Perverse Sheaves and Fundamental Lemmas

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1 Introduction

The topic of the lecture is the application of tools from algebraic geometry to problems in harmonic analysis. This has seen the most success in the context of the Langlands program, in which many technical problems amount to proving equalities between integers arising from harmonic analysis. These identities don't seem to yield to analytical methods, but (surprisingly!) have been established by algebro-geometric techniques. Unfortunately, we won't have time to explain the motivation for the questions from within the Langlands program.

1.1 Grothendieck's function-sheaf dictionary

Grothendieck's function-sheaf dictionary provides translation from problems in analysis to problems in geometry.

I'll start with something very basic. Let X be an algebraic variety over $k = \mathbb{F}_q$. We think of X as being defined by a collection of equations, e.g. $X = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We think of $X(k)$ as the solutions to these equations with coordinates in k:

$$X(k) = \{(x_1, \ldots, x_n) \in k^n \mid P_i(x_1, \ldots, x_n) = 0 \text{ for all } i\}.$$

For any field extension k'/k, we can consider solutions with coordinates in k':

$$X(k') = \{(x_1, \ldots, x_n) \in (k')^n \mid P_i(x_1, \ldots, x_n) = 0 \text{ for all } i\}.$$

In particular, for the algebraic closure $\overline{k}$ we can consider

$$X(\overline{k}) = \{(x_1, \ldots, x_n) \in \overline{k}^n \mid P_i(x_1, \ldots, x_n) = 0 \text{ for all } i\}.$$

Now, we have Frobenius $\sigma_q \in \operatorname{Gal}(\overline{k}/k)$ acting on $X(\overline{k})$ by

$$(x_1, \ldots, x_n) \mapsto (x_1^q, \ldots, x_n^q).$$

The fact that this defines an action on $X(\overline{k})$ boils down to the observation that σ_q preserves the set of solutions to the equations, since it preserves their coefficients. We clearly see that

$$\boxed{\operatorname{Fix}(\sigma_q, X(\overline{k})) = X(k).}$$

More generally, if k'/k is the (unique) field extension of degree m, then

$$X(k') = \operatorname{Fix}(\sigma^m, X(\overline{k})).$$

ℓ-adic sheaves on X.

Let X be a variety over $\mathbb{F}_q$.

Definition 1.1. We say that $\mathcal{F}$ is an *ℓ-adic sheaf* on X if it is constructible, i.e. if there is a stratification $X = \bigsqcup X_\alpha$, where each X_α is a locally closed subvariety, such that $\mathcal{F}|_{X_\alpha}$ is a $\mathbb{Q}_\ell$-local system.

Note that a local system is equivalent to a representation of the fundamental group,

$$\mathcal{F}|_{X_\alpha} \leftrightarrow \rho_{\mathcal{F}_\alpha} \colon \pi_1(X_\alpha) \to \mathrm{GL}_n(\mathbb{Q}_\ell).$$

Example 1.2. Let $X = \operatorname{Spec} k$. Then an ℓ-adic sheaf on Spec k is a continuous representation of $\mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}(n, \mathbb{Q}_\ell)$. (There are variations with $\mathbb{Q}_\ell$ replaced by some extension, often $\overline{\mathbb{Q}_\ell}$.)

Since $\mathrm{Gal}(\overline{k}/k) \cong \widehat{\mathbb{Z}}$, generated by σ, such a representation is determined by the image $\sigma \mapsto P(\sigma)$. Then we have $\mathrm{tr}(\rho(\sigma)) \in \overline{\mathbb{Q}_\ell}$.

A point $x \in X(k)$ corresponds to a map

$$x \colon \operatorname{Spec} k \to X$$

Thus, given an ℓ-adic sheaf $\mathcal{F}$ on X we can pull it back to an ℓ-adic sheaf $x^*\mathcal{F}$ on Spec k. Then we can form $\mathrm{tr}(\sigma, x^*\mathcal{F})$. Thus, from $\mathcal{F}$ we get a function $t_{\mathcal{F}} \colon X(k) \to \overline{\mathbb{Q}_\ell}$ by

$$t_{\mathcal{F}}(x) = \mathrm{tr}(\sigma, x^*\mathcal{F}).$$

This is the *function-sheaf correspondence.*

The dictionary includes relations between the standard manipulations on functions and sheaves. For example:

1. *Tensor product.* If $\mathcal{F}, \mathcal{G}$ are ℓ-adic sheaves on X, then we can form the sheaf $\mathcal{F} \otimes \mathcal{G}$. The corresponding function

$$t_{\mathcal{F}\otimes\mathcal{G}}(x) = t_{\mathcal{F}}(x) t_{\mathcal{G}}(x)$$

   because the representation corresponding to the tensor product is the tensor product of representations.

