SESSION 2, 2 1/2: ESTIMATE OF KLOOSTERMAN SUMS

Exercice 1 : Lang Isogeny Let G_0 a connected commutative group over $\mathbb{F}_p$, $\mathcal{L} : G_0 \to G_0$ the Lang isogeny.

(i) Show that :

$$\mathcal{L}_* \overline{\mathbb{Q}}_l \simeq \bigoplus_{\chi \in \widehat{G_0(\mathbb{F}_q)}} \mathcal{F}(\chi).$$

(ii) Deduce that for any $\chi \in \widehat{G_0(\mathbb{F}_q)}$, $H_c^*(G, \mathcal{F}(\chi)) = 0$.

Exercice 2 : Generalized Kloosterman sums (after Deligne) Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_q$, for $a \in \mathbb{F}$, we consider the hypersurface V_a on $\mathbb{A}^n$ defined by $x_1 x_2 \dots x_n = a$. Let $\psi : \mathbb{F}_q \to \mathbb{C}^*$ be a non-trivial additive character. We want to study the Kloostermann sums :

$$K_{n,a} = \sum_{x_1 \dots x_n = a} \psi(x_1 + \cdots + x_n)$$

and give some estimates. We note $\pi, \sigma : \mathbb{A}^n \to \mathbb{A}^1$ the product and sum of coordinates. In order to do that, we want prove the following theorem :

Theorem 1. *The cohomology groups of V_a with coefficients in $\mathcal{F}(\psi\sigma)$ verify*

(i) $H_c^i = 0$, *for* $i \neq n-1$,

(ii) $H_c^\bullet \simeq H^\bullet$,

(iii) *For* $a \neq 0$, $\dim H^{n-1} = n$,

(iv) *If* $a = 0$, $H_c^{n-1} \simeq \overline{\mathbb{Q}}_l$ *canonically.*

This theorem has a global analog we will prove alongside

Theorem 2. *We have the following assertions :*

(i) *The sheaf* $R^{n-1}\pi_! \mathcal{F}(\psi\sigma)$ *is smooth of rank* n *over* $\mathbb{A}^1 - \{0\}$,

(ii) *The extension by 0 of* $R^{n-1}\pi_! \mathcal{F}(\psi\sigma)$ *from* $\mathbb{A}^1$ *to* $\mathbb{P}^1$ *is the same as the direct image of the restriction to* $\mathbb{G}_m$.

(iii) *At 0, the monodromy is tame.*

(iv) *At* ∞, *the wild inertia acts whithout non zero fixed point and the Swan conductor equals 1.*

(v) *We have* $R\pi_! \mathcal{F}(\psi\sigma) \simeq R\pi_* \mathcal{F}(\psi\sigma)$.

We proceed by induction and show that

$$1(n) \Rightarrow 2(n) \Rightarrow 1(n+1).$$

We assume 1(n) :

(i) Using Ex.1, show 2(n)(i).

(ii) Let $\mathcal{G}$ the direct image of the restriction of $R^{n-1}\pi_!\mathcal{F}(\sigma\psi)$ to $\mathbb{G}_m$. Using an appropriate exact sequence, show first that $\mathcal{G} = R^{n-1}\pi_!\mathcal{F}(\sigma\psi)$.

(iii) We consider Δ the cone of the map $j_!R\pi_!\mathcal{F}(\sigma\psi) \to j_*R\pi_*\mathcal{F}(\sigma\psi)$, where $j : \mathbb{A}^1 \to \mathcal{P}^1$.Show that Δ has finite support and compute $H^*(\mathcal{P}^1, \Delta)$. Deduce 2(ii) and (v).

Computation of Swan conductors : Let D the Galois group of a nonarchimedean local field K, I the inertia group and P the wild inertia. Let V be a finite dimensional $\overline{\mathbb{Q}}_l$-representation of D

(i) Show that : $\mathrm{Swan}_0(R^{n-1}\pi_!\mathcal{F}(\psi\sigma)) + \mathrm{Swan}_\infty(R^{n-1}\pi_!\mathcal{F}(\psi\sigma)) = 1$

(ii) Using the fact that if $V^I = 0$ and $(V \otimes \chi)^I = 0$ for every character χ of I/P, then V is wildly ramified, show that $R^{n-1}\pi_!\mathcal{F}(\psi\sigma)) =$ is tamely ramified at 0.

We can now deduce 1(n+1) for $a \neq 0$ ($a = 0$ is treated separetly).

(i) Using a cohomological reflection of the identity :

$$K_{n+1,a} = \sum_{x \in \mathbb{F}_q^*} \psi(x) K_{n,a/x},$$

and Leray exact sequence, show 1(ii).

(ii) Deduce 1(i).

(iii) By spectral sequence argument, obtain 1(iii).

Exercice 3 : Estimates Show from the previous exercices that, when $a \neq 0$:

$$|K_{n,a}| \leq nq^{n-1/2}$$