k}}$,
(3) $\tau' \colon V^{\sigma^d} \xrightarrow{\sim} V$,

satisfying the following conditions:

(1) ("commutativity") The following diagram commutes:

$$\begin{array}{ccc} V^{\sigma^{d+1}} & \xrightarrow{\sigma^d(\tau)} & V^{\sigma^d} \\ {\scriptstyle \sigma(\tau')}\downarrow & & \downarrow{\scriptstyle \tau'} \\ V^\sigma & \xrightarrow{\ \tau\ } & V \end{array}$$

(2) For $x \notin T$, (V_x, τ_x) is isomorphic to the trivial isocrystal $(G(F_x \widehat{\otimes}_{\mathbf{F}_q} \overline{\mathbf{F}}_q), \mathrm{Id} \widehat{\otimes}_{\mathbf{F}_q} \sigma)$.
(3) For $x \in T$, (V_x, τ'_x) is isomorphic to the trivial isocrystal $(G(F_x \widehat{\otimes}_{\mathbf{F}_q} \overline{\mathbf{F}}_q), \mathrm{Id} \widehat{\otimes}_{\mathbf{F}_{q^d}} \sigma^d)$.

The automorphisms of (V, τ, τ') are automorphisms of $\mathcal{E}$ commuting with τ and τ'.

The operation of "taking the generic fiber" defines a functor [NND08, §5.2]

$$\mathcal{C}(\alpha, \beta; T, T'; d) \to C(T, T'; d).$$

5.3.1. *Kottwitz triples.* Recall that a *Kottwitz triple* is a datum $(\gamma_0, (\gamma_x)_{x \notin T}, (\delta_x)_{x \in T})$ where:

- γ_0 is a stable conjugacy class of $G(F)$,
- γ_x is a conjugacy class of $G(F_x)$ for each $x \notin T$, and is stably conjugate to γ_0,
- δ_x is a σ-conjugacy class of $G(F_x \widehat{\otimes}_{\mathbf{F}_q} \mathbf{F}_{q^d})$, whose norm

$$N(\delta_x) := \delta_x \cdot \sigma(\delta_x) \cdot \ldots \cdot \sigma^{d-1}(\delta_x)$$

  is stably conjugate to γ_0.

Construction 5.6. We now recall from [NND08, §6.1] how to attach to each $(V, \tau, \tau') \in C(T, T'; d)$ a Kottwitz triple $(\gamma_0, (\gamma_x)_{x \notin T}, (\delta_x)_{x \in T})$.

First a remark on notation: for a map $\tau \colon V^\sigma \to V$, we denote by $\tau^n \colon V^{\sigma^n} \to V$ the map $\tau \circ \sigma(\tau) \circ \ldots \circ \sigma^{n-1}(\tau)$.

(1) *Definition of γ_0.* Since $F_{\overline{k}}$ has cohomological dimension 1, the G-torsor V is split over $F_{\overline{k}}$. Consider $\gamma = \tau^d (\tau')^{-1}$, which is a *linear* automorphism of V. Using the "commutativity" axiom we find that

$$\sigma(\gamma) = \sigma(\tau) \circ \sigma^2(\tau) \circ \ldots \circ \sigma^d(\tau) \circ \sigma(\tau')^{-1} = \tau^{-1} \gamma \tau.$$

This shows that the conjugacy class of γ is stable under σ, hence defined over F. Since G was assumed to be quasi-split with simply connected derived subgroup, this conjugacy class must then contain an F-point. Thus, we have an element $\gamma_0 \in G(F)$ whose stable conjugacy class is well-defined.

(2) *Definition of γ_x, $x \notin T$.* By assumption, we can pick an isomorphism

$$(V_x, \tau_x) \cong (G(F_x \widehat{\otimes}_{\mathbf{F}_q} \overline{\mathbf{F}}_q), \mathrm{Id} \, \widehat{\otimes}_{\mathbf{F}_q} \sigma).$$

Since τ and τ' commute, so do τ_x and τ'_x, so that τ'_x defines an automorphism of $(G(F_x \widehat{\otimes}_{\mathbf{F}_q} \overline{\mathbf{F}}_q), \mathrm{Id} \, \widehat{\otimes}_{\mathbf{F}_q} \sigma)$. We can then write $\tau'_x = \gamma_x^{-1} \otimes \sigma^d$, for some $\gamma_x \in G(F_x)$ which is stably conjugate to γ_0. (The point is that picking this trivialization of τ_x amounts to setting "$\tau_x = \mathrm{Id}$" in the equation $\gamma_x = \tau_x^d (\tau'_x)^{-1}$.)

(3) *Definition of δ_x, $x \in T$.* By assumption, we can pick an isomorphism

$$(V_x, \tau'_x) \cong (G(F_x \widehat{\otimes}_{\mathbf{F}_q} \overline{\mathbf{F}}_q), \mathrm{Id} \, \widehat{\otimes}_{\mathbf{F}_q^d} \sigma^d).$$

Since τ and τ' commute, so do τ_x and τ'_x, so that τ_x defines an automorphism of $(G(F_x \widehat{\otimes}_{\mathbf{F}_q} \overline{\mathbf{F}}_q), \mathrm{Id} \, \widehat{\otimes}_{\mathbf{F}_q} \sigma)$. We can then write $\tau_x = \delta_x \otimes \sigma$, for some $\delta_x \in G(F_x)$, well-defined up to σ-conjugacy, whose norm $N(\delta_x) = \delta_x \cdot \sigma(\delta_x) \cdot \ldots \cdot \sigma^{r-1}(\delta_x)$ is stably conjugate to γ_0.

Definition 5.7. We say that $(V, \tau, \tau') \in C(T, T'; d)$ is *semisimple* if γ_0 is semisimple, and *elliptic* if γ_0 is elliptic.

We say $(\mathcal{E}, t, t') \in \mathcal{C}(\alpha, \beta; T, T'; d)$ is *semisimple* (resp. *elliptic*) if the associated (V, τ, τ') is semisimple (resp. elliptic).

5.3.2. *The Kottwitz invariant.* Following [NND08, §6.2] we can attach to the Kottwitz triple $(\gamma_0, (\gamma_x), (\delta_x))$ a character $\mathrm{inv}(\gamma_0, (\gamma_x), (\delta_x)) \in X^*(Z(\widehat{G}_{\gamma_0})^\Gamma)$. Briefly, this is done as follows.

- For $x \notin T$, since γ_x and γ_0 are stably conjugate, by a theorem of Steinberg we can find $g \in G(F_x \widehat{\otimes}_k \overline{k})$ such that

$$g\gamma_0 g^{-1} = \gamma_x.$$

Then (using that $\gamma_0 \in G(F)$) we have

$$g\gamma_0 g^{-1} = \gamma_x = \sigma(\gamma_x) = \sigma(g)\gamma_0\sigma(g)^{-1}.$$

This shows that $g^{-1}\sigma(g)$ is in the centralizer of γ_0 in $G(F_x \widehat{\otimes}_k \overline{k})$, hence defines a class in $B(G_{\gamma_0,x}) = G_{\gamma_0}(F_x \widehat{\otimes}_k \overline{k})/\sigma$-conjugacy.
- For $x \in T$, we have $g \in G(F_x \widehat{\otimes}_k \overline{k})$ such that $N\delta_x = g\gamma_0 g^{-1}$. Then $g^{-1}\sigma^d(g)$ is in $G_{\gamma_0}(F_x \widehat{\otimes}_k \overline{k})$ and defines a class in $B(G_{\gamma_0,x})$.

For each x, we apply the map $B(G_{\gamma_0,x}) \to X^*(Z(\widehat{G}_{\gamma_0})^{\Gamma_x})$ of [Kot90, §6] to get a local character $\mathrm{inv}_x(\gamma_0, (\gamma_x), (\delta_x)) \in X^*(Z(\widehat{G}_{\gamma_0})^{\Gamma_x})$. Almost all of the resulting characters are trivial, so that it makes sense to sum the restrictions of all these characters to $Z(\widehat{G}_{\gamma_0})^{\Gamma}$, and we define this sum to be $\mathrm{inv}(\gamma_0, (\gamma_x), (\delta_x))$.

Proposition 5.8 ([NND08] Proposition 7.1). *For elliptic $(V, \tau, \tau') \in C(T, T'; d)$, and $(\gamma_0, (\gamma_x), (\delta_x))$ the associated Kottwitz triple, if γ_0 is semisimple then we have $\mathrm{inv}(\gamma_0, (\gamma_x), (\delta_x)) = 0$.*

Proposition 5.9. *There exists $(V, \tau, \tau') \in C(T, T'; d)$ having a given elliptic Kottwitz triple $(\gamma_0, (\gamma_x), (\delta_x))$ if and only if $\mathrm{inv}(\gamma_0, (\gamma_x), (\delta_x)) = 0$. If the set of such is non-empty, then the number of isogeny classes within $C(T, T'; d)$ having the same Kottwitz triple is the cardinality of*

$$\ker^1(F, G_{\gamma_0}) := \ker(H^1(F, G_{\gamma_0}) \to \prod_x H^1(F_x, G_{\gamma_0})).$$

Proof. This follows from the proof of [NND08, Proposition 11.1] combined with [ND13, Proposition 4.3]. $\square$

5.3.3. *Automorphisms of the generic fiber.* Let $(V, \tau, \tau') \in C(T, T'; d)$ with elliptic Kottwitz triple $(\gamma_0, (\gamma_x), (\delta_x))$. By [ND13, §3.9], the automorphisms of (V, τ, τ') are the F_x-points of an inner form J_{γ_0} of G_{γ_0} defined over F.

Remark 5.10. As pointed out in [ND13, §3.9], the Hasse principle implies that J_{γ_0} is determined by its local components:

- For $x \notin T$, $J_{\gamma_0,x}$ is the centralizer of γ_x in $G(F_x)$,
- For $x \in T$, $J_{\gamma_0,x}$ is the twisted centralizer of δ_x in $G(F_x \otimes_{\mathbf{F}_q} \mathbf{F}_{q^d})$.

5.4. **Counting lattices.** We now study the fibers of the functor $\mathcal{C}(\alpha, \beta; T, T'; d) \to C(T, T'; d)$.

Proposition 5.11. *Fix an elliptic $(V, \tau, \tau') \in C(T, T'; d)$. Suppose $(\mathcal{E}, t, t') \in \mathcal{C}(\alpha, \beta; T, T'; d)$ lies over (V, τ, τ'). The size of the isogeny class of $(\mathcal{E}, t, t')$ is*

$$\mathrm{vol}(\Xi \cdot J_{\gamma_0}(F)\backslash G_{\gamma_0}(\mathbf{A})) \cdot \prod_{x \notin T} \mathrm{O}_{\gamma_x}(\phi_{\beta_x}) \prod_{x \in T} \mathrm{TO}_{\delta_x \sigma}(\phi_{\alpha_x}).$$

Here we normalize Haar measures as in §1.3.1.

Proof. Promoting (V, τ, τ') to $(\mathcal{E}, t, t')$ amounts to choosing a $\mathcal{G}_x \widehat{\otimes}_k \overline{k}$-bundle over $\mathrm{Spec}\, \mathcal{O}_x$ for all $x \in X$, plus an I-level structure, such that

- for $v \notin |T|$, $\mathcal{E}_x$ is fixed by τ,

- for $v \notin |T|'$, $\mathcal{E}_x$ is fixed by τ',
- for $x \in |T|$, the relative position of $\mathcal{E}_x$ and $\tau(\mathcal{E}_x)$ is given by α_x,
- for $v \in |T|'$ (hence outside $|T|$), the relative position of $\mathcal{E}_v$ and $\tau'(\mathcal{E}_v)$ is given by β_v.