2. *Pullback.* If $f \colon X \to Y$ is a morphism of algebraic varieties and $\mathcal{F}$ is an ℓ-adic sheaf on Y, then we can form an ℓ-adic sheaf $f^*\mathcal{F}$ on X. At the level of functions, this induces

$$t_{f^*\mathcal{F}}(x) = t_{\mathcal{F}}(f(x)).$$

3. *Pushforward.* This operation is a little subtler. Let X be an algebraic variety over k and $\mathcal{F}$ an ℓ-adic sheaf on X. Then we get a function $t_{\mathcal{F}} \colon X(k) \to \mathbb{Q}_\ell$. Since $X(k)$ is finite, we can consider the sum.

$$\sum_{x \in X(k)} t_{\mathcal{F}}(x).$$

   The Grothendieck-Lefschetz trace formula gives a *cohomological* formula for this:

Theorem 1.3 (Grothendieck-Lefschetz)**.** *We have*

$$\sum_{x \in X(k)} t_{\mathcal{F}}(x) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{tr}(\sigma, H^i_c(X \otimes_k \overline{k}, \mathcal{F})).$$

This is motived by the topological Lefschetz trace formula for the number of fixed points of a map $X \to X$.

Now, if $f : X \to Y$ is a morphism and $\mathcal{F}$ is an ℓ-adic sheaf on X, then for all $y \in Y(k)$ we can consider

$$\sum_{\substack{x \in X(k) \\ f(x) = y}} t_{\mathcal{F}}(x)$$

i.e. the function which assigns to $y \in Y$ the summ over its pre-images. It turns out that this is equal to the trace of the proper pushforward:

$$\sum_{\substack{x \in X(k) \\ f(x) = y}} t_{\mathcal{F}}(x) = t_{f_! \mathcal{F}}(y)$$

where the proper pushfoward may be described as $(f_! \mathcal{F})_y = R\Gamma_c(X_y \otimes \overline{k}, \mathcal{F})$. Now, $f_! \mathcal{F}$ is not necessarily a sheaf: it could be a complex.

1.2 Character sums

We illustrate the power of the function-sheaf dictionary with some examples.

Let G be a commutative algebraic group over k (think $G = \mathbb{G}_m$ or $\mathbb{G}_a$). Consider the *Lang map*

$$L_G : G \xrightarrow{L_G} G$$

defined by $L_G(g) = \sigma(g) g^{-1}$. Since G is commutative, this is a homomorphism of groups, which is even an étale isogeny (since σ has vanishing differential). The kernel is evidently $G(k)$, so we have a short exact sequence

$$0 \to G(k) \to G \xrightarrow{L_G} G \to 0.$$

Example 1.4. If $G = \mathbb{G}_m$ then $L_G(x) = x^{q-1}$, the *Kummer isogeny*, and we obtain the short exact sequence

$$0 \to \mu_{q-1} \to \mathbb{G}_m \xrightarrow{x \mapsto x^{q-1}} \mathbb{G}_m \to 0$$

If $G = \mathbb{G}_a$, then $L_G(x) = x^q - q$, the *Artin-Schreier isogeny*, and the kernel is just k:

$$0 \to k \to \mathbb{G}_a \xrightarrow{x \mapsto x^q - x} \mathbb{G}_a \to 0.$$

The Lang isogeny induces a map $\pi_1(G, e) \to G(k)$ (viewing it as a fibration over G with fiber $G(k)$). Thus, given a character $\chi \colon G(k) \to \mathbb{Q}_\ell^\times$ we can compose to get a character of the fundamental group. We denote the corresponding local system by L_χ.

$$\begin{array}{ccc} \pi_1(G,e) & \longrightarrow & G(k) \\ & \searrow & \downarrow \\ & & \mathbb{Q}_\ell^\times \end{array}$$

Example 1.5. See a book of Katz for a reference.

- If $G = \mathbb{G}_m$, and $\chi \colon k^\times \to \mathbb{Q}_\ell^\times$ then $\mathcal{L}_\chi$ is called a *Kummer sheaf.*
- If $G = \mathbb{G}_a$ and $\psi \colon k \to \mathbb{Q}_\ell^\times$ then $\mathcal{L}_\chi$ is called an *Artin-Schreier sheaf.*

Example 1.6. A basic fact is that if ψ is non-trivial, then $\sum_{x\in k} \psi(x) = 0$. According to the function-sheaf dictionary, this means that

$$\sum_{i=0}^{k} (-1)^i \operatorname{tr}(\sigma_i, H^i_c(\mathbb{G}_a \otimes_k \overline{k}, \mathcal{L}_\psi)) = 0.$$

This looks non-trivial. In fact, on the geometric side we know more: *all* the cohomology groups $H^i_c(\mathbb{G}_a \otimes_k \overline{k}, \mathcal{L}_\psi)$ are 0. The proof is basically a cohomological analogue of the usual argument for proving the vanishing for the sum.

