SESSION 3: HITCHIN FIBRATION AND SUPPORT THEOREM FOR SL_2

We will consider a smooth connected projective curve over $\mathbb{C}$.

(i) Describe for SL_2 the Hitchin fibration

$$f : \mathcal{M}_D \to \mathbb{A}_D.$$

(ii) For $a \in H^0(X, 2D)$, we consider the spectral curve X_a defined on the total space of $\mathcal{O}_X(D)$ over X by the equation :

$$t^2 - a = 0.$$

We suppose that a is generically non-zero, we denote by $\mathbb{A}_D^{\heartsuit}$ this open locus. Prove that if at every closed point $v \in X$, $\mathrm{val}_v(a) \leq 1$, then the spectral curve is smooth and connected and describe the fiber $\mathcal{M}_a$. Describe the Picard stack which acts $\mathcal{P}_a$ Describe the group scheme J_a over X such that $\mathrm{Bun_J} = \mathcal{P}_a$.

(iii) Prove that $\mathcal{M}_D^{\heartsuit}$ is smooth for $\deg(D) \geq 2g - 1$, we will compute its tangent complex.

(iv) For $a \in \mathbb{A}_D^{\heartsuit}$, let U_a be the locus where the morphism $\pi_a : X_a \to X$ is étale, we get in particular a map $\pi_1(U_a, a) \to W$. Show that we have an exact sequence :

$$H^0(X, \bigoplus_{x \in X - U_a} \mathbb{Z}/2\mathbb{Z}) \longrightarrow \pi_0(\mathcal{P}'_a) \longrightarrow \pi_0(\mathcal{P}_a) \longrightarrow 0$$

where $\mathcal{P}'_a$ is the stack of J_a^0-torsors.

(v) We admit that $\pi_0(\mathcal{P}'_a) = (X_*(T))_{\pi_1(U_a, a)}$. (*)Show that $\pi_0(\mathcal{P}_a) = 0$ only if at least one ramification point is unibranch and equals $\mathbb{Z}/2\mathbb{Z}$ otherwise.

Now we assume that $\pi_0(\mathcal{P}_a) = \mathbb{Z}/2\mathbb{Z}$.

(i) Prove that the normalisation of the spectral curve is an étale cover of X.

(ii) Show that there exists an effective divisor D' on X such that :

$$\mathrm{div}(a) = 2D'$$

with D' such that $\mathcal{L}_\rho := \mathcal{O}_X(D' - D)$ is of trivial square, associated to ρ.

(iii) Deduce that there exists a section $b \in H^0(X, \mathcal{L}_\rho \otimes D)$ such that $b^2 = a$.

(iv) Now, for every point $\mathcal{L}_\rho \in \mathrm{Pic}_X[2]^*$, we can associate an elliptic torus H_ρ on X associated to the representation $\rho : \pi_1(X, x) \to \mathbb{Z}/2\mathbb{Z}$. Let $\mathbb{A}_{H_\rho, D} = H^0(X, \mathcal{L}_\rho \otimes D)$ its Hitchin base and $\mathbb{S}_\rho$ it's image in $\mathbb{A}_D$.

Show that we have an action of $\mathbb{Z}/2\mathbb{Z}$ on $f_*\overline{\mathbb{Q}}_l$ and identify the support of the « minus »part.