Singularities in formal arc spaces and harmonic analysis over non-archimedean fields

Ngô Bau Châu
Notes by Tony Feng

I'm going to talk about a basic set of questions in harmonic analysis, which have an interpretation in terms of varieties and formal arc spaces. As usual, any new theory comes with a new set of questions. We do know how to do some calculations in this area, but on the other hand we lack even some basic definitions. I hope some people will get interested in these sorts of questions.

1 Formal arc spaces

1.1 Arc space and formal models

Let X be an algebraic variety over a field k.

Definition 1.1. There is an nth *jet space* $J_n(X)$ with the property that

$$J_nX(k) = X(k[t]/t^{n+1}).$$

The *formal arc space* LX is the projective limit of the jet spaces, and satisfies

$$LX(k) = X(k[[t]]).$$

If X is smooth, then J_nX and LX are all smooth. In general, the singularities of LX have finite-dimensional formal models, by results of Drinfeld and Grinberg-Kazhdan. Namely, for any non-degenerate point $x \in LX$, there exists an isomorphism of formal neighborhoods

$$(LX)_x \cong Y_y \times \mathbb{D}^\infty$$

where Y is a finite-dimensional k-scheme.

One of the main problems with this result is that you have it only for formal neighborhoods, whereas you'd like it for (say) étale neighborhoods.

Perverse sheaves. What we really want is to define the notion of perverse sheaf on these formal arc spaces. In particular, can we define the notion of intersection complexes?

At the moment, the answers to these questions are not known. However, we do at least know that we can define "trace functions."

Assume that $k = \mathbb{F}_q$ is a finite field. Then is is a theorem that

$$x \mapsto \mathrm{tr}(\sigma_q, IC_y(Y))$$

is well-defined, independent of the formal model Y used to model a neighborhood of x.

1.2 Reductive monoids

Definition 1.2. A *reductive monoid* is a $G \times G$-equivariant normal embedding $j \colon G \hookrightarrow M$ of a reductive group G.

Semisimple groups have no $G \times G$-equivariant normal embedding. You need some $\mathbb{G}_m$ in the center to make a normal embedding.

***L*-monoids.** We describe a construction for L-functions attached to certain reductive monoids.

Vinburg gave many constructions of reductive monoids. For instance, one can construct M_ρ out of the data:

- A short exact sequence
$$0 \to G' \to G \to \mathbb{G}_m \to 0.$$
- Taking Langlands dual of the above sequence, we get
$$0 \to \mathbb{G}_m \to \widehat{G} \to \widehat{G'} \to 0.$$
- A representation $\rho \colon \widehat{G} \to \mathrm{GL}(V_\rho)$ such that the restriction to $\mathbb{G}_m$ gives rise to the scalar multiplication on V_ρ. The is the setting of defining automorphic L-functions in Langlands' theory.

In the classical setting, Langlands showed that you get all the local L-functions by integrating over integral matrices. In the reductive monoids setting, one replaces integral matrices by

$$L^\circ M_\rho(k) := M_\rho(O) \cap G(F).$$

Here F is a local field, O its ring of integers, and k the residue field. We don't know how to define IC sheaves, but we can define the associated trace function ψ_ρ on $G(F)$, which is $G(O) \times G(O)$-invariant.

Theorem 1.3 (Bouthier-Ngô-Sakellaridis)**.**

$$\mathrm{tr}(\psi_\rho, \pi \otimes |\det|^s) = L(s, \pi, \rho)$$

for all unramified representations π of $G(k((t)))$.

We have $\psi = \sum_n \psi_n$ where ψ_n is the spherical Hecke function whose Satake transform is the trace of the nth symmetric power of ρ. The functions ψ_n have disjoint support.

1.3 Braverman-Kazhdan conjecture

Conjecture. There exists a Schwartz space $\mathcal{S}_\rho(G(F))$ with ψ_ρ as a typical member, equipped with a Fourier-type transform that locally satisfies the Poisson summation formula.

From this the analytic continuation and functional equation of automorphic L-functions would follow.

Another expectation is that $S_\rho(G)$ should be generated by trace functions of perverse sheaves on $L^\circ M_\rho$. In fact, it should be the space of compactly supported sections of some sheaf.

1.4 Use of quotient stacks

A standard trick is to replace the formal arc space by some stack, and then replace the formal disk with some curve.

The quotient stack $[M_\rho/G]$ contains the point G/G as an open substack. Instead of $L^\circ M_\rho$, we can consider the space of maps $x\colon \mathbb{D} \to [M_\rho/G]$ sending the generic point to $[G/G]$.

We can globalize this: let C be a smooth projective curve and $\mathcal{T}$ a line bundle on C. Let $\mathcal{M}$ be the space of maps $\phi\colon C \to [M/G] \wedge^{\mathbb{G}_m} \mathcal{T}$ mapping the generic point of C in $[G/G]$. Then $\mathcal{M}$ is a finite-dimensional model of singularities of $L^\circ M$. In fact, this space is very well known: it is an invariant closed subscheme of the Beilinson-Drinfeld Grassmannian. The calculation of the trace function for its IC sheaves is standard.