Example 1.7. *(Gauss sums)* Given a multiplicative character $\chi \colon \mathbb{F}_q^\times \to \mathbb{Q}_\ell^\times$, and a non-trivial additive character $\psi \colon \mathbb{F}_q \to \mathbb{Q}_\ell^\times$, the associated *Gauss sum* is

$$\sum_{x\in\mathbb{F}_q^\times} \chi(x)\psi(x)$$

which by the function-sheaf dictionary is

$$\sum_{x\in\mathbb{F}_q^\times} \chi(x)\psi(x) = \sum_{i=0}^{2} (-1)^i \operatorname{tr}(\sigma, H^i_c(\mathbb{G}_m, \mathcal{L}_\chi \otimes j^*\mathcal{L}_Y)).$$

where $j \colon \mathbb{G}_m \hookrightarrow \mathbb{G}_a$ is the obvious inclusion.

It is an elementary fact that the Gauss sum is some complex number with absolute value $\sqrt{q}$. Therefore, by the theory of weights we know that $H^0_c = H^2_c = 0$ and $H^1_c = 1$.

Example 1.8. (Kloosterman sum) Let $\psi \colon \mathbb{F}_q \to \mathbb{Q}_\ell^\times$ be an additive character and $a \in \mathbb{F}_q^\times$. Then the *Kloosterman sum* associated to χ and a is

$$\mathrm{Kl}(a) = \sum_{\substack{x,y\in k^\times \\ xy=a}} \psi(x)\psi(y).$$

This appears a lot in classical analytic number theory.

The sum looks like it should come from the function-sheaf correspondence for a variety like $\mathbb{G}_m$, but not quite the same: it is $K_a = \operatorname{Spec} k[x,y]/(xy-a)$.

We have a map $l\colon K_a \to \mathbb{G}_a$ by $l(x,y) = x+y$. Then it is easy to see that

$$\mathrm{Kl}(a) = \sum (-1)^i \operatorname{tr}(\sigma, H^i_c(K_a \otimes_k \overline{k}, l^* \mathcal{L}_\psi)).$$

From geometric considerations, we can see that $H^0_c = H^2_c = 0$ and $\dim H^1_c = 2$. So that tells us that $\mathrm{Kl}(a) = \operatorname{tr}(\sigma, H^1_c) = \alpha+\beta$ where $|\alpha| = |\beta| = \sqrt{q}$, so $\alpha+\beta \leq 2\sqrt{q}$. This gives a highly non-trivial bound on Kloosterman sums!

We don't have time to justify some of these computations like the B etti numbers - see the problem sheet. In many cases, we will be interested in X an affine curve. In this case we know that $H^0_c = H^2_c = 0$, so there is only H^1_c. The value $\dim H^1_c$ was computed by Ogg-Grothendieck-Shafarevich. It is called the"Swan conductor."

One of the most interesting applications of this dictionary is the *Hasse-Davenport identity for Kloosterman sums.* We have a *twisted Kloosterman sum* for k'/k a quadratic extension,

$$\mathrm{Kl}'(a \in \mathbb{F}_q^\times) = \sum_{\substack{x \in (k')^\times \\ \operatorname{Nm} x = a}} \psi(\operatorname{tr} x).$$

Theorem 1.9. *We have* $\mathrm{Kl}(a) = -\mathrm{Kl}'(a)$.

You might be able to prove this by bare hands, but let me explain the geometric interpretation, which is very nice.

Proof. Notice that these two sums look very similar. The left hand side is attached to the variety $K_a = \operatorname{Spec} k[x,y]/(xy-a)$. This has an involution $\tau(x,y) = (y,x)$. We know that

$$\operatorname{Fix}(\sigma, K_a(\overline{k})) = K_a(k) = \{(x,y) \in K^2 \mid xy = a\}.$$

If we twist σ by τ, then we get

$$\operatorname{Fix}(\sigma \circ \tau, K_a(\overline{k})) = \{x \in k' \mid \operatorname{Nm}(x) = a\}.$$

It was Deligne-Lustzig's idea to apply the Grothendieck-Lefschetz trace formula for σ and $\sigma \circ \tau$, getting

$$\begin{aligned} \mathrm{Kl}'(a) &= \operatorname{tr}(\sigma \circ \tau, R\Gamma_c(K_a \otimes_k \overline{k}, l^* \mathcal{L}_\psi)) \\ \mathrm{Kl}(a) &= \operatorname{tr}(\sigma, R\Gamma_c(K_a \otimes_k \overline{k}, l^* \mathcal{L}_\psi)). \end{aligned}$$

We claim that τ acts on $R\Gamma_c(K_a \otimes_k \overline{k}, l^* \mathcal{L}_\psi)$ by -1, which will obivously establish the result. In this case we could prove this by hand, since we have explicit coordinates.