As is well-known (cf. [NND08, §9] or [ND13, §5] for the present situation; the earliest reference we know is [Kot80]), this is counted by

$$\int_{\Xi \cdot J_{\gamma_0}(F) \backslash \prod_{x \in T} G(F_x \otimes_{\mathbf{F}_q} \mathbf{F}_{q^d}) \times G(\mathbf{A}^T)} \bigotimes_{x \in T} \phi_{\alpha_x}(h_x^{-1} \delta_x \sigma(h_x)) \bigotimes_{x \notin T} \phi_{\beta_x}(h_x^{-1} \gamma_x h_x) \, dh.$$

Here we use Remark 5.10 to view $J_{\gamma_0}(F)$ as a subset of $G(F_x)$. The statement of the proposition is a straightforward rewriting of this formula. $\qquad\square$

5.5. **Count of elliptic elements.** Define

$$\#\mathcal{C}(\alpha, \beta; T, T'; d)^{\text{ell}} := \sum_{(\mathcal{E}, t, t') \text{ elliptic}} \frac{1}{\# \operatorname{Aut}(\mathcal{E}, t, t')}.$$

Combining Proposition 5.9, §5.3.3, and Proposition 5.11, we obtain:

Theorem 5.12. *We have*

$$\#\mathcal{C}(\alpha, \beta; T, T'; d)^{\text{ell}} = \sum_{\substack{(\gamma_0, (\gamma_x), (\delta_x)) \\ \operatorname{inv}(\gamma_0, (\gamma_x), (\delta_x)) = 0 \\ \gamma_0 \ elliptic}} \ker^1(F, G_{\gamma_0}) \cdot \operatorname{vol}(\Xi \cdot J_{\gamma_0}(F) \backslash J_{\gamma_0}(\mathbf{A}_F)) \cdot dg(K)^{-1}$$

$$\cdot \left(\prod_{x \notin T} O_{\gamma_x}(f_{\beta_v}) \right) \prod_{x \in |T|} \operatorname{TO}_{\delta_x \sigma}(f_{\alpha_x})$$

Remark 5.13. This is the analogue of [ND13, Théorème 5.1]. The expressions look almost the same, but one should keep in mind that in our applications $T = \{x_0\}$, and $f_{\alpha_{x_0}}$ should be thought of as an indicator function on a partial affine flag variety rather than an affine Grassmannian. In addition, the measure of K_{x_0} is adapted accordingly.

Corollary 5.14. *Let $\operatorname{Sht}_{\mathcal{G}}^{\leq \mu}/a^{\mathbf{Z}}$ be the moduli stack of $\mathcal{D}$-shtukas with parahoric level structure at x_0, as in §3.5. Then*

$$\# \operatorname{Fix}(h_{\beta, T'} \circ \operatorname{Frob}, \operatorname{Sht}_{\mathcal{G}}^{=\mu}/a^{\mathbf{Z}}|_{x_0}) = \sum_{\substack{(\gamma_0, (\gamma_x), (\delta_x)) \\ \operatorname{inv}(\gamma_0, (\gamma_x), (\delta_x)) = 0}} \ker^1(F, G_{\gamma_0}) \cdot \operatorname{vol}(a^{\mathbf{Z}} \cdot J_{\gamma_0}(F) \backslash J_{\gamma_0}(\mathbf{A}_F)) \cdot$$

$$\cdot dg(K)^{-1} \cdot \left(\prod_{x \notin T} O_{\gamma_x}(f_{\beta_v}) \right) \cdot \operatorname{TO}_{\delta_{x_0} \sigma}(f_\mu).$$

Proof. This is immediate from Lemma 5.4 and the observation that every non-zero element of a division algebra is elliptic, since the associated group of units is anisotropic mod center. (As mentioned at the beginning of §5, the results used from [NND08] and [ND13] also apply to $\mathcal{D}$-shtukas, and in fact were originally proved for this case in [Ngo06, §4].) $\qquad\square$

6. Geometrization of base change for Hecke algebras

In this section we present a geometric interpretation of the base change homomorphism for spherical Hecke algebras, and then for the center of parahoric Hecke algebras. The results here are a generalization to arbitrary split reductive groups G of results from [Ngo99], which proved the result for GL_n.

Using the work of Gaitsgory on realizing central sheaves on the affine flag variety as nearby cycles, we then deduce a geometric interpretation of base change for the center of the parahoric Hecke algebra.

6.1. Definition of base change homomorphism.

For this section only, we let F be a local field and G be a reductive group over F. Given a compact open subgroup $H \subset G(F)$, we have the *Hecke algebra*

$$\mathcal{H}_{G,H} := \mathrm{Fun}_c(H \backslash G(F)/H, \overline{\mathbf{Q}}_\ell).$$

We begin by defining base change homomorphisms for some Hecke algebras with respect to a degree r unramified extension of local fields E/F.

For simplicity *we assume that G is split over F.* (Our results should extend at least to quasi-split G without much difficulty.) We let E/F be the unramified extension of degree r.

Definition 6.1 ([Hai09, §1]). Let $K \subset G(F)$ be a hyperspecial maximal compact subgroup. The *base change homomorphism for spherical Hecke algebras* (with respect to E/F) is the homomorphism of **C**-algebras

$$\mathcal{H}_{G(E),K} \to \mathcal{H}_{G(F),K}$$

characterized by the following property. Let W_F be the Weil group of F. For an admissible unramified homomorphism $\psi \colon W_F \to {}^L G$ let $\psi' \colon W_E \to {}^L G$ denote the restriction to $W_E \subset W_F$. Let π_ψ and $\pi_{\psi'}$ denote the corresponding representations of $G(E)$ and $G(F)$ under the Local Langlands Correspondence. Then for any $\phi \in \mathcal{H}_{G(E),K}$ we have

$$\langle \mathrm{trace}\ \pi_{\psi'}, \phi \rangle = \langle \mathrm{trace}\ \pi_\psi, b(\phi) \rangle$$

Definition 6.2 ([Hai09, §1]). Let $J \subset K$ be a parahoric subgroup and $\mathbb{I}_K$ denote the characteristic function of $K \subset G(F)$. By a theorem of Bernstein (cf. [Hai09, Theorem 3.1.1]), convolution with $\mathbb{I}_K$ defines an isomorphism

$$- *_J \mathbb{I}_K \colon Z(\mathcal{H}_{G,J}) \xrightarrow{\sim} \mathcal{H}_{G,K}.$$

We define the *base change homomorphism for parahoric Hecke algebras* to be the homomorphism

$$b \colon \mathcal{H}_{G(E),J} \to \mathcal{H}_{G(F),J}$$

making the following diagram commute:

$$
\begin{array}{ccc}
Z(\mathcal{H}_{G(E),J}) & \xrightarrow{\ b\ } & Z(\mathcal{H}_{G(F),J}) \\
\sim \Big\downarrow{\scriptstyle -*_J \mathbb{I}_K} & & \sim \Big\downarrow{\scriptstyle -*_J \mathbb{I}_K} \\
\mathcal{H}_{G(E),K} & \xrightarrow{\ b\ } & \mathcal{H}_{G(F),K}
\end{array}
\qquad (6.1)
$$

6.1.1. *Interpretation under Satake isomorphism.* Let $T \subset G$ be a maximal split torus. We have the Satake isomorphism

$$\mathcal{S} \colon \mathcal{H}_{G,K} \xrightarrow{\sim} \overline{\mathbf{Q}}_\ell[X_*(T)]^W$$

where W is the Weyl group of G relative to T. We also have the Bernstein isomorphism

$$B \colon Z(\mathcal{H}_{G,J}) \xrightarrow{\sim} \mathcal{H}_{G,K}.$$

We can define the base change homomorphism on the Satake side as follows. We define the *norm homomorphism*

$$N \colon \overline{\mathbf{Q}}_\ell[X_*(T_E)]^W \to \overline{\mathbf{Q}}_\ell[X_*(T_F)]^W$$

to be that induced by the norm $T_E \to T_F$. Since we are working in the split setting, this simply corresponds to multiplication by r on $X_*(T_E) \xrightarrow{\sim} \mathbf{Z}^n$.

Then $b \colon \mathcal{H}_{G(E),J} \to \mathcal{H}_{G(F),J}$ is determined by the commutativity of the following diagram ([Hai09, §3.2])

$$
\begin{array}{ccc}
Z(\mathcal{H}_{G(E),J}) & \xrightarrow{\ b\ } & Z(\mathcal{H}_{G(F),J}) \\
\sim \downarrow {\scriptstyle -*_J \mathbb{I}_K} & & \sim \downarrow {\scriptstyle -*_J \mathbb{I}_K} \\
\mathcal{H}_{G(E),K} & \xrightarrow{\ b\ } & \mathcal{H}_{G(F),K} \\
\sim \downarrow {\scriptstyle S} & & \sim \downarrow {\scriptstyle S} \\
\overline{\mathbf{Q}}_\ell[X_*(T_E)]^W & \xrightarrow{\ N\ } & \overline{\mathbf{Q}}_\ell[X_*(T_F)]^{W(F)}
\end{array}
\qquad (6.2)
$$

For more on the base change homomorphism, see [Hai09, §3].

6.2. Geometrization of the Satake transform.

In this section we will recall a geometric interpretation of the Satake transform.

6.2.1. *The classical Satake transform.* We first review the Satake transform [Gro98]. Let $\mathcal{H}_{G,K}$ be the spherical Hecke algebra of $G(F)$ with respect to a hyperspecial maximal compact subgroup K. The Hecke algebra for T may be identified as $\mathcal{H}_T = \overline{\mathbf{Q}}_\ell[X_*(T)]$.

We choose a Borel subgroup B containing T and let N be its unipotent radical. There is a Satake transform $\mathcal{S} \colon \mathcal{H}_{G,K} \to \mathcal{H}_T$ given by

$$\mathcal{S}f(t) = \delta(t)^{1/2} \int_N f(tx) dx,$$

where the Haar measure dx is normalized to assign volume 1 to $N(\mathcal{O}_t)$.

Theorem 6.3 (Satake). *The Satake transform gives a ring isomorphism*

$$\mathcal{S} \colon \mathcal{H}_{G,K} \cong \mathcal{H}_T^W \cong R(\widehat{G}),$$

where $R(\widehat{G})$ is the representation ring of $\widehat{G}$ with $\overline{\mathbf{Q}}_\ell$-coefficients.

For $\lambda \in X_*(T)$, write t^λ for the corresponding element of $X_*(T)$. Viewing $\mathcal{S}f \in \mathcal{H}_T \cong \overline{\mathbf{Q}}_\ell[X_*(T)]$ as functions on $X_*(T)$, we may write

$$\mathcal{S}f(t^\lambda) = \delta(t^\lambda)^{1/2} \int_N f(t^\lambda x) dx. \qquad (6.3)$$

6.2.2. *Interpretation via semi-infinite orbits.* We will interpret the function (6.3) geometrically, as the trace function associated to a certain subscheme S_λ, studied by Mirkovic-Vilonen in [MV07, §3], in the sense that if $f_\mathcal{F}$ is the trace function associated to $\mathcal{F}$, then

$$\mathcal{S}f_\mathcal{F}(t^\lambda) = q^{-\rho(\lambda)} \operatorname{Tr}(\operatorname{Frob}, R\Gamma_c(S_\lambda \otimes_k \overline{k}, \mathcal{F}))$$

where ρ is the usual half sum of the positive roots.

Following the notation of [MV07], for $\lambda \in X_*(T)$ we let $L_\lambda = t^\lambda G(\mathcal{O})$ denote the image of λ in the affine Grassmannian Gr_G, and S_λ be its orbit under N_F.

