TERM PAPER FOR 18.784, FALL 2021

Information. The term paper will consist of a mathematical essay of about 10 single-spaced pages in 12-point font, written in a professional style in $\LaTeX$, on a topic related to p-adic numbers. In addition, you will give a 20-minute talk during the last three weeks of class on the subject of your paper.

Guidelines. The paper should be written for an audience at the level of your peers. Terms which have not been introduced in class should be defined (it is okay to give informal definitions occasionally), theorems that have not been stated in class should be explained, etc. As with your presentations, it is not necessary to present complete proofs for every statement; the main goal is to convey the key ideas.

The paper will be evaluated on the quality and originality of exposition. You may go beyond 10 pages if needed, but additional length will not by itself earn favorable marks. (In fact, it is less painful to read a short poorly-written paper than a long poorly-written paper.) Also do not equate difficulty with quality – a poorly written paper on very difficult subject will earn few points. On the other hand, many easy topics are already covered well by existing literature, and a well-written paper which looks very similar to existing well-written papers will also receive few points.

Submissions. Solutions must be typed in $\LaTeX$ and submitted online before 11:59 P.M. on the due date. **Late submissions will not be accepted.**

Incremental progress. You are required to submit partial work towards the term paper at various times during the semester. These works will not be graded individually, but part of your paper grade comes from keeping up with these assignments.

- Reading assignment (due Sep 30). Read the assigned short essay, and come prepared to discuss it in class.
- Topic preferences (due Oct 8). List five, starting with your first choice. If you are proposing your own topic, mention some references or otherwise describe in a little detail what you have in mind.
- References and Outline (due Oct 18). List the books and articles that you will base your term paper on. Describe what subtopics will be discussed. List the names of the sections of your paper, with a sentence or two for each explaining what it will contain.
- First third of paper (due Nov 1). 3-4 pages at the level of your first draft.
- First draft of paper (due Nov 22).
- Peer revision comments (due Dec 3). You will be split into groups of 3(ish, depending on the total number of students). You will give feedback on your two classmates' first drafts.
- Final version of paper (due Dec 9).

What does it mean to write a mathematical paper? You are not expected to obtain any original results in your paper. However, this does not mean that your paper should consist of material "copied" from references. Rather, you are expected to put significant work into creating an original *exposition*. Your mathematical personality enters in deciding what and how much motivation to include, how to balance technicalities with examples, and so on.

Think of an expository paper as being like a short story – your mission is to craft an engaging narrative. A good way to accomplish this is to distance yourself as much as possible from the references. Do not look at any of your references as you write, or at notes that you took on the references, etc. Try to understand the material so well that you can tell it completely from memory; you will find that the process of internalizing it forces your mind to shape the material into a coherent and compelling story.

Getting started. Starting by trying to get a broad sense of the topic, and finding appropriate references. For the suggested topics I have given at least one initial reference. Wikipedia and Google are useful for this purpose, although you should avoid citing results from Wikipedia. The department also has many knowledgeable graduate students, and I am happy to discuss in office hours.

Some possible topics. Below is a list of possible topics, along with a first reference to get you started. **The listed sources are not required or necessary and may not be sufficient.** You are expected to find additional references on your own – this is an important part of the research process! You can also propose a topic that is not on this list at all, but check with me first to make sure that it is at an appropriate level.

(1) Weierstrass Preparation Theorem. [G20, §6.2, §6.3].
(2) Newton polygons. [G20, §6.4].

The above two topics are covered in Gouvea's book, although we will not get to them in class. To do a good job on these you will need to go beyond the book, and also show significant innovation in your presentation.

(3) Quadratic forms and the Hasse invariant. [S73, Chapters II-IV].
(4) The Brauer group of a p-adic field. [§IV of Milne's notes.]
(5) Lubin-Tate theory. [LT65]
(6) Statements of local class field theory. [C09, §7]

The next two topics require background in algebraic geometry.

(7) Elliptic curves over p-adic numbers. [S09, §7]
(8) Tate curves. [S94, Chapter 5]

The next two topics require background in analysis.

(9) Fourier analysis over p-adic numbers / Local aspects of Tate's thesis. [T67] (This reference needs to be supplemented, but fortunately there are many resources on it.)
(10) p-adic zeta functions (this also requires background in modular forms). [Ser]

The next three backgrounds are about some applications of p-adic numbers.

(11) Monsky Theorem. [Start with the Wikipedia entry]
(12) Skolem-Mahler-Lech. [Start with Terry Tao's blog post.]
(13) Applications of p-adic numbers in cryptography, e.g. this paper.

Final presentation. Your final talk with be a solo, 20-minute blackboard presentation plus an additional 5 minutes of fielding questions, during the last three weeks of class. This is not meant to be a summary of your entire paper. You may decide to spend all your time on motivation, or to just get to the statement of the main theorem without any proofs, or to illustrate one clever proof idea. The material is up to your discretion, but keep in mind that "more is not better".

Tips on mathematical writing. This document by Halmos (click on the name for the hyperlink), also summarized at this post, contains useful guidance on mathematical writing. See also the list of practical suggestions by Bjorn Poonen, and `https://mathcomm.org/writing/teaching-writing/` for more resources.

Grading. The presentation will be graded according to the same rubric as the collaborative presentations, and the paper will be graded according to the following rubric.

- **Mathematical correctness and synthesis of sources (45%).** I am looking to see that the math and its motivation should be correct, sufficiently rigorous, and demonstrate a solid understanding of the material. The paper should provide readers with greater insight than they would receive simply by reading the paper's sources: for example, although the paper may not present original results, it should successfully synthesize material from several sources to create a focused, cohesive narrative.
- **Exposition (45%).** I am looking to see that the paper is carefully crafted to ease reading and understanding for the target audience (peers of the author). For example, the paper should be consistent and cohesive; new ideas should be concisely introduced or motivated before being used; displays and examples should be carefully crafted to aid understanding; citations should clearly acknowledge any sources used; writing should be appropriately concise and carefully formatted and proofread.
- **Writing Process (10%).** I am looking to see that sufficient effort was put into the incremental drafts, and that feedback was understood and taken into account.

References

[C09] Childress, Nancy. Class field theory. Universitext. Springer, New York, 2009. x+226 pp.

[G20] Gouvêa, Fernando Q. p-adic numbers. An introduction. Third edition of [1251959]. Universitext. Springer, Cham, [2020]. vi+373 pp.

[LT65] Lubin, Jonathan; Tate, John. Formal complex multiplication in local fields. Ann. of Math. (2) 81 (1965), 380–387.

[S73] Serre, J.-P. A course in arithmetic. Translated from the French. Graduate Texts in Mathematics, No. 7. Springer-Verlag, New York-Heidelberg, 1973. viii+115 pp.

[Ser] Serre, Jean-Pierre. Formes modulaires et fonctions zta p-adiques. (French) Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), pp. 191–268. Lecture Notes in Math., Vol. 350, Springer, Berlin, 1973.

[S09] Silverman, Joseph H. The arithmetic of elliptic curves. Second edition. Graduate Texts in Mathematics, 106. Springer, Dordrecht, 2009. xx+513 pp.

[S94] Silverman, Joseph H. Advanced topics in the arithmetic of elliptic curves. Graduate Texts in Mathematics, 151. Springer-Verlag, New York, 1994. xiv+525 pp.

[T67] Tate, J. T. Fourier analysis in number fields, and Hecke's zeta-functions. Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), 305–347, Thompson, Washington, D.C., 1967.