2 Relative situation: loop geometry

We are interested in the space of maps $\mathbb{D} \to [M/G]$ containing the point as an open subset. We might be interested in $f : Y \to X$ where Y is an Artin stack, X is a scheme, and f is generically an isomorphism. We want to study the space of maps $\mathbb{D} \to Y$ over a given map $\mathbb{D} \to X$.

We think of Y being "slightly different" from X, and this relative problem as capturing this difference between Y and X.

2.1 Jacquet-Rallis integral

Let $G = \mathrm{GL}_n$ and $\mathfrak{g} = \mathrm{Lie}(G)$, $H = \mathrm{GL}_{n-1}$ a subgroup of G acting on $\mathfrak{g}$ by the adjoint representation. We say that $x \in \mathfrak{g}$ is *H-regular* if $\mathrm{Stab}_X(H)$ is trivial and Hx is closed. Let $\mathcal{S}$ be the space of locally constant functions with compact support on $\mathfrak{g}(F)$. For every $\phi \in \mathcal{S}$, we want to understand the function

$$JR(x) := \int_H \phi(\mathrm{ad}(h)x)\, dh$$

on the set of H-regular elements of $\mathfrak{g}(F)$.

The quotient stack formalism. The natural setting for the computation is the stack $\mathcal{Y}$ classifying quadruples $(V, x, v, v^\vee)$ where V is an n-dimensional vector space, $x \in \mathrm{End}(V)$, and $v \in V, v^\vee \in V^\vee$. Then $[\mathfrak{g}/H]$ is the closed substack of $\mathcal{Y}$ defined by the equation $\langle v^\vee, v\rangle = 1$.

We have a map $f\colon \mathcal{Y} \to \mathfrak{b} = \mathbb{A}^n \times \mathbb{A}^n$, with coordinates $a_i = \mathrm{tr}(\wedge^i x)$ and $b_j = \langle v^\vee, x^j v\rangle$.

If $\phi = 1_{\mathfrak{g}(O)}$, then we expect that

$$JR_x(\phi) = \sum_{i=0}^{n} \psi_i(x)$$

where ψ_i is the trace function on the closed subset of $L^\circ\mathfrak{b}$ defined by $\mathrm{val}(\gamma) \geq i$. A similar equality in the global setting was proved by Zhiwei Yun. We expect that for every $\phi \in \mathcal{S}$, then function $x \mapsto JR_x(\phi)$ has an asymptotic of the same form.

2.2 The adjoint action

Let $G = \mathrm{GL}_n$ act on $\mathfrak{g} = \mathfrak{gl}_n$ by the adjoint action. Invariant theory provides a G-invariant map $f\colon \mathfrak{g} \to \mathfrak{a}$ where $\mathfrak{a} = \mathbb{A}^n$. The quotient $[\mathfrak{g}/G] \to \mathfrak{a}$ is not an isomorphism because the stabilizers are large (even the generic one).

There exists an open subshceme a^{rss} (regular semisimple subset) of $\mathfrak{a}$ that is the complement of the discriminant divisor. An element $x \in \mathfrak{g}^{rss} := f^{-1}(\mathfrak{a})$ is a regular semisimple matrix. The centralizer G_x is a torus depending only on a.

For $\phi \in C_c^\infty$, we introduce the orbital integral

$$O_x(\phi) = \int_{G(F)/G_x(F)} \phi(\mathrm{ad}(g)^{-1}x)dg/dt.$$

This is a coset space, so you have to put a measure on the centralizer. We won't go into this, but it is possible. There is a naïve way which is traditionally used, but there is also a better way.

When $x \to 0$, there exists a germ expansion

$$O_x(\phi) = \sum_{\xi} \Gamma_\xi(x) O_\xi(\phi)$$

where ξ ranges over the set of nilpotent orbits. You want to think of $O_x(\phi)$ as a sheaf, and $x \to 0$ is like taking the stalk. This is called the *Shalika expansion*. A priori this is only a germ, but you can extend it to define a *function* $\Gamma_\xi\colon L^\circ\mathfrak{a}(k) \to \mathbb{C}$. Note that these *do not depend* on ϕ.

Question: Are these functions connected to perverse sheaves on $L^\circ\mathfrak{a}$?

I believe that the answer is yes. Of course, we expect to see non-trivial local systems, because we have stabilizers.

Let $a\colon \mathbb{D} \to \mathfrak{a}$ send the generic point of $\mathbb{D}$ to $\mathfrak{a}^{rss}$. The stack of maps $a\colon \mathbb{D} \to [\mathfrak{g}/G]$ lying over a is not locally of finite type.

The advantage of the global situation is that if C is a smooth projective curve and $\mathcal{T}$ a $\mathbb{G}_m$-torsor over C, then the stack of maps $a\colon C \to [\mathfrak{g}/G] \wedge^{\mathbb{G}_m} \mathcal{T}$ is locally of finite type. (The twist is needed for the resulting stack to be geometrically nice.) The *Hitchin fibration* is a special case of this.