Lemma 6.4. *Let $\mathcal{F}$ be a sheaf on Gr_G defined over k, and $f_\mathcal{F}$ its associated trace function (cf. (4.2)). Then we have*

$$\mathcal{S}f_\mathcal{F}(t^\lambda) = q^{-\rho(\lambda)} \operatorname{Tr}(\operatorname{Frob}, R\Gamma_c(S_\lambda \otimes_k \overline{k}, \mathcal{F})).$$

Proof. We study the rational points $S_\lambda(k)$, in preparation for an application of the Grothendieck-Lefschetz trace formula. The stabilizer of t^λ is

$$\begin{aligned}
\operatorname{Stab}_{N(F)}(L_\lambda) &= \{n \in N(F) \colon nt^\lambda G(\mathcal{O}) = t^\lambda G(\mathcal{O})\} \\
&= \{n \in N(F) \colon t^{-\lambda}nt^\lambda \in G(\mathcal{O})\}.
\end{aligned}$$

This says that the map $n \mapsto t^{-\lambda}nt^\lambda$ defines a bijection between $N(\mathcal{O})$ and $\operatorname{Stab}_{N(F)}(L_\lambda)$. Therefore the map $n \mapsto t^\lambda n$ defines a bijection between $S_\lambda = [N(F)/\operatorname{Stab}_{N(F)}(L_\lambda)] \cdot L_\lambda$ and $N(F)/N(\mathcal{O})$. An application of the Grothendieck-Lefschetz trace formula then yields

$$\operatorname{Tr}(\operatorname{Frob}, R\Gamma_c(S_\lambda \otimes_k \overline{k}, \mathcal{F})) = \sum_{n \in S_\lambda(k)} f_\mathcal{F}(n) = \sum_{n \in N(F)/N(\mathcal{O})} f_\mathcal{F}(t^\lambda \cdot n)$$

which is exactly what was claimed upon recalling that $\delta(t^\lambda) = q^{\rho(\lambda)}$. $\square$

Theorem 6.5 (Mirkovic-Vilonen)**.** *There is a natural equivalence of functors*

$$H^*(-) \cong \bigoplus_{\lambda \in X_*(T)} H_c^{2\rho(\lambda)}(S_\lambda, -) \colon P_{G_\mathcal{O}}(\operatorname{Gr}_G, \overline{\mathbf{Q}}_\ell) \to \operatorname{Mod}_{\overline{\mathbf{Q}}_\ell}.$$

Proof. In the required generality, this is a consequence of [HaRi, Theorem 3.16]. For some historical perspective, this is a result of Mirkovic-Vilonen for the affine Grassmannian *over* **C** [MV07, Theorem 3.6]. See also [Zhu, Theorem 5.3.9] and the references indicated in [Zhu, Remark 5.3.10]. $\square$

Proposition 6.6. *We have*

$$H_c^*(S_\lambda, \mathcal{F}_\mu * \mathcal{F}_{\mu'}) \cong \bigoplus_{\lambda_1 + \lambda_2 = \lambda} H_c^*(S_{\lambda_1}, \mathcal{F}_\mu) \otimes H_c^*(S_{\lambda_2}, \mathcal{F}_{\mu'})$$

Proof. In the required generality, this is a consequence of [HaRi, Theorem 3.16]. For some historical perspective, this statement is proved implicitly in [MV07, Proposition 6.4] for the affine Grassmannian over **C**. In fact, in view of Theorem 6.5 it is equivalent to [MV07, Proposition 6.4]. It is formulated in the general setting in [Zhu, Proposition 5.3.14], and a proof is sketched there.

$\square$

6.3. **Base change for spherical Hecke algebras.** For $r \in \mathbb{N}$, let k_r be the (unique) extension of k of degree r and F_r denote the unique unramified field extension of F of degree r. For each $\mu \in X_*(T)$, we let $\mathrm{Sat}_{\mathrm{Gr}_G, r}(\mu)$ be the associated perverse sheaf on Gr_{G, k_r} and we let $\psi_{r, \mu}$ be the trace function on $\mathrm{Gr}_G(k_r)$ associated to $\mathrm{Sat}_{\mathrm{Gr}_G, r}(\mu)$. We will give a geometric interpretation for $b(\psi_{r, \mu})$.

6.3.1. *Weil restriction.* It will be useful to adopt a different perspective on the Hecke algebra $\mathcal{H}_{G(F_r), K_r}$, where K_r denotes the maximal compact subgroup of $G(F_r)$ corresponding to our chosen hyperspecial vertex. As is usual, we can think of $\mathcal{H}_{G(F_r), K_r}$ as functions on $\mathrm{Gr}_G(k_r) \to \overline{\mathbf{Q}}_\ell$ which are invariant with respect to the left K_r-action. However, using the identification

$$\mathrm{Gr}_G(k_r) = (\mathrm{Res}_{k_r/k} \mathrm{Gr}_{G, k_r})(k)$$

we can instead consider $\mathcal{H}_{G(F_r), K_r}$ as functions on $(\mathrm{Res}_{k_r/k} \mathrm{Gr}_{G, k_r})(k)$. Let τ be the cyclic permutation on Gr_G^r given by

$$\tau(y_1, \ldots, y_r) = (y_r, y_1, \ldots, y_{r-1}).$$

By the definition of Weil restriction, we have a canonical bijection

$$\mathrm{Gr}_G(k_r) \xrightarrow{\sim} \mathrm{Fix}(\mathrm{Frob} \circ \tau, \mathrm{Gr}_G^r) \quad \text{sending} \quad y \mapsto (y, \mathrm{Frob}(y), \ldots, \mathrm{Frob}^{r-1}(y)).$$

Definition 6.7. For each $\mu \in X_*(T)$, we let $\mathcal{F}_\mu := \mathrm{Sat}_{\mathrm{Gr}_G}(\mu)$ be the perverse sheaf on Gr_G. Consider the perverse sheaf

$$\mathcal{F}_\mu^{(1)} \boxtimes \ldots \boxtimes \mathcal{F}_\mu^{(r)} \in D_c^b(\mathrm{Gr}_G^r)$$

where $\mathcal{F}_\mu^{(i)} \cong \mathcal{F}_\mu$; the superscripts are just labellings. The endomorphism τ on Gr_G^r lifts to an endomorphism $\widetilde{\tau}$ of $\mathcal{F}_\mu^{(1)} \boxtimes \ldots \boxtimes \mathcal{F}_\mu^{(r)}$ in an obvious way. Define a function

$$\zeta_{r, \mu} \colon \mathrm{Fix}(\mathrm{Frob} \circ \tau, \mathrm{Gr}_G^r) \to \overline{\mathbf{Q}}_\ell$$

by

$$\zeta_{r, \mu}(y) := \mathrm{Tr}(\mathrm{Frob} \circ \widetilde{\tau}, (\mathcal{F}_\mu^{(1)} \boxtimes \ldots \boxtimes \mathcal{F}_\mu^{(r)})_{\overline{y}}).$$

Proposition 6.8. *The $\zeta_{r, \mu}$ form a basis for $\mathcal{H}_{G(F_r), K_r}$.*

Proof. This is well-known. It amounts to the fact that the change-of-basis matrix between the standard (double coset) basis of the Hecke algebra and the basis consisting of the $\zeta_{r, \mu}$ is upper-triangular for the Bruhat order. $\square$

6.3.2. *Convolution product.* Recall that there is a convolution product $*$ on $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ [Zhu, §5].

Definition 6.9. Consider rth convolution product

$$\mathcal{F}_\mu^{*r} := \mathcal{F}_\mu^{(1)} * \ldots * \mathcal{F}_\mu^{(r)}$$

as a perverse sheaf on Gr_G. This has an automorphism κ' of order r given by the composition

$$\mathcal{F}_\mu^{(1)} * \ldots * \mathcal{F}_\mu^{(r)} \xrightarrow{\kappa} \mathcal{F}_\mu^{(r)} * \mathcal{F}_\mu^{(1)} * \ldots * \mathcal{F}_\mu^{(r-1)} \xrightarrow{\iota} \mathcal{F}_\mu^{(1)} * \ldots * \mathcal{F}_\mu^{(r)}$$

where

- κ is the cyclic permutation obtained from the commutativity constraint $\mathcal{F} * \mathcal{B} \cong \mathcal{B} * \mathcal{F}$ of Geometric Satake [Zhu, Proposition 5.2.6], and
- ι is the tautological isomorphism $\mathcal{F}_\mu^{(i \bmod r)} \cong \mathcal{F}_\mu^{(i+1 \bmod r)}$ coming from the fact that all the $\mathcal{F}_\mu^{(i)}$ are defined to be the same perverse sheaf.

Define $\phi_{r,\mu}\colon \mathrm{Fix}(\mathrm{Frob}, \mathrm{Gr}_g(\overline{k})) \to \overline{\mathbf{Q}}_\ell$ by

$$\phi_{r,\mu}(y) = \mathrm{Tr}(\mathrm{Frob} \circ \kappa', (\mathcal{F}_\mu^{(1)} * \ldots * \mathcal{F}_\mu^{(r)})_y).$$

6.3.3. *The base change identity.* We now explain the relationship between these functions and the base change homomorphism.

Proposition 6.10. *We have $b(\zeta_{r,\mu}) = \phi_{r,\mu}$.*

Proof. By Theorem 6.3 it suffices to equate the Satake transforms of both sides. In other words, we must prove the following identity:

$$\int_{N(F)} b(\zeta_{r,\mu})(t^\lambda x)\,dx = \int_{N(F)} \phi_{r,\mu}(t^\lambda x)\,dx \text{ for all } \lambda \in X_*(T).$$

By (6.2), this is equivalent to establishing the two equations

$$\int_{N(F_r)} \zeta_{r,\mu}(t^\lambda x_r)\,dx_r = \int_{N(F)} \phi_{r,\mu}(t^{r\lambda}x)\,dx \tag{6.4}$$

$$\int_{N(F)} \phi_{r,\mu}(t^\lambda x)\,dx = 0 \quad \text{if } \lambda \notin r \cdot X_*(T)\ . \tag{6.5}$$

To do this we use the Lefschetz trace formula[4]:

$$\sum_{y \in \mathrm{Fix}(\mathrm{Frob} \circ \tau, Y(\overline{k}))} \mathrm{Tr}(\mathrm{Frob} \circ \widetilde{\tau}, \mathcal{F}_y) = \mathrm{Tr}(\mathrm{Frob} \circ \widetilde{\tau}, R\Gamma_c(Y \otimes_k \overline{k}, \mathcal{F})). \tag{6.6}$$

$$\sum_{y \in \mathrm{Fix}(\mathrm{Frob} \circ \kappa', Y(\overline{k}))} \mathrm{Tr}(\mathrm{Frob} \circ \kappa', \mathcal{F}_y) = \mathrm{Tr}(\mathrm{Frob} \circ \kappa', R\Gamma_c(Y \otimes_k \overline{k}, \mathcal{F})). \tag{6.7}$$

Applying (6.6) to $Y := S_\lambda \times \ldots \times S_\lambda$ and $\mathcal{F} := \mathcal{F}_\mu^{(1)} \boxtimes \ldots \boxtimes \mathcal{F}_\mu^{(r)}$ and using Lemma 6.4 plus the Künneth theorem gives

$$\int_{N(F_r)} \zeta_{r,\mu}(t^\lambda x_r)dx_r = q^{-\rho(\lambda)}\,\mathrm{Tr}(\mathrm{Frob} \circ \widetilde{\tau}, R\Gamma_c(S_\lambda \otimes_k \overline{k}, \mathcal{F}_\mu)^{\otimes r}).$$

We note that here $\widetilde{\tau}$ acts by cyclically permuting the tensor factors $R\Gamma_c(S_\lambda \otimes_k \overline{k}, \mathcal{F}_\mu)^{\otimes r}$, and Frob acts factorwise.