The map $l\colon K_a \to \mathbb{G}_a$ given by $l(x,y) = x+y$ is a ramified double covering. Then $l_* \mathbb{Q}_\ell$ has an action of τ because K_a does, and we can split

$$l_* \mathbb{Q}_\ell = (l_* \mathbb{Q}_\ell)_+ \oplus (l_* \mathbb{Q}_\ell)_-$$

(as τ acts on the fibers by swapping x and y). It turns out that $(l_*\mathbb{Q}_\ell)_+ \cong \mathbb{Q}_\ell$. The second factor is something else, which we don't really care about, but it turns out to be a local system of rank 1 on $\mathbb{A}^1$ minus the ramification points.

From the discussion above, we know that

$$\begin{aligned} H^1_c(K_a, l^*\mathcal{L}_\psi) &= H^1_c(\mathbb{G}_a, \ell_*\mathbb{Q}_\ell \otimes \mathcal{L}_\psi) \\ &= \underbrace{H^i_c(\mathbb{G}_a, (\ell_*\mathbb{Q}_\ell)_+ \otimes \mathcal{L}_\psi)}_{\tau=1} \oplus \underbrace{H^i_c(\mathbb{G}_a, (\ell_*\mathbb{Q}_\ell)_- \otimes \mathcal{L}_\psi)}_{\tau=-1} \end{aligned}$$

We want to prove that $H_c(\mathbb{G}_a, (\ell_*\mathbb{Q}_\ell)_+ \otimes \mathcal{L}_\psi) = 0$. But since $(\ell_*\mathbb{Q}_\ell)_+ \cong \mathbb{Q}_\ell$, that is $H^1_c(\mathbb{G}_a, \mathcal{L}_\psi) = 0$. □

This is a powerful illustration of high a tricky calculation can be reduced to straightforward operations in geometry.

One advantage of the geometric approach is that instead of handling one sum at a time, you can do things in *families*. You can then try to use the compatibility within families to deduce something.

Example 1.10. For the map $\mathbb{A}^2 \to \mathbb{G}_a$ sending $(x, y) \mapsto x + y$. Inside $\mathbb{A}^2$ we have the "singular sum" $xy = 0$. We can try to deform it off to something "smooth." Indeed, the above identity is true for $a = 0$. Try to do it by hand, then argue by "continuity" in families.

2 The Perverse Continuation Principle

2.1 Purity and duality

There is only so much mileage that you can get out of the function-sheaf corresponding using the sheaf operations $f_!$ and f^*. The advantage of working with ℓ-adic sheaves is that there are *more* operations. Two of particular importance, which we discuss presently, are *purity* and *duality*.

Example 2.1. Recall the Kloosterman sum associated to an additive character $\psi\colon k \to \overline{\mathbb{Q}_\ell}^\times$,

$$\mathrm{Kl}(a) = \sum_{\substack{x,y\in k \\ xy=a}} \psi(x+y).$$

By the Grothendieck-Lefschetz trace formula, we identified

$$\mathrm{Kl}(a) = \sum_{i=0}^{2} (-1)^i \operatorname{tr}(\sigma, H^i_c(K_a \otimes_k \overline{k}, l^*\mathcal{L}_\psi))$$

where $K_a = \operatorname{Spec} k[x,y]/(xy-a)$ and $l\colon K_a \to \mathbb{G}_a$ sends $(x,y) \mapsto x+y$. As mentioned previously, it turns out that $H^0_c = H^2_c = 0$ and $\dim H^1_c = 2$. Deligne's Purity Theorem says that the complex absolute values of the eigenvalues of σ acting on H^1_c are both $q^{1/2}$, where $q = |k|$. That is what allowed us to deduce a bound on the Kloosterman sum.

Definition 2.2. Let X be an algebraic variety over k and $\mathcal{F}$ an ℓ-adic sheaf on X. We say that $\mathcal{F}$ is *mixed of weight* ≤ 0 if for all $x \in |X|$, the complex absolute values of the eigenvalues of σ_x acting on the stalks $\mathcal{F}_{\overline{x}}$ are at most 1.

Theorem 2.3 (Deligne's purity theorem)**.** *Let X be an algebraic variety over k and $\mathcal{F}$ an ℓ-adic sheaf on X which is pure of weight ≤ 0. The complex absolute values of the eigenvalues of σ acting on $H^i_c(X \otimes_k \overline{k}, \mathcal{F})$ are $\leq q^{i/2}$.*

This is in Deligne's Weil I. For our purposes, we need to know a little more, concerning *complexes* of sheaves.

Denote by D the Verdier duality functor

$$D\colon D^b_c(X) \to D^b_c(X)$$

Definition 2.4. We say that $\mathcal{F}$ is *pure of weight* 0 if $\mathcal{F}$ and $D(\mathcal{F})$ are both mixed of weight ≤ 0.

