18.784: SEMINAR IN NUMBER THEORY
PROBLEM SET 2
DUE BY 11:59PM FRIDAY OCT 1

1. (1) Prove what Jeffery called the "strongest wins principle": if $x, y \in \mathbf{Q}$ are such that $|x|_p \neq |y|_p$, then $|x+y|_p = \max(|x|_p, |y|_p)$.

(2) Prove that multiplication and addition are continuous functions $\mathbf{Q} \times \mathbf{Q} \to \mathbf{Q}$ for the p-adic topology (i.e., the topology induced by the p-adic metric).

2. Recall that for $a \in \mathbf{Q}$ and $r \in \mathbf{R}$, Tristan defined the "open ball" $B(a,r) := \{x \in \mathbf{Q} \colon d_p(a,x) < r\}$ and the "closed ball" $\overline{B}(a,r) := \{x \in \mathbf{Q} \colon d_p(a,x) \leq r\}$.

(1) Tristan said that "every point of $B(a,r)$ is a center". Formulate rigorously what this means and then prove it.

(2) Show that for every r, there exists r' such that $B(a,r) = \overline{B}(a,r')$. Is it true that for every r', there exists r such that $\overline{B}(a,r') = B(a,r)$? Prove it or give a counterexample.

3. Suppose that p and ℓ are two different primes. Show that the p-adic absolute value is not equivalent to the ℓ-adic absolute value.

4. Consider the sequence of integers (written in base 10) $4, 44, 444, 4444, 44444, \ldots$. Zawad showed in class that this is a Cauchy sequence for the 5-adic norm. Find, with proof, its limiting value.