Applying (6.7) to $Y = S_\lambda$ and $\mathcal{F} = \mathcal{F}_\mu^{*r} = \mathcal{F}_\mu^{(1)} * \ldots * \mathcal{F}_\mu^{(r)}$ and using Lemma 6.4 gives

$$\int_{N(F)} \phi_{r,\mu}(t^\lambda x)dx = q^{-\rho(\lambda)}\,\mathrm{Tr}(\mathrm{Frob} \circ \kappa', R\Gamma_c(S_\lambda \otimes_k \overline{k}, \mathcal{F}_\mu^{*r})). \tag{6.8}$$

We first digest the expression (6.8). By Proposition 6.6 we have

$$R\Gamma_c(S_\lambda \otimes_k \overline{k}, \mathcal{F}_\mu^{(1)} * \ldots * \mathcal{F}_\mu^{(r)}) \cong \bigoplus_{\lambda_1 + \ldots + \lambda_r = \lambda} \bigotimes_{i=1}^r R\Gamma_c(S_{\lambda_i} \otimes_k \overline{k}, \mathcal{F}_\mu).$$

Let's try to understand the action of κ', which is a composition $\kappa' = \iota \circ \kappa$. The map κ acts by cyclic permutation of both the spaces and sheaves, so it induces the permutation

$$\kappa^*\colon H_c^*(S_{\lambda_1} \otimes_k \overline{k}, \mathcal{F}_\mu^{(1)}) \otimes H_c^*(S_{\lambda_2} \otimes_k \overline{k}, \mathcal{F}_\mu^{(2)}) \otimes \ldots \otimes R\Gamma_c(S_{\lambda_r} \otimes_k \overline{k}, \mathcal{F}_\mu^{(r)})$$

$$\to H_c^*(S_{\lambda_r} \otimes_k \overline{k}, \mathcal{F}_\mu^{(r)}) \otimes H_c^*(S_{\lambda_1} \otimes_k \overline{k}, \mathcal{F}_\mu^{(1)}) \otimes \ldots \otimes R\Gamma_c(S_{\lambda_{r-1}} \otimes_k \overline{k}, \mathcal{F}_\mu^{(r-1)}).$$

---

[4]Note that this is applicable because $\mathrm{Frob} \circ \tau$ is the Frobenius for a twisted form of Y: see the discussion beginning in the last sentence of [Ngo99, p. 651].

Next, the map ι relabels the sheaves only, so the conclusion is that κ' induces

$$(\kappa')_* \colon H_c^*(S_{\lambda_1} \otimes_k \overline{k}, \mathcal{F}_\mu^{(1)}) \otimes H_c^*(S_{\lambda_2} \otimes_k \overline{k}, \mathcal{F}_\mu^{(2)}) \otimes \ldots \otimes R\Gamma_c(S_{\lambda_r} \otimes_k \overline{k}, \mathcal{F}_\mu^{(r)})$$

$$\to H_c^*(S_{\lambda_r} \otimes_k \overline{k}, \mathcal{F}_\mu^{(1)}) \otimes H_c^*(S_{\lambda_1} \otimes_k \overline{k}, \mathcal{F}^{(2)}) \otimes \ldots \otimes R\Gamma_c(S_{\lambda_{r-1}} \otimes_k \overline{k}, \mathcal{F}_\mu^{(r)}).$$

In particular, we emphasize that the composition, at the level of cohomology, effects a permutation of the *spaces*. Now, Frobenius preserves each tensor and summand. Therefore, $\mathrm{Frob} \circ \kappa'$ acts on the summands of the form

$$\bigoplus_{j=0}^{r-1} \bigotimes_{i=1}^{r} H_c^*(S_{\lambda_{i+j} \pmod r} \otimes_k \overline{k}, \mathcal{F}_\mu)$$

by a cyclic permutation followed by a factorwise endomorphism. From this form we see that if $\lambda_1, \ldots, \lambda_r$ are not all equal then $\mathrm{Frob} \circ \kappa'$ permutes the summands freely, so $\mathrm{Frob} \circ \kappa'$ has trace 0. In particular, if λ is not divisible by r then the λ_i cannot be all equal, so that the trace is 0. This establishes (6.5).

On the other hand, our analysis above implies that if $\lambda = r\lambda'$ then the contribution to $\mathrm{Tr}(\mathrm{Frob} \circ \kappa')$ all comes from the terms with all $\lambda_1 = \ldots = \lambda_r = \lambda'$, and we have

$$\mathrm{Frob} \circ \kappa'|_{H_c^*(S_{\lambda'} \otimes_k \overline{k}, \mathcal{F}_\mu)^{\otimes r}} = \mathrm{Frob} \circ \widetilde{\tau}|_{H_c^*(S_{\lambda'} \otimes_k \overline{k}, \mathcal{F}_\mu)^{\otimes r}}$$

so that

$$\int_{N(F_{r,t})} \psi_{r,\mu}(t^{\lambda'} x_r) dx_r = \int_{N(F_t)} \phi_{r,\mu}(t^\lambda x) \, dx$$

which establishes (6.4). □

Lemma 6.11. *We have $\zeta_{r,\mu} = \psi_{r,\mu}$.*

Proof. By Theorem 6.3 it suffices to check equality of the Satake transforms of both sides. Using Lemma 6.4, this amounts to showing

$$\mathrm{Tr}(\mathrm{Frob}^r, R\Gamma_c(S_\lambda \otimes_k \overline{k}, \mathcal{F}_\mu)) = \mathrm{Tr}(\mathrm{Frob} \circ \widetilde{\tau}, R\Gamma_c(S_\lambda \otimes_k \overline{k}, \mathcal{F}_\mu)^{\otimes r}).$$

This follows from the general linear algebra fact asserted in Lemma 6.12 below. □

Lemma 6.12 (Saito-Shintani[5])**.** *Let V be a finite-dimensional representation over a field k. Let τ be the endomorphism of $V^{\otimes r}$ defined by*

$$v_1 \otimes \ldots \otimes v_r \mapsto v_r \otimes v_1 \otimes \ldots \otimes v_{r-1}.$$

Then for all $T_1, \ldots, T_r \in \mathrm{End}(V)$, we have

$$\mathrm{Tr}(T_1 \ldots T_r, V) = \mathrm{Tr}((T_1 \otimes \ldots \otimes T_r)\tau, V^{\otimes r}).$$

Proof. If $\{e_i\}$ is a basis for V, then

$$(T_1 \otimes \ldots \otimes T_r)\tau \cdot (e_{i_1} \otimes \ldots \otimes e_{i_r}) = T_1(e_{i_r}) \otimes \ldots \otimes T_r(e_{i_{r-1}}).$$

We expand out both sides of the desired equality:

$$\mathrm{Tr}((T_1 \otimes \ldots \otimes T_r)\tau) = \sum_{i_1, \ldots, i_r} \langle e_{i_1} \otimes \ldots \otimes e_{i_r}, T_1(e_{i_r}) \otimes \ldots \otimes T_r(e_{i_{r-1}}) \rangle$$

$$= \sum_{i_1 \ldots, i_r} \langle e_{i_1}, T_1(e_{i_r}) \rangle \ldots \langle e_{i_r}, T_r(e_{i_{r-1}}) \rangle$$

---

[5]We first found this explicitly stated, without proof, in [Ngo06], where it is said to be implicit in the work of Saito and Shintani on base change.

and
$$\mathrm{Tr}(T_1 \dots T_r) = \sum_i \langle e_i, T_1 \cdots T_r e_i \rangle.$$

Thus want to show that
$$\sum_{i_1,\dots,i_r} \langle e_{i_1}, T_1(e_{i_r}) \rangle \dots \langle e_{i_r}, T_r(e_{i_{r-1}}) \rangle = \sum_i \langle e_i, T_1 \cdots T_r e_i \rangle.$$

This follows by repeated iteration of the following more general identity.   □

Lemma 6.13. *Let V be a finite-dimensional vector space with basis $\{e_i\}$. For any $T \in \mathrm{End}(V)$, we have*
$$\sum_j \langle x, T e_j \rangle \langle e_j, y \rangle = \langle x, Ty \rangle. \tag{6.9}$$

Proof. It suffices to establish the equation for y ranging over a basis of V; taking $y = e_i$ the left hand side is $\langle x, T e_i \rangle$, and so is the right hand side.   □

Combining these results, we obtain the main formula of interest:

Theorem 6.14. *We have $b(\psi_{r,\mu}) = \phi_{r,\mu}$.*

Proof. This follows immediately upon combining Proposition 6.10 and Lemma 6.11.
     □

6.4. **Base change for the centers of parahoric Hecke algebras.** We now establish an identity for central functions of parahoric Hecke algebras analogous to Theorem 6.14. This is based on a degeneration from the spherical case.

6.4.1. *Setup.* We first set some notation. Pick a smooth global curve $X/\mathbf{F}_q$ (not necessarily projective) with a rational point $x_0 \in X(\mathbf{F}_q)$. (The reader may imagine that X, x_0 are as previously fixed, but this discussion applies more generally.) Let $\mathcal{G} \to X$ be a parahoric group scheme, such that $\mathcal{G}|_{X-x_0} \cong G \times X$ and $\mathcal{G}(\mathcal{O}_{x_0}) = J$ is a parahoric subgroup. We form the affine Grassmannian
$$\pi\colon \mathrm{Gr}_{\mathcal{G}} \to X.$$
Note that for $x \in X - x_0$ we have
$$\mathrm{Gr}_{\mathcal{G}}|_x \cong \mathrm{Gr}_G \times_k k(x).$$

For each $\mu \in X_*(T)$, we let $\mathcal{F}_\mu := \mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu)$ be the (shifted) perverse sheaf on $\mathrm{Gr}_{\mathcal{G}}$ and $\mathcal{F}_{\mu,x_0}$ be the restriction to $\mathrm{Gr}_{\mathcal{G},x_0}$. (We have normalized our shifts so that $\mathcal{F}_{\mu,x_0}$ is perverse.) We let $\psi'_{r,\mu} \in \mathcal{H}_{G(F_{x_0} \otimes_{\mathbf{F}_q} \mathbf{F}_{q^r}),J}$ be the function as in Definition 4.6.

6.4.2. *Convolution product.* By Theorem 4.12, $R\Psi(\mathcal{F}_\mu) := R\Psi_{x_0}(\mathcal{F}_\mu)$ is a central sheaf on $\mathrm{Gr}_{\mathcal{G},x_0}$. We therefore have, as in Definition 6.9, an automorphism
$$\kappa'\colon R\Psi(\mathcal{F}_\mu)^{(1)} * \dots * R\Psi(\mathcal{F}_\mu)^{(r)} \xrightarrow{\kappa} R\Psi(\mathcal{F}_\mu)^{(r)} * R\Psi(\mathcal{F}_\mu)^{(1)} * \dots * R\Psi(\mathcal{F}_\mu)^{(r-1)}$$
$$\xrightarrow{\iota} R\Psi(\mathcal{F}_\mu)^{(1)} * \dots * R\Psi(\mathcal{F}_\mu)^{(r)}.$$

Definition 6.15. Let
$$R\Psi(\mathcal{F}_\mu)^{*r} := \underbrace{R\Psi(\mathcal{F}_\mu)^{(1)} * \dots * R\Psi(\mathcal{F}_\mu)^{(r)}}_{r \text{ times}}.$$

Define $\phi'_\mu \colon \mathrm{Fix}(\mathrm{Frob}, \mathrm{Gr}_{\mathcal{G},x_0}(\overline{k})) \to \overline{\mathbf{Q}}_\ell$ by

$$\phi'_\mu(y) = \mathrm{Tr}(\mathrm{Frob} \circ \kappa', (R\Psi(\mathcal{F}_\mu)^{*r})_y).$$

Theorem 6.16. *We have $b(\psi'_{r,\mu}) = \phi'_\mu$.*

Proof. By Theorem 4.12, $\phi'_{r,\mu}(y)$ is in the center of the Iwahori-Hecke algebra. (Since $R\Psi(\mathcal{F}_\mu)$ is central by Theorem 4.12, $\phi'_{r,\mu}(y)$ clearly commutes with all the other functions of the form $\phi'_{r,\nu}(y)$; then use that such things form a basis for the Iwahori Hecke algebra as ν runs over the extended affine Weyl group.)