We now come to one of the main results of Deligne's Weil II:

Theorem 2.5 (Deligne's purity theorem)**.** *Let $f\colon X \to Y$ be a proper morphism and $\mathcal{F} \in D^b_c(X)$ pure of weight weight 0. Then $f_*\mathcal{F}$ is pure of weight 0.*

Remark 2.6. The properness comes into play in identifying f_* and $f_!$.

The theory of perverse sheaves supplies "a lot" of pure sheaves. Let's try to get some feel for what pure sheaves are, before we discuss perverse sheaves.

Example 2.7. Let X/k be a *smooth* algebraic variety and $\mathcal{F}$ a local system on X. Assume that for every $x \in |X|$, the complex absolute values of the eigenvalues of σ_x acting on $\mathcal{F}_{\overline{x}}$ are 1. Then $\mathcal{F}$ is pure.

The point is that $\mathcal{F}$ is mixed of weight ≤ 0, by the assumption. Then $D(\mathcal{F})$ is simply the dual local system $\mathcal{F}^\vee$, and its stalk-wise eigenvalues are inverse to those of $\mathcal{F}$, hence also have absolute value 1.

2.2 Intersection cohomology and perverse sheaves

If X were not smooth in the preceding example, then we would have no idea what $D(\mathcal{F})$ is in general. A class of sheaves for which the description of the Verderi dual is fairly nice is the *intersection cohomology sheaves* introduced by Goresky, MacPherson, and Deligne.

If $j\colon U \hookrightarrow X$ is the inclusion of a smooth open subset and $\mathcal{F}$ is a local system on U, then $IC_X(\mathcal{F})$ is the *middle-extension sheaf* $j_{!*}\mathcal{F}$. This has the property that

$$D(j_{!*}\mathcal{F}) = j_{!*}(D\mathcal{F}).$$

Although it is difficult to say what the stalks of $j_{!*}\mathcal{F}$ are at all points, we do know:

Theorem 2.8. *If $\mathcal{F}$ is pure, then $j_{!*}\mathcal{F}$ is pure.*

It is known that $j_{!*}(\mathcal{F})$ lies in $\mathcal{P}(X)$, the category of *perverse sheaves* on X. This is an abelian full subcategory of $D^b_c(X)$. It is the heart of a certain t-structure.

We won't give a full treatment of perverse sheaves here, but we mention that there is a cohomological functor

$$^{\mathrm{p}}H^i\colon D^b_c(X) \to \mathcal{P}(X)$$

sending $\mathcal{F} \mapsto {}^{\mathrm{p}}\mathcal{H}^i(X)$ taking exact triangles to long exact sequences.

Simple objects. What are the *simple* objects of $\mathcal{P}(X)$? If $i\colon Z \hookrightarrow X$ is a an irreducible subvariety and $j\colon U \hookrightarrow Z$ is smooth open subset, and $\mathcal{F}$ is a local system on U, then we can form the perverse sheaf

$$\mathcal{K} := i_* j_{!*}\mathcal{F}[\dim Z] \in \mathcal{P}(X).$$

Theorem 2.9. *All simple objects of $\mathcal{P}(X)$ arise in this way.*

Theorem 2.10 (BBDG)**.** *If $\mathcal{F}$ is a pure object of $D^b_c(X)$ then over $X \otimes_k \overline{k}$, $\mathcal{F}$ is (non-canonically) isomorphic to a direct sum of simple perverse sheaves (with shifts).*

Now we combine this with Deligne's work from Weil II.

Example 2.11. Let $f\colon X \to Y$ be a proper morphism with X smooth. Let $\mathcal{F}$ be a pure local system on X, e.g. $\mathcal{F} = \mathbb{Q}_\ell$. We know that

$$f_*\mathbb{Q}_\ell = \bigoplus_i {}^{\mathrm{p}}\mathcal{H}^i(f_*\mathbb{Q}_\ell)[-i].$$

By the BBDG theorem, the summands can be further decomposed as

$${}^{\mathrm{p}}\mathcal{H}^i(f_*\mathbb{Q}_\ell) = \bigoplus_\alpha K_\alpha$$

where the K_α are simple perverse sheaves. Combining these two observations, we find that

$$f_*\mathbb{Q}_\ell \cong \bigoplus_{i,\alpha} K_\alpha[-i]. \tag{1}$$

Moreover, by the classification of simple perverse sheaves we know that $K_\alpha = i_{\alpha*} j_{\alpha!*}\mathcal{F}_\alpha$ where $\mathcal{F}_\alpha$ is a local system on some U_α, such that $\operatorname{supp} K_\alpha = Z_\alpha$ is an irreducible closed subset of $X \otimes_k \overline{k}$.