Now the argument is essentially the same as for Corollary 4.14. Consider the map

$$\mathrm{pr} \colon \mathrm{Gr}_{\mathcal{G}}^{\leq \mu} \to \mathrm{Gr}_{G \times X}^{\leq \mu}$$

induced by forgetting the level structure at x_0. Since pr is proper, by Lemma 4.9 and the fact that pr is an isomorphism away from x_0 we have

$$\mathrm{pr}_! R\Psi_{x_0}^{\mathrm{Gr}_{\mathcal{G}}}(\mathcal{F}_\mu) = R\Psi_{x_0}^{\mathrm{Gr}_{G \times X}}(\mathrm{Sat}_{\mathrm{Gr}_{G \times X}}(\mu)).$$

By Lemma 4.11 and the fact that $\mathrm{Gr}_{G \times X} \to X$ is smooth, we have

$$R\Psi_{x_0}^{\mathrm{Gr}_{G \times X}}(\mathrm{Sat}_{\mathrm{Gr}_{G \times X}}(\mu)) \cong \mathrm{Sat}_{\mathrm{Gr}_{G \times X}}(\mu).$$

Since $\mathrm{pr}_!$ corresponds to $- *_J \mathbb{I}_K$ at the level functions, this implies

$$\phi'_\mu *_J \mathbb{I}_K = \phi_\mu.$$

Thus by Theorem 6.14 and (6.1), we have that

$$\phi'_\mu *_J \mathbb{I}_K = \phi_\mu = b(\psi_{r,\mu}) = b(\psi'_{r,\mu}) *_J \mathbb{I}_K. \qquad (6.10)$$

In view of the Bernstein isomorphism (Theorem 4.5), the fact that ϕ'_μ and $b(\psi'_{r,\mu})$ are central plus (6.10) implies that they are equal. $\qquad \square$

6.4.3. *A global reformulation.* We now recast Theorem 6.16 into a form that will be more suitable for our eventual needs.

Let $\mathcal{G}$ and $\mathrm{Gr}_{\mathcal{G}}$ be as in §6.4.1. We first recall a construction of the convolution product $\mathcal{F}_\mu * \mathcal{F}_{\mu'}$. Recall the iterated global affine Grassmannian $\mathrm{Gr}_{\mathcal{G},X^2}$ from §3.4.1. We can form the twisted tensor product $\mathcal{F}_\mu \widetilde{\boxtimes} \mathcal{F}_{\mu'} := \mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G},X^2}}(\mu,\mu')$ on $\mathrm{Gr}_{\mathcal{G},X^2}$ [Zhu, §A], which is supported on the Schubert variety $\mathrm{Gr}_{\mathcal{G},X}^{\leq (\mu,\mu')}$.

Restricting to the diagonal $X \subset X^2$, we have the multiplication map

$$m \colon \mathrm{Gr}_{\mathcal{G},X^2}^{\leq (\mu,\mu')}|_\Delta \to \mathrm{Gr}_{\mathcal{G}}^{\leq \mu+\mu'}$$

defined on points by

$$(x, x, \mathcal{E}_1 \xdashrightarrow{\varphi} \mathcal{E}_2 \xdashrightarrow{\beta} \mathcal{E}^0) \mapsto (x, \mathcal{E}_1 \xdashrightarrow{\beta \circ \varphi} \mathcal{E}^0).$$

Then the convolution product is defined by (cf. [MV07, §4] or [Zhu, §5.1])

$$\mathcal{F}_\mu * \mathcal{F}_{\mu'} := Rm_!(\mathcal{F}_\mu \widetilde{\boxtimes} \mathcal{F}_{\mu'}) \in \mathrm{Perv}_{L^+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}^{\leq \mu+\mu'}). \qquad (6.11)$$

Let us now write down our particular situation of interest. Consider the diagram

$$
\begin{array}{ccc}
\mathrm{Gr}_{\mathcal{G},X^r}^{\leq (\mu_1,\ldots,\mu_r)}|_\Delta & \longrightarrow & \Delta(X) \subset X^r \\
\downarrow{\scriptstyle m} & & \downarrow \\
\mathrm{Gr}_{\mathcal{G},X}^{\leq \mu_1+\ldots+\mu_r} & \longrightarrow & X
\end{array}
$$

Then by (6.11) we have

$$\mathcal{F}(\mu_1) * \ldots * \mathcal{F}(\mu_r) = Rm_!(\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}, X^r}(\mu_1, \ldots, \mu_r))$$

Now Theorem 6.16 can be reformulated as follows, using Corollary 4.9 to commute $Rm_!$ and nearby cycles.

Proposition 6.17. *Let* $r.\mu := (\mu, \ldots, \mu) \in X_*(T)^r_+$. *Let* $f_{\nu, x_0} \in \mathcal{H}_{G(F_{x_0}), J}$ *be the function* f_ν *viewed in the parahoric Hecke algebra of* F_{x_0}, *and define* $\psi'_{r\mu, x_0} \in Z(\mathcal{H}_{G(F_{x_0}), J})$ *similarly. Then we have*

$$b(\psi'_{r, r\mu, x_0}) = \sum_{\nu \leq r\mu} \mathrm{Tr}(\sigma \circ \kappa', R\Psi^X_{x_0}(Rm_! \, \mathrm{Sat}_{\mathrm{Gr}_{\widetilde{\mathcal{G}}}^{\leq r.\mu}, X^r}(r.\mu))_\nu) f_{\nu, x_0}.$$

7. Comparison of two moduli problems

7.1. Setup. We now let G be the group scheme of units of a global algebra $\mathcal{D}$ as in §3.5 and $\mathcal{G}$ a parahoric group scheme corresponding to some choice of level structure at x_0, so $\mathrm{Sht}_{\mathcal{G}}$ are the $\mathcal{D}$-shtukas studied in §3.5. We continue to assume, that $\mathcal{G}$ is reductive away from x_0 (though we could avoid this simply by shrinking X° to remove the points where $\mathcal{G}$ is not reductive).

Let $Z \subset X$ be the set of places of ramification for $\mathcal{D}$. *We assume throughout that* $\#Z \geq n^2(\|\mu_1\| + \ldots + \|\mu_r\|)$, *so as to be able to apply Proposition 3.19.*

Let $X^\circ := (X - Z - \{x_0\})$. We will now define and compare two different moduli stacks of shtukas.

7.2. Situation A. Let

$$\mathrm{Sht}^\mu_A := (\mathrm{Sht}^{\leq \mu}_{\mathcal{G}, X} / a^{\mathbf{Z}})^r.$$

We have a map

$$\pi_A \colon \mathrm{Sht}^\mu_A |_{(X-Z)^r} \to (X - Z)^r.$$

By Proposition 3.19 the restriction $\pi^\circ_A := \pi_A|_{(X^\circ)^r}$ is proper.

Definition 7.1. Let $r.\mu = \underbrace{(\mu, \ldots, \mu)}_{r \text{ times}}$. We define $\mathcal{A}^\mu_r \in D^b_c((X^\circ)^r)$ as follows:

$$\mathcal{A}^\mu_r := R\pi^\circ_{A*}(\mathrm{Sat}_{\mathrm{Sht}^\mu_A}(r.\mu)).$$

We have the following easy but crucial property.

Proposition 7.2. *The complex* $\mathcal{A}^\mu_r \in D^b_c((X^\circ)^r)$ *is locally constant on* $(X^\circ)^r$, *in the sense that each* $R^i \pi^\circ_{A*}(\mathrm{Sat}_{\mathrm{Sht}^\mu_A}(r.\mu))$ *is a local system.*

Proof. By the properness of π°_{A*} we know that $R^i \pi^\circ_{A*}(\mathrm{Sat}_{\mathrm{Sht}^\mu_A}(r.\mu))$ is constructible, and the local acyclicity from Proposition 3.18 then implies that it is locally constant. $\square$

Note that by the Künneth formula, we have

$$\mathcal{A}^\mu_r \cong (\mathcal{A}^\mu_1)^{\boxtimes r}$$

Choose a basepoint $x \in X^\circ$, and let $x^r \in (X^\circ)^r$ denote the diagonal point $(x, \ldots, x)$. Then the symmetric group S_r acts on $(X^\circ)^r$, hence also $\pi_1((X^\circ)^r, x^r)$. This lifts to an S_r-equivariant structure on the local system $\mathcal{A}^\mu_r$, i.e. an action of $\pi_1((X^\circ)^r, x^r) \rtimes S_r$ on $(\mathcal{A}^\mu_r)_{x^r}$, commuting with the action of the global Hecke algebra $\mathcal{H}^{\otimes r}$.

7.3. **Situation B.** Let

$$\mathrm{Sht}_B^\mu := \mathrm{Sht}_{\mathcal{G}, X^r}^{\leq r.\mu} / a^{\mathbf{Z}}$$

where $r.\mu = \underbrace{(\mu, \ldots, \mu)}_{r \text{ times}}$. We have a map

$$\pi_B \colon \mathrm{Sht}_B^\mu |_{(X-Z)^r} \to (X - Z)^r.$$

By the assumption that $\mathcal{D}$ is totally ramified at sufficiently many places, the map $\pi_B^\circ := \pi_B|_{(X^\circ)^r}$ is proper by Proposition 3.19.

Definition 7.3. We define $\mathcal{B}_r^\mu \in D_c^b((X^\circ)^r)$ as follows:

$$\mathcal{B}_r^\mu := R\pi_{B*}^\circ(\mathrm{Sat}_{\mathrm{Sht}_B^\mu}(r.\mu)).$$

Proposition 7.4. *The complex* $\mathcal{B}_r^\mu \in D_c^b((X^\circ)^r)$ *is locally constant on* $(X^\circ)^r$, *in the sense that each* $R^i\pi_{B*}^\circ(\mathrm{Sat}_{\mathrm{Sht}_B^\mu}(r.\mu))$ *is a local system.*

Proof. The proof is the same as for Proposition 7.2. $\qquad\qquad\square$

Again, we have commuting actions of the Hecke algebra $\mathcal{H}$ and $\pi_1((X^\circ)^r, x^r) \rtimes S_r$ on $\mathcal{B}_r^\mu$.

7.4. **The comparison theorem.**

Theorem 7.5 ([Ngo06])**.** *Let* $\tau \in S_r$ *be an* r-*cycle, i.e.* $\langle\tau\rangle \cong \mathbf{Z}/r\mathbf{Z}$. *For* $g \in \pi_1((X^\circ)^r, x^r)$ *and* $h \in \mathcal{H}$ *we have*

$$\mathrm{Tr}((h \otimes 1 \ldots \otimes 1)g\tau, (\mathcal{A}_r^\mu)_{x^r}) = \mathrm{Tr}(hg\tau, (\mathcal{B}_r^\mu)_{x^r}).$$

Proof. This is [Ngo06, §3.3 Theorem 1]. Since this is really crucial for us, we outline for the sake of exposition how the proof goes. Keep in mind that $\mathcal{A}_r^\mu$ and $\mathcal{B}_r^\mu$ are both local systems.

By an application of the Cebotarev density theorem, it suffices to prove the equality for $g = \mathrm{Frob}_{(x_1, \ldots, x_r)}$ for a dense open subset of X^r, and in particular on the locus $(x_1, \ldots, x_r)$ where the x_i are pairwise distinct. On this locus (and under a certain further restriction on the points $(x_1, \ldots, x_r)$) Ngô independently computes both sides of the equation, following the Langlands-Kottwitz paradigm, and verifies that they are equal by direct comparison.

Let us say a little more about this computation, which is carried out in [Ngo06, §5]. Using the Grothendieck-Lefschetz trace formula to re-express both sides, there are two main inputs: (1) a count of fixed points, and (2) a computation of the trace of Frobenius on the stalks of the relevant sheaves. The counting step is done as in §5, and the analysis of the stalks enters via results as in §6.3. The interesting feature is that the pairwise distinctness of the points $(x_1, \ldots, x_r)$, plus the extra restriction that we have omitted, turns out to imply that the point counting formulas involve *no twisted orbital integrals*. Therefore, no fundamental lemma is required to prove the desired equality.

$$\square$$

Remark 7.6. For a heuristic that underlies the theorem, coming from a conjectural description of the cohomology of shtukas, see [Ngo06, §2.2, 3.3]. The punchline is that after admitting this conjectural description, the identity in Theorem 7.5 reduces to Lemma 6.12.

Definition 7.7. Let

$$R\Psi_{x_0^r}(\mathcal{A}_r^\mu) := R\pi_{A!}R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Sht}_A^\mu}(r.\mu)|_{\Delta(X^\circ)}) \in D_c^b(x_0)$$

be the cohomology of nearby cycles at $x_0^r \in \Delta(X - Z)$, and let

$$R\Psi_{x_0^r}(\mathcal{B}_r^\mu) := R\pi_{B!}R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Sht}_B^\mu}(r.\mu)|_{\Delta(X^\circ)}) \in D_c^b(x_0)$$

be the cohomology of nearby cycles at $x_0^r \in \Delta(X - Z)$.[6] Again we apply Remark 4.8 to equip $R\Psi_{x_0^r}(\mathcal{A}_r^\mu)$ and $R\Psi_{x_0^r}(\mathcal{B}_r^\mu)$ with $\mathbf{F}_q$-structures.

Thanks to Proposition 7.2 and Proposition 7.4, the complex $R\Psi_{x_0^r}(\mathcal{A}_r^\mu)$ is equipped with commuting actions of $\pi_1((X^\circ)^r, x^r) \rtimes S_r$ and $(\mathcal{H})^{\otimes r}$, while $R\Psi_{x_0^r}(\mathcal{B}_r^\mu)$ is equipped with commuting actions of $\pi_1((X^\circ)^r, x^r) \rtimes S_r$ and $\mathcal{H}$.

Corollary 7.8. *For $g \in \pi_1((X^\circ)^r, x^r)$ and $h \in \mathcal{H}$ we have*

$$\mathrm{Tr}((h \otimes 1 \ldots \otimes 1) \circ \mathrm{Frob} \circ \tau, R\Psi_{x_0^r}(\mathcal{A}_r^\mu)) = \mathrm{Tr}(h \circ \mathrm{Frob} \circ \tau, R\Psi_{x_0^r}(\mathcal{B}_r^\mu)). \quad (7.1)$$

Proof. This immediate from Theorem 7.5, Proposition 3.20 plus Corollary 4.10, and the Cebotarev density theorem. □

8. Calculation of traces on the cohomology of nearby cycles

Our next step is to combine the work of §4, §5 and §6 to prove Kottwitz-style formulas for both sides of (7.1). We maintain the notation of those preceding sections.

8.1. Calculating the trace in situation A.

Definition 8.1. For a Kottwitz triple $(\gamma_0, (\gamma_x), (\delta_x))$ write

$$c(\gamma_0, (\gamma_x), (\delta_x)) := \ker^1(F, G_{\gamma_0}) \cdot \mathrm{vol}(\Xi \cdot J_{\gamma_0}(F)\backslash J_{\gamma_0}(\mathbf{A}_F)) \cdot dg(K)^{-1}$$

where the notation is as in §5.

Theorem 8.2. *Let $T' \subset |X^\circ|$. Assume that $K_v := \mathcal{G}(\mathcal{O}_v)$ is spherical at all $v \in T'$. Let*

$$\beta = (\beta_v \in K_v\backslash G(F_v)/K_v)_{v \in T'}$$

and $h_\beta \in \mathcal{H}$ be the corresponding Hecke operator. Let $R\Psi_{x_0^r}(\mathcal{A}_r^\mu)$ be as in Definition 7.7, let τ be as in Theorem 7.5, let f_{β_v} be as in Definition 4.1, and let $\psi_{r,\mu'}$ be as in §6.4.1. Then we have

$$\mathrm{Tr}((h_\beta \otimes 1 \ldots \otimes 1) \circ \mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(\mathcal{A}_r^\mu))$$

$$= \sum_{\substack{(\gamma_0, (\gamma_x), (\delta_x)) \\ \mathrm{inv}(\gamma_0, (\gamma_x), (\delta_x))=0}} c(\gamma_0, (\gamma_x), (\delta_x)) \cdot \left(\prod_{v \neq x_0} \mathrm{O}_{\gamma_v}(f_{\beta_v})\right) \mathrm{TO}_{\delta_{x_0}\sigma}(\psi'_{r,\mu})$$

Proof. We'll use the Lefschetz trace formula to rewrite the trace in terms of a sum of traces over fixed points. The effect of $\sigma\tau$ on a point of Sht_A is illustrated below:

$$\begin{Bmatrix} {}^\sigma\mathcal{E}_1|_{X-x_0} \xrightarrow{\leq\mu} \mathcal{E}_1|_{X-x_0} \\ {}^\sigma\mathcal{E}_2|_{X-x_0} \xrightarrow{\leq\mu} \mathcal{E}_2|_{X-x_0} \\ \vdots \\ {}^\sigma\mathcal{E}_r|_{X-x_0} \xrightarrow{\leq\mu} \mathcal{E}_r|_{X-x_0} \end{Bmatrix} \xrightarrow{\sigma\tau} \begin{Bmatrix} {}^{\sigma^2}\mathcal{E}_r|_{X-x_0} \xrightarrow{\leq\mu} {}^\sigma\mathcal{E}_r|_{X-x_0} \\ {}^{\sigma^2}\mathcal{E}_1|_{X-x_0} \xrightarrow{\leq\mu} {}^\sigma\mathcal{E}_1|_{X-x_0} \\ \vdots \\ {}^{\sigma^2}\mathcal{E}_{r-1}|_{X-x_0} \xrightarrow{\leq\mu} {}^\sigma\mathcal{E}_{r-1}|_{X-x_0}. \end{Bmatrix}.$$

---

[6]This is a small abuse of the notation used in §4.2.

Therefore, a fixed point of the correspondence $(h_\beta \otimes 1 \ldots \otimes 1) \circ \mathrm{Frob}_{x_0} \circ \tau$ corresponds to a point as above such that

$$\mathcal{E}_2 = {}^\sigma \mathcal{E}_1$$

$$\vdots$$

$$\mathcal{E}_r = {}^\sigma \mathcal{E}_{r-1}$$

$$\mathcal{E}_1 \xrightarrow{=\beta} {}^\sigma \mathcal{E}_r.$$

By substitution this can be rewritten in terms of $\mathcal{E}_1$, and we find that a fixed point is equivalent to the data of commuting modifications

$$t \colon {}^\sigma \mathcal{E}_1|_{X-x_0} \xrightarrow{\leq \mu} \mathcal{E}_1|_{X-x_0}$$

$$t' \colon {}^{\sigma^r} \mathcal{E}_1|_{X-T'} \xrightarrow{=\beta} \mathcal{E}_1|_{X-T'}.$$

Hence in the notation of §5 we see that

$$\mathrm{Fix}((h_\beta \otimes 1 \ldots \otimes 1) \circ \mathrm{Frob}_{x_0} \circ \tau) = \bigcup_{\nu \leq \mu} \mathcal{C}(\nu_{x_0}, \beta; x_0, T'; r).$$

By Lemma 4.9 plus Proposition 3.20, we have

$$R\Psi_{x_0^r}(\mathcal{A}_r^\mu) \cong R\pi_{A!}(R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Sht}_A}(r.\mu))).$$

Now invoking the Grothendieck-Lefschetz trace formula, we have

$$\mathrm{Tr}((h_\beta \otimes 1 \ldots \otimes 1) \circ \mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(\mathcal{A}_r^\mu))$$

$$= \sum_{\nu \leq \mu} \sum_{\xi \in \mathcal{C}(\nu_{x_0}, \beta; x_0, T'; r)} \mathrm{Tr}((h_\beta \otimes 1 \ldots \otimes 1) \circ \mathrm{Frob} \circ \tau, R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Sht}_A}(r.\mu))_\xi).$$

By Corollary 4.23, for all $\xi \in \mathcal{C}(\nu_{x_0}, \beta; x_0, T'; r)$ we have

$$R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Sht}_A}(r.\mu))_\xi = R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}^r}(r.\mu))_\nu.$$

Now using Corollary 5.14, we can rewrite our formula as

$$\mathrm{Tr}((h_\beta \otimes 1 \ldots \otimes 1) \circ \mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(\mathcal{A}_r^\mu))$$

$$= \sum_{\substack{(\gamma_0, (\gamma_x), (\delta_x)) \\ \mathrm{inv}(\gamma_0, (\gamma_x), (\delta_x)) = 0}} c(\gamma_0, (\gamma_x), (\delta_x)) \cdot \prod_{v \neq x_0} O_{\gamma_v}(f_{\beta_v})$$

$$\cdot \sum_{\nu \leq \mu} \mathrm{TO}_{\delta_{x_0}\sigma}(f_\nu) \cdot \mathrm{Tr}((h_\beta \otimes 1 \ldots \otimes 1) \circ \mathrm{Frob} \circ \tau, R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}^r}(r.\mu))_\nu).$$

$$(8.1)$$

Since the Hecke operator h_β supports a modification at T', which is disjoint from x_0, it acts trivially on all the stalks lying over x_0^r, so we may ignore it when computing the trace in (8.1). Since

$$R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}^r}(r.\mu)) \cong R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Gr}_\mathcal{G}}(\mu))^{\boxtimes r}, \qquad (8.2)$$

the trace of $\mathrm{Frob} \circ \tau$ coincides with the trace of Frobenius for the Satake sheaf on the Weil restriction $\mathrm{Res}_{k_r/k}(\mathrm{Gr}_\mathcal{G}^{\leq \mu} \otimes_{\mathbf{F}_q} \mathbf{F}_{q^r})$. Therefore, by [Ngo06, §5.2 Proposition

3], we have[7]

$$\mathrm{Tr}(\mathrm{Frob}\circ\tau, R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}^r}(r.\mu))_\nu) = \mathrm{Tr}(\mathrm{Frob}^r, R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu))_\nu). \qquad (8.3)$$

Substituting (8.3) into (8.1) we arrive at

$$\mathrm{Tr}((h_\beta \otimes 1 \ldots \otimes 1) \circ \mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(\mathcal{A}_r^\mu))$$

$$= \sum_{\substack{(\gamma_0,(\gamma_x),(\delta_x)) \\ \mathrm{inv}(\gamma_0,(\gamma_x),(\delta_x))=0}} c(\gamma_0,(\gamma_x),(\delta_x)) \cdot \prod_{v \neq x_0} \mathrm{O}_{\gamma_v}(f_{\beta_v})$$

$$\cdot \sum_{\nu \leq \mu} \mathrm{TO}_{\delta_{x_0}\sigma}(f_\nu) \cdot \mathrm{Tr}(\mathrm{Frob}^r, R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu))_\nu). \qquad (8.4)$$

By Lemma 4.17 we have the following identity:

$$\psi'_{r,\mu} = \sum_{\nu \leq \mu} \mathrm{Tr}(\mathrm{Frob}^r, R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu))_\nu)f_\nu.$$

Substituting this in (8.4), we finally find

$$\mathrm{Tr}((h_\beta \otimes 1 \ldots \otimes 1) \circ \mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(\mathcal{A}_r^\mu))$$

$$= \sum_{\substack{(\gamma_0,(\gamma_x),(\delta_{x_0})) \\ \mathrm{inv}(\gamma_0,(\gamma_x),(\delta_x))=0}} c(\gamma_0,(\gamma_x),(\delta_x)) \cdot \left(\prod_{v \neq x_0} \mathrm{O}_{\gamma_v}(f_{\beta_v}) \right) \cdot \mathrm{TO}_{\delta_{x_0}\sigma}(\psi'_{r,\mu})$$

which is what we wanted to show. □

8.2. Calculating the trace in situation B.

We now want to prove an analogous formula for the trace in situation B. The computation in this case is a little more involved. The main reason is that the action of S_r on $\mathcal{B}_{x_0^r}$ is difficult to express explicitly, since it is obtained by "continuation" from a locus where it is described explicitly. More precisely, it was obtained from the fact that $\mathcal{B}|_U$ was a local system, so that we could extend it over X. However, this process obfuscates the geometric meaning of this action, and we will need to use the results of §6, particularly the geometric model of base change studied in §6.4.3, in order to understand it.