Definition 2.12. We define $\operatorname{supp} f$ to be the set of Z_α which appear as the support of some K_α in the decomposition (1). So $\operatorname{supp} f = \{Z_\alpha\}$ is a finite set of irreducible closed subschemes of $Y \otimes_k \overline{k}$.

You can think of the complex $f_*\mathbb{Q}_\ell$ as describing the geometry of the fibers of f (since its stalk at $y \in Y$ is the cohomology of the fiber $f^{-1}(y)$). Then the subvarieties Z_α are the loci where the fibers of f undergo "significant change."

If f is proper and smooth, so Y is also smooth, then ${}^{\mathrm{p}}\mathcal{H}^i(f_*\mathbb{Q}_\ell) = H^i(f_*\mathbb{Q}_\ell)$ is a local system, so we see only $\{Z_\alpha\} = \{Y\}$.

2.3 The perverse continuation principle

The perverse continuation principle describes a nice "continuity" that holds for perverse sheaves, reflected in the following theorem.

Theorem 2.13. *Suppose we have two proper morphisms $X_1 \xrightarrow{f_1} Y$ and $X_2 \xrightarrow{f_2} Y$ with X_1, X_2 both smooth:*

$$X_1 \xrightarrow{f_1} Y \xleftarrow{f_2} X_2$$

Assume that $\operatorname{supp} f_1 = \operatorname{supp} f_2 = \{Y\}$.

If there exists an open subset $U \subset Y$ such that for all closed points $y \in |U|$, say $y \in U(k')$, we have that

$$\#f_1^{-1}(y)(k') = \#f_2^{-1}(y)(k') \tag{2}$$

then in fact (2) *holds for all $y \in |Y|$.*

This is saying that if we know that the fibers over some open subset of Y have the same size, then this holds as well over the boundary.

Proof. By the decomposition theorem,

$$f_{1*}\mathbb{Q}_\ell = \bigoplus_{i,\alpha_1} K_\alpha[-i]$$

and

$$f_{2*}\mathbb{Q}_\ell = \bigoplus_{i,\alpha_2} K_{\alpha_2}[-i]$$

The support condition implies that $K_\alpha = j_{!*}j^*K_\alpha$, i.e. all the sheaves appearing here are determined by their restrictions to U as the middle-extension sheaf.

Then the assumption (2) implies that

$$\sum(-1)^i K_{\alpha_1}|_U = \sum(-1)^i K_{\alpha_2}|_U \tag{3}$$

in the Grothendieck group of $\mathcal{P}(U)$. Indeed, the hypothesis (2) is an equality of the trace of Frobenius at the stalks of both sides of (3) at the points $y \in U$. By restricting further we may assume that everything is a local system. Then we want to know that two (semisimple) representations (of the étale fundamental group) agree if their trace functions agree, which follows from a kind of Chebotarev principle. ♠♠♠ TONY: [not sure about this actually] Basically what we're saying is that an element in the Grothendieck group is completely determined by its trace function.

Thus $j_{!*}$ extends to an equality in $K_0(\mathcal{P}(X))$, and in particular an equality of the trace of σ on fibral cohomology over *all* points of Y. We are then done because the Grothendieck-Lefschetz Trace formula describes the rational points of the fibers in terms of the traces on cohomology of the fibers.

□

Typically you cannot expect to deduce equalities over the whole space when you only know equalities on some open subset. The assumptions are obviously crucial.

♠♠♠ TONY: [what's a "simple" example where $X_1 \neq X_2$?]

Example 2.14. The Jacquet-Ye fundamental lemma, the Jacquet-Rallis Fundamental Lemma, and the Langlands-Shelstad Fundamental Lemma are applications of this.

How could we ever achieve the control of the support as demanded in the hypothesis? There are three tools:

1. The Fourier-Deligne transform,

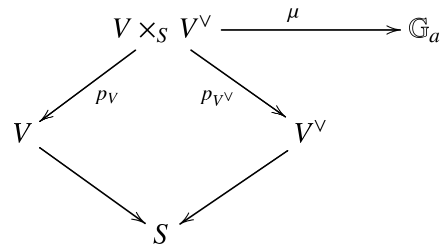
2. small maps,

3. support theorem for abelian fibrations.

These are the three tools that we use for the three fundamental lemmas, respectively.

2.3.1 Fourier-Deligne transform

Let S be an algebraic variety over k and $V \to S$ be a vector bundle of rank n. Denote by $V^\vee \to S$ the dual vector bundle. Then there is a diagram



Suppose we have an additive character ψ on $\mathbb{G}_a$, hence an Artin-Schreier sheaf $\mathcal{L}_\psi$ on $\mathbb{G}_a$.

Definition 2.15. The Fourier-Deligne transform is

$$FD\colon D_c^b(V) \to D_c^b(V^\vee)$$

taking $\mathcal{F} \mapsto p_{V^\vee,!}(p_V^*\mathcal{F} \otimes \mu^*\mathcal{L}_\psi)[n]$.