Theorem 8.3. *Let $T' \subset |X^\circ|$. Assume that $K_v := \mathcal{G}(\mathcal{O}_v)$ is spherical at all $v \in T'$. Let*

$$\beta = (\beta_v \in K_v \backslash G(F_v)/K_v)_{v \in T'}$$

and $h_\beta \in \mathcal{H}$ be the corresponding Hecke operator. Let $R\Psi_{x_0^r}(\mathcal{B}_r^\mu)$ be as in Definition 7.7, let τ be as in Theorem 7.5, and let $\psi_{r,\mu'}$ be as in §6.4.1. Then we have

$$\mathrm{Tr}(h_\beta \circ \mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(\mathcal{B}_r^\mu))$$

$$= \sum_{\substack{(\gamma_0,(\gamma_x),(\delta_x)) \\ \mathrm{inv}(\gamma_0,(\gamma_x),(\delta_x))=0}} c(\gamma_0,(\gamma_x),(\delta_x)) \cdot \left(\prod_{v \neq x_0} \mathrm{O}_{\gamma_v}(f_{\beta_v}) \right) \cdot \mathrm{O}_{\gamma_{x_0}}(b(\psi'_{r,\mu})).$$

---

[7]The proof of this formula, which is not explicitly written in [Ngo06], goes as follows. By (8.2) we have

$$\mathrm{Tr}(\mathrm{Frob}\circ\tau, R\Psi_{x_0^r}(\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}^r}(r.\mu))_\nu) = \mathrm{Tr}(\mathrm{Frob}\circ\tau, R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu))_\nu^{\boxtimes r})$$

$$= \mathrm{Tr}(\mathrm{Frob}^r, R\Psi_{x_0}(\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu))_\nu),$$

where in the last equality we used Lemma 6.12.

Proof. Let $X^{(r)} = \mathrm{Sym}^r(X)$ by the rth symmetric power of X, and

$$\mathrm{Sht}^\mu_{B,X^{(r)}} := \mathrm{Sht}^{\leq r.\mu}_{\mathcal{G},X^{(r)}} / a^{\mathbf{Z}}.$$

The idea here is to push down the computation from X^r to $X^{(r)}$, which trivializes the S_r action on the fiber $\mathrm{Sht}_{B,X^{(r)}}$ over a point, transferring the effect of this action completely to the sheaf theory, where it was studied in §6. Consider the following commutative diagram, in which the front cartesian square is the fiber of the back cartesian square over the diagonal copy of X in $X^{(r)}$.

$$(8.5)$$

The map $\mathrm{add} \colon X^r \to X^{(r)}$ is totally ramified over the diagonal $\Delta(X) \subset X^r$, so for any étale sheaf $\mathcal{F}$ on X^r the stalk of $\mathcal{F}$ at $x^r \in X^r$ is canonically identified with the stalk of $\mathrm{add}_*\mathcal{F}$ at $\mathrm{add}(x^r) \in X^{(r)}$. Therefore, from the front cartesian square we have

$$\mathrm{Tr}(h_\beta \circ \mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(\mathcal{B}_r^\mu)) = \mathrm{Tr}(h_\beta \circ \mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(R\pi_!^\circ Rm_! \, \mathrm{Sat}_{\mathrm{Sht}_B^\mu}(r.\mu)))$$

where π° is the restriction of π to the fiber over X°. We now proceeding as before, using the Grothendieck-Lefschetz trace formula to rewrite the trace in terms of a sum of traces over fixed points. We begin by describing the fixed points of $h_\beta \circ \mathrm{Frob}_{x_0} \circ \tau$ on $\mathrm{Sht}^{\leq r\mu}_{\mathcal{G},X} / a^{\mathbf{Z}}$.

Now, on $\mathrm{Sht}^\mu_{B,X^{(r)}}$ the permutation τ evidently acts trivially. A point of $\mathrm{Sht}_{\mathcal{G},X} / a^{\mathbf{Z}}|_{x_0}$ is a modification

$$\sigma \mathcal{E}_1|_{X-x_0} \xrightarrow{\leq r\mu} \mathcal{E}_1|_{X-x_0}$$

occurring over x_0. The map σ takes this to

$$\sigma^2 \mathcal{E}_1|_{X-x_0} \xrightarrow{\leq r\mu} \sigma \mathcal{E}_1|_{X-x_0}.$$

Therefore, a fixed point of the correspondence $h_\beta \circ \mathrm{Frob}_{x_0} \circ \tau$ is equivalent to the data of commuting modifications

$$t \colon \sigma \mathcal{E}_1|_{X-x_0} \xrightarrow{\leq r\mu} \mathcal{E}_1|_{X-x_0}$$

$$t' \colon \sigma \mathcal{E}_1|_{X-T'} \xrightarrow{=\beta} \mathcal{E}_1|_{X-T'}.$$

Hence we see that (remember that x_0 is assumed to have degree 1)

$$\mathrm{Fix}(h_\beta \circ \mathrm{Frob}_{x_0} \circ \tau) = \bigcup_{\nu \leq r\mu} \mathcal{C}(\nu_{x_0}, \beta; x_0, T'; 1)$$

so by the Grothendieck-Lefschetz trace and arguing as in §8.1 for situation A,

$$\mathrm{Tr}(h_\beta \circ \mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(\mathcal{B}_r^\mu)) = \sum_{\nu \leq r\mu} \#\mathcal{C}(\nu_{x_0}, \beta; x_0, T'; 1)$$
$$\cdot \mathrm{Tr}(h_\beta \circ \mathrm{Frob} \circ \tau, R\Psi_{x_0^r}(Rm_! \,\mathrm{Sat}_{\mathrm{Sht}_B^\mu}(r.\mu))_\nu).$$

Using Corollary 5.14, we rewrite this as

$$\mathrm{Tr}(h_\beta \circ \mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(\mathcal{B}_r^\mu))$$
$$= \sum_{\substack{(\gamma_0,(\gamma_x),(\delta_x)) \\ \mathrm{inv}(\gamma_0,(\gamma_x),(\delta_x))=0}} c(\gamma_0, (\gamma_x), (\delta_x)) \cdot \prod_{v \neq x_0} \mathrm{O}_{\gamma_v}(f_{\beta_v})$$
$$\cdot \sum_{\nu \leq \mu} \mathrm{TO}_{\delta_{x_0}}(f_\nu) \cdot \mathrm{Tr}(h_\beta \circ \mathrm{Frob} \circ \tau, R\Psi_{x_0^r}(Rm_! \,\mathrm{Sat}_{\mathrm{Sht}_B^\mu}(r.\mu))_\nu).$$
$$(8.6)$$

As in the previous calculation for situation A, the Hecke operator acts trivially on the stalk at x_0 because is supported on a disjoint set of points. We use the affine Grassmannian as a local model to calculate $\mathrm{Tr}(\mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(Rm_!\mathrm{Sat}_{\mathrm{Sht}_B}(r.\mu))_\nu)$. By Theorem 4.19, we have

$$\mathrm{Tr}(\mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(Rm_! \,\mathrm{Sat}_{\mathrm{Sht}_B^\mu}(r.\mu))_\nu) = \mathrm{Tr}(\mathrm{Frob}_{x_0} \circ \kappa', R\Psi_{x_0^r}(Rm_! \,\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G},X^r}^{\leq r.\mu}}(r.\mu))_\nu),$$
$$(8.7)$$

where the notation on the right hand side is as in Corollary 6.17, if we can show that the permutation τ on the left side is carried by Theorem 4.19 to the permutation κ' studied in §6.4.3. (The same issue is raised and explained in [Ngo06, §5.6 Proposition 3].) To prove it, consider the diagram

Here $\widetilde{W}_{X^r}^{\leq r.\mu}$ expresses $\mathrm{Gr}_{\mathcal{G},X^r}^{\leq r.\mu}$ as a local model for $\mathrm{Sht}_B^{\leq r.\mu}|_\Delta$ and $\widetilde{W}_{X^{(r)}}^{\leq r\mu}$ expresses $\mathrm{Gr}_{\mathcal{G},X^{(r)}}^{\leq r\mu}$ as a local model for $\mathrm{Sht}_{B,X^{(r)}}^\mu$. The existence of such a commutive diagram is immediate from the proof of Theorem 4.19. The claim is then that the permutation actions on $R\Psi_{x_0^r}(Rm_! \,\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G},X^r}}(r.\mu))$ and $R\Psi_{x_0^r}(Rm_!\mathrm{Sat}_{\mathrm{Sht}_B}(r.\mu))$, induced by middle extension from $(X^\circ)^r$ to $(X - Z)^r$, are compatible. This is clear from the diagram and the fact that the identity can be checked on the locus where the points $(x_1, \ldots, x_r)$ are distinct, where it is evidently given by the same geometric permutation action in both cases.

Now combining Proposition 6.17, Corollary 4.23, and Lemma 4.17, we have

$$b(\psi_{r,\mu}') = \sum_{\nu \leq r\mu} \mathrm{Tr}(\mathrm{Frob} \circ \kappa', R\Psi_{x_0^r}(Rm_! \,\mathrm{Sat}_{\mathrm{Gr}_{\mathcal{G}}}(\mu))_\nu) f_\nu. \qquad (8.8)$$

Putting together (8.6), (8.7), and (8.8) gives

$$\mathrm{Tr}((h_\beta \otimes 1 \ldots \otimes 1) \circ \mathrm{Frob}_{x_0} \circ \tau, R\Psi_{x_0^r}(\mathcal{B}_r^\mu))$$

$$= \sum_{\substack{(\gamma_0, (\gamma_x), (\delta_x)) \\ \mathrm{inv}(\gamma_0, (\gamma_x), (\delta_x))=0}} c(\gamma_0, (\gamma_x), (\delta_x)) \cdot \left(\prod_{v \neq x_0} \mathrm{O}_{\gamma_v}(f_{\beta_v}) \right) \mathrm{TO}_{\delta_{x_0}}(b(\psi'_{r,\mu})),$$

which is what we wanted to show. $\qquad\square$

8.3. The base change fundamental lemma for parahoric Hecke algebras.

We can now deduce some cases of the base change fundamental lemma.

Corollary 8.4. *Let $T' \subset |X^\circ|$. Assume that $K_v := \mathcal{G}(\mathcal{O}_v)$ is spherical at all $v \in T'$. Let*

$$\beta = (\beta_v \in K_v \backslash G(F_v)/K_v)_{v \in T'}$$

and $h_\beta \in \mathcal{H}$ be the corresponding Hecke operator. Let $\psi_{r,\mu'}$ be as in §6.4.1 and the base change homomorphim $\psi_{r,\mu'} \mapsto b(\psi_{r,\mu'})$ be as in §6.1. Then we have

$$\sum_{\substack{(\gamma_0, (\gamma_x), (\delta_x)) \\ \mathrm{inv}(\gamma_0, (\gamma_x), (\delta_x))=0}} c(\gamma_0, (\gamma_x), (\delta_x)) \cdot \left(\prod_{v \neq x_0} \mathrm{O}_{\gamma_0}(f_{\beta_v}) \right) \cdot \mathrm{TO}_{\delta_{x_0}\sigma}(\psi'_{r,\mu})$$

$$= \sum_{\substack{(\gamma_0, (\gamma_x), (\delta_{x_0})) \\ \mathrm{inv}(\gamma_0, (\gamma_x), (\delta_x))=0}} c(\gamma_0, (\gamma_x), (\delta_x)) \cdot \left(\prod_{v \neq x_0} \mathrm{O}_{\gamma_0}(f_{\beta_v}) \right) \cdot \mathrm{O}_{\gamma_0}(b(\psi'_{r,\mu}))$$

Proof. This follows immediately from substituting Theorem 8.2 and Theorem 8.3 into Corollary 7.8, and the following comment about changing the γ_v to γ_0: since by definition of $X - Z$ we have that $\mathcal{G}(F_x) \cong \mathrm{GL}_n(F_x)$ for all $x \in X - Z$, the notion of stable conjugacy coincides with the notion of conjugacy. $\qquad\square$

It seems to be "well-known" how to deduce a fundamental lemma from a statement such as Corollary 8.4.[8] Nevertheless, let us give a proof for completeness, following [Ngo06, §5.7 Théorème 1]. First we introduce a piece of notation.

Definition 8.5. *For $\mu = (\mu_1, \ldots, \mu_n) \in X_*(\mathrm{GL}_n) \cong \mathbf{Z}^n$, we define*

$$|\mu| := \mu_1 + \ldots + \mu_n.$$

The stack $\mathrm{Sht}_{\mathcal{G}}^{\leq \mu}$ is non-empty if and only if $|\mu| = 0$, because a $\mathcal{G}$-bundle has the notion of degree on X, which is preserved by the Frobenius twist σ on S. Let

$$Z(\mathcal{H}_{G(F_{x_0,r}),J})_0 \subset Z(\mathcal{H}_{G(F_{x_0}),J})$$

be the subspace generated by the $\psi'_{r,\mu}$ with $|\mu| = 0$, which is the same as the subspace generated by the $\psi_{r,\mu}$ with $|\mu| = 0$.