Theorem 2.16 (Katz-Laumon)**.** *The Fourier-Deligne transform of a perverse sheaf is perverse.*

Moreover, suppose that $j\colon S' \hookrightarrow S$ is the inclusion of an open subset, and denote by $j_V\colon V' \to V$ and $j_{V^\vee}\colon (V')^\vee \to V^\vee$ the inclusion of the pullbacks of the vector bundles to S. If $\mathcal{F} \in \mathcal{P}(\mathcal{V})$ is such that

$$\mathcal{F} = j_{V!*}j_V^*\mathcal{F}$$

then the same is true for $FD(\mathcal{F})$.

♠♠♠ TONY: [In particular, this tells us that the Fourier-Deligne transform of an IC sheaf is an IC sheaf... right?]

2.3.2 Small resolutions

Definition 2.17. If $f\colon X \to Y$ is a surjective, proper, generically finite morphism with X smooth, then we say that f is *small* if

1. $\dim(X \times_Y X) \le \dim X$ and
2. if Z is a component of $X \times_Y X$ such that $\dim Z = \dim X$, then $Z \to X$ is surjective.

If f is small, then $f_*\mathbb{Q}_\ell$ is a perverse sheaf and for all inclusions of an open subset $j\colon U \hookrightarrow Y$, we have

$$j_{!*}(j^* f_*\mathbb{Q}_\ell) = f_*\mathbb{Q}_\ell$$

This will imply the Jacquet-Rallis Fundamental Lemma (by an argument due to Zhiwei Yun).

2.3.3 Support theorem for abelian fibration

This is too complicated to really explain (see Harris's volume for a reference).

The situation is a degeneration of abelian varieties.

Theorem 2.18. *Let $f\colon X \to S$ be a proper morphism, with generic fiber an abelian variety. Let $g\colon P \to S$ be a (smooth) group scheme such that P_s acts on X_s with affine stabilizers. (We need some additional technical assumptions that we won't write down.)*

If X_s is irreducible, then $\operatorname{supp} f = \{S\}$. More generally, $Z \in \operatorname{supp} f$ if $\#Irred(f_s)$ has a jump at Z.

Remark 2.19. If you were to look and Zhiwei's and my papers, then they would seem quite difficult. That is just because of technical limitations. The same geometric idea underlies them both, but we lack the technical ability to make a uniform argument.

3 The Jacquet-Ye Fundamental Lemma

3.1 Overview

We're going to spend the next three talks discussing proofs of analytic identities using the geometric ideas. First let us give a blueprint of the argument.

Suppose we have a space X admitting an action of G. Then we get an action on $X(F)$ by $G(F)$. Assume that $X(F)$ is locally compact (it will be in all cases of interest), and consider $\varphi \in C_c^\infty(X(F))$. We want to define an integral of φ along an orbit for $G(F)$: for $x \in X(F)$, this

$$\theta_\varphi(x) := \int_{G(F)\cdot x} \varphi(y)\, dy.$$

We can think of this as a function on X/G. More algebro-geometrically, suppose that X is an affine variety. Then we can define the invariant quotient $(X//G) = \operatorname{Spec} k[X]^G$. For $a \in (X//G)(F)$, we can view the above as a function of a:

$$\theta_\varphi(a) := \int_{G(F)\cdot x} \varphi(y)\, dy.$$

Geometric setup. We are actually going to consider the *stack* $[X/G]$, which admits a map to $X//G$. Let $\varphi = 1_{X(\mathcal{O})}$, where $\mathcal{O}$ is the ring of functions on the formal disk. Then a factors through a map from $\mathbb{D} = \operatorname{Spec} \mathcal{O} \to [X/G]$:

$$\begin{array}{ccc} & & [X/G] \\ & \overset{y}{\nearrow} & \downarrow \\ \mathbb{D} & \xrightarrow[a]{} & X//G \end{array}$$

More globally, we can replace the formal disk $\mathbb{D}$ by a curve C:

$$\begin{array}{ccc} & & [X/G] \\ & \overset{y}{\nearrow} & \downarrow \\ C & \xrightarrow[a]{} & X//G \end{array}$$

3.2 Kloosterman integrals

Our setup is with $X = \mathfrak{g} = \mathfrak{gl}(n)$ and $G = U \times U$, where U is the group of upper triangular unipotent matrices. The G-action on X is by

$$x \mapsto {}^t u_- x u_+.$$

Invariant functions. What is $k[\mathfrak{g}]^{U\times U}$? Identify $V = k^n$ via a basis $v_1, v_2, \ldots, v_n$, with dual basis $v_1^\vee, \ldots, v_n^\vee$. Let $e_i \colon \mathfrak{g} \to \mathbb{G}_a$ be the morphism defined by

$$e_i(x) = \langle v_1^\vee \wedge \ldots \wedge v_i^\vee, x(v_1 \wedge \ldots \wedge v_i)\rangle$$

(i.e. taking x to the top left $i \times i$ minor of $\mathfrak{g}$), which is invariant because if you replace x with ${}^t u_- x u_+$, then u_+ fixes $v_1 \wedge \ldots \wedge v_n$ and ${}^t u_-$ fixes $v_1^\vee \wedge \ldots \wedge v_i^\vee$.