Theorem 8.6. *Let F_t be a local field of characteristic p, and $F_{t,r}/F_t$ the unramified extension of degree r. Let δ be a σ-conjugacy class in $\mathrm{GL}_n(F_{t,r})$, with norm $N\delta_t = \gamma_t \in \mathrm{GL}_n(F_t)$. Assume γ_t is regular semisimple and separable. If $\phi \in Z(\mathcal{H}_{G(F_{t,r}),J})_0$, then we have*

$$\mathrm{TO}_{\delta_t\sigma}(\phi) = \mathrm{O}_{\gamma_t}(b(\phi)).$$

---

[8]It is remarked on p.84 of the Arxiv version 2 of [Ngo06] that this is "standard", and a reference is given to [Clo90].

Proof. Let $\mathbf{F}_q$ be the residue field of F_t. Choose a global curve X over $\mathbf{F}_q$ having a rational point x_0, and function field F, so that $F_{x_0} \cong F_t \cong \mathbf{F}_q((t))$. Choose a division algebra D as in §7.1, and define G and $\mathcal{G}$ as in §3.5. We can then apply Corollary 8.4.

For a fixed function $h \in \mathcal{H}_{G,K}(\mathbf{A})$ the orbital integrals and twisted orbital integrals are locally constant near regular semisimple separable elements. Therefore, by weak approximation we can choose $\widetilde{\gamma} \in G(F)$ close enough to γ_t in the t-adic topology so that $\widetilde{\gamma} = N(\widetilde{\delta}_{x_0}) \in G(F_t \otimes_{\mathbf{F}_q} \mathbf{F}_{q^r})$ for some $\widetilde{\delta}_{x_0} \in G(F_t \otimes_{\mathbf{F}_q} \mathbf{F}_{q^r})$, and such that

$$\mathrm{O}_{\gamma_{x_0}}(b(\psi_\mu)) = \mathrm{O}_{\widetilde{\gamma}}(b(\psi_\mu))$$
$$\mathrm{TO}_{\delta_{x_0}\sigma_{x_0}}(\psi_\mu) = \mathrm{TO}_{\widetilde{\delta}_{x_0}\sigma_{x_0}}(\psi_\mu).$$

We can choose an appropriate Hecke operator $h = (h_v) \in \mathcal{H}_G(\mathbf{A})$ so that $\mathrm{O}_{\widetilde{\gamma}}(h_v) \neq 0$ for $v \neq x$.

Because a fixed choice of h is the identity at all but finitely many places, any Kottwitz triple for which the product of orbital integrals is non-zero forces the γ_v to be in K_v at all but finitely many v. Then by [Kot86b, Proposition 7.1] there are only finitely many possibilities for the Kottwitz triple, as all γ_v outside a fixed finite set must be (rationally) conjugate to γ. (Technically this discussion is unnecessary here because we are only dealing with GL_n at this point.) Therefore, since the support of any adelic Hecke operator is compact open in $G(\mathbf{A})$, while $G(F)$ is discrete, for any fixed $h \in \mathcal{H}_{G,K}(\mathbf{A})$ there are only finitely many non-zero summands in Corollary 8.4.

For GL_n, different conjugacy classes of γ_0 are also different stable conjugacy classes. Hence we may choose the Hecke operator at an unramified auxiliary place appropriately to ensure that

$$\left(\prod_{v \neq x_0} \mathrm{O}_{\gamma_0}(h_v) \right) \cdot \mathrm{TO}_{\delta_{x_0}\sigma}(\psi'_{r,\mu}) \quad \text{and} \quad \left(\prod_{v \neq x_0} \mathrm{O}_{\gamma_0}(h_v) \right) \cdot \mathrm{O}_{\gamma_0}(b(\psi'_{r,\mu}))$$

vanish except for the chosen γ_{x_0}. Then we have

$$c(\gamma_0, (\gamma_x), (\delta_x)) \cdot \left(\prod_{v \neq x_0} \mathrm{O}_{\gamma_0}(h_v) \right) \cdot \mathrm{TO}_{\delta_{x_0}\sigma}(\psi'_{r,\mu})$$

$$= c(\gamma_0, (\gamma_x), (\delta_x)) \cdot \left(\prod_{v \neq x_0} \mathrm{O}_{\gamma_0}(h_v) \right) \cdot \mathrm{O}_{\gamma_0}(b(\psi'_{r,\mu}))$$

Since $|\mu| = 0$, $\mathrm{Sht}_{\mathcal{G}}^{\leq \mu}$ is non-empty so these terms are not 0. Dividing out by the common (necessarily non-zero) factor $c(\gamma_0, (\gamma_x), (\delta_x)) \cdot \left(\prod_{v \neq x_0} \mathrm{O}_{\gamma_0}(h_v) \right)$ then yields the desired equality for all $\psi'_{r,\mu}$ with $|\mu| = 0$. We conclude by observing that these span $Z(\mathcal{H}_{G(F_{x_0,r}),J})_0$. $\square$

REFERENCES

[Bro13]   Michael Broshi, *G-torsors over a Dedekind scheme*, J. Pure Appl. Algebra **217** (2013), no. 1, 11–19. MR 2965898

[Clo90]   Laurent Clozel, *The fundamental lemma for stable base change*, Duke Math. J. **61** (1990), no. 1, 255–302. MR 1068388

[Del73]   P. Deligne, *Séminaire de géométrie algébrique du Bois Marie: Groupes de Monodromie en Géometrie Algébrique (SGA 7 II)*., Lecture Notes in Mathematics, vol. 569, Springer-Verlag, 1973.

[Dri87]   V. G. Drinfel'd, *Cohomology of compactified moduli varieties of F-sheaves of rank* 2, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **162** (1987), no. Avtomorfn. Funkts. i Teor. Chisel. III, 107–158, 189. MR 918745

[Gai01]   D. Gaitsgory, *Construction of central elements in the affine Hecke algebra via nearby cycles*, Invent. Math. **144** (2001), no. 2, 253–280. MR 1826370

[Gro98]   Benedict H. Gross, *On the Satake isomorphism*, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser., vol. 254, Cambridge Univ. Press, Cambridge, 1998, pp. 223–237. MR 1696481

[Hai09]   Thomas J. Haines, *The base change fundamental lemma for central elements in parahoric Hecke algebras*, Duke Math. J. **149** (2009), no. 3, 569–643. MR 2553880

[HaRi]   Thomas Haines and Timo Richarz, *The test function conjecture for parahoric local models*, Preprint available at `https://arxiv.org/pdf/1801.07094.pdf`.

[HaRa]   Urs Hartl and Esmail Arasteh Rad, *Langlands-rapoport conjecture over function fields*, Preprint available at `https://arxiv.org/pdf/1605.01575.pdf`.

[Kot80]   Robert Edward Kottwitz, *Orbital integrals on* GL$_3$, Amer. J. Math. **102** (1980), no. 2, 327–384. MR 564478

[Kot86a]   Robert E. Kottwitz, *Base change for unit elements of Hecke algebras*, Compositio Math. **60** (1986), no. 2, 237–250. MR 868140

[Kot86b]   ———, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), no. 3, 365–399. MR 858284

[Kot90]   ———, *Shimura varieties and λ-adic representations*, Automorphic forms, Shimura varieties, and *L*-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 161–209. MR 1044820

[Kot92]   ———, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), no. 2, 373–444. MR 1124982

[KP]   Mark Kisin and Georgios Pappas, *Integral models of Shimura varieties with parahoric level structure*, Preprint available at `https://arxiv.org/pdf/1512.01149.pdf`.

[KW01]   Reinhardt Kiehl and Rainer Weissauer, *Weil conjectures, perverse sheaves and l'adic Fourier transform*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 42, Springer-Verlag, Berlin, 2001. MR 1855066

[Lab90]   J.-P. Labesse, *Fonctions élémentaires et lemme fondamental pour le changement de base stable*, Duke Math. J. **61** (1990), no. 2, 519–530. MR 1074306

[Laf97]   Laurent Lafforgue, *Chtoucas de Drinfeld et conjecture de Ramanujan-Petersson*, Astérisque (1997), no. 243, ii+329. MR 1600006

[Laf02]   ———, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. **147** (2002), no. 1, 1–241. MR 1875184

[Laf18]   Vincent Lafforgue, *Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale*, J. Amer. Math. Soc. **31** (2018), no. 3, 719–891. MR 3787407

[Lau04]   Eike Sören Lau, *On generalised D-shtukas*, Bonner Mathematische Schriften [Bonn Mathematical Publications], vol. 369, Universität Bonn, Mathematisches Institut, Bonn, 2004, Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2004. MR 2206061

[MV07]   I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143. MR 2342692

[ND13]   Tuan Ngo Dac, *Comptage des G-chtoucas: la partie elliptique*, Compos. Math. **149** (2013), no. 12, 2169–2183. MR 3143709

[ND15]   ———, *On a counting problem for G-shtukas*, Arithmetic and geometry, London Math. Soc. Lecture Note Ser., vol. 420, Cambridge Univ. Press, Cambridge, 2015, pp. 318–350. MR 3467129

[Ngo99]   Bao Châu Ngo, *Faisceaux pervers, homomorphisme de changement de base et lemme fondamental de Jacquet et Ye*, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 5, 619–679. MR 1710755

[Ngo06]   _____, *D-chtoucas de Drinfeld à modifications symétriques et identité de changement de base*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 2, 197–243. MR 2245532

[NND08]   Bao Châu Ngô and Tuân Ngô Dac, *Comptage de G-chtoucas: la partie régulière elliptique*, J. Inst. Math. Jussieu **7** (2008), no. 1, 181–203. MR 2398149

[PR08]   G. Pappas and M. Rapoport, *Twisted loop groups and their affine flag varieties*, Adv. Math. **219** (2008), no. 1, 118–198, With an appendix by T. Haines and Rapoport. MR 2435422

[PR10]   Georgios Pappas and Michael Rapoport, *Some questions about G-bundles on curves*, Algebraic and arithmetic structures of moduli spaces (Sapporo 2007), Adv. Stud. Pure Math., vol. 58, Math. Soc. Japan, Tokyo, 2010, pp. 159–171. MR 2676160

[PZ13]   G. Pappas and X. Zhu, *Local models of Shimura varieties and a conjecture of Kottwitz*, Invent. Math. **194** (2013), no. 1, 147–254. MR 3103258

[RD]   Walter Randolph Ray-Dulany, *Base change for the iwahori-hecke algebra of gl_2*, Ph.D. thesis. Available at `https://drum.lib.umd.edu/bitstream/handle/1903/10963/RayDulany_umd_0117E_11632.pdf?sequence=1&isAllowed=y`.

[RH]   Esmail Arasteh Rad and Somayeh Habibi, *Local Models for the Moduli Stacks of Global G-shtukas*, Preprint available at `https://arxiv.org/pdf/1605.01588.pdf`.

[Ric16]   Timo Richarz, *Affine Grassmannians and geometric Satake equivalences*, Int. Math. Res. Not. IMRN (2016), no. 12, 3717–3767. MR 3544618

[RZ15]   Timo Richarz and Xinwen Zhu, *Construction of the full Langlands dual group via the geometric Satake correspondence (appendix to The geometric Satake correspondence for ramified groups)*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 2, 409–451. MR 3346175

[Var04]   Yakov Varshavsky, *Moduli spaces of principal F-bundles*, Selecta Math. (N.S.) **10** (2004), no. 1, 131–166. MR 2061225

[Zhu]   Xinwen Zhu, *An introduction to affine Grassmannians and the geometric Satake equivalence*, To appear in Proceedings of the IAS/Park City Mathematics Series. Preprint available at `https://arxiv.org/pdf/1603.05593.pdf`.

[Zhu14]   _____, *On the coherence conjecture of Pappas and Rapoport*, Ann. of Math. (2) **180** (2014), no. 1, 1–85. MR 3194811