♠♠♠ TONY: [this isn't actually a homomorphism..?]

This induces a map

$$f \colon \mathfrak{g} \to \mathcal{E} = \operatorname{Spec} k[e_1, \ldots, e_n] \tag{4}$$

which factors through the stack quotient $[\mathfrak{g}/U \times U]$:

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & [\mathfrak{g}/U \times U] \\ & \searrow^{f} & \downarrow \\ & & \mathcal{E} \end{array}$$

It turns out that $\mathcal{E}$ is the invariant quotient of $[\mathfrak{g}/U \times U]$.

The space $\mathcal{E}$ contains the open subset $\mathcal{E}^\circ = \operatorname{Spec} k[e_i^{\pm 1}]$. We have a section $\mathcal{E}^\circ \to \mathfrak{g}$ sending

$$(e_1, \ldots, e_n) \mapsto \operatorname{diag}(e_1, e_1^{-1} e_2, \ldots, e_{n-1}^{-1} e_n).$$

Proposition 3.1. *The map $[\mathfrak{g}/U \times U] \to \mathcal{E}$ is an isomorphism over $\mathcal{E}^\circ$. In other words,*

$$\begin{array}{c} f^{-1}(\mathcal{E}^\circ) = {}^t U A U \\ \downarrow \\ \mathcal{E}^\circ \end{array}$$

is a $U \times U$ -torsor.

Remark 3.2. This is basically Bruhat decomposition.

Kloosterman integrals. Now we can define Kloosterman integrals. Let $\psi \colon k \to \overline{\mathbb{Q}_\ell}^\times$ be an additive character. Let $a \in A(F)$ where $F = k((t))$. Note that we have a homomorphism $U \to \mathbb{G}_a^{n-1}$ by sending

$$u \mapsto \sum_{i=1}^{n-1} u_{i,i+1}$$

i.e. the sum of the superdiagonal entries.Then we define

$$\psi_U \colon U(F) \to \mathbb{Q}_\ell^\times$$

by

$$\psi_U(u) = \psi\left(\sum_{i=1}^{n-1} \operatorname{Res}_0(u_{i,i+1}\, dt) \right).$$

Definition 3.3. Set $\varphi := 1_{\mathfrak{g}(O)}$. For $a \in A$, we define the *Kloosterman integral*

$$\operatorname{Kl}(a) = \int_{(U \times U)(F)} \varphi({}^t u_- a u_+) \psi_U(u_+) \psi_U(u_-)\, du.$$

It comes up in the "Jacquet trace formula."

Note that ${}^tU(F)aU(F) = f^{-1}(a)(F)$, where f is as in (4). This is closed in $\mathfrak{g}(F)$. If you restrict a compactly supported function to a closed subset then you still get compact support, of course. So integral is actually a (complicated) finite sum.

Remark 3.4. The Kloosterman integral vanishes trivially (i.e. the sum is empty) unless $f(a) = (a_1, \ldots, a_n)$ with each $a_i \in \mathcal{O}$.

Definition 3.5. Let F'/F be an unramified quadratic extension. Define a *twisted Kloosterman integral*

$$\mathrm{Kl}'(a) = \int_{U(F')} 1_{\mathfrak{g}(\mathcal{O})}({}^t\overline{u}au)\psi_{U'}(u)\,du.$$

Theorem 3.6. *If $f(a) = (a_1, \ldots, a_n) \in \mathcal{O}^{\oplus n}$, then*

$$\mathrm{Kl}(a) = (-1)^{r(a)}\,\mathrm{Kl}(a')$$

where $r(a) = \mathrm{val}(a_1 \ldots a_{n-1})$.

Example 3.7. Let $G = \mathrm{GL}(2)$ and $a = \mathrm{diag}(t, t^{-1})$. We are considering the orbit

$$\begin{pmatrix} 1 & \\ x_- & 1 \end{pmatrix}\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}\begin{pmatrix} 1 & x_+ \\ & 1 \end{pmatrix} = \begin{pmatrix} t & tx^+ \\ tx_- & tx_-x_+ + t^{-1} \end{pmatrix}.$$

The Kloosterman sum is over the integral elements, which forces $x_+, x_- \in t^{-1}\mathcal{O}$ and $x_-x_+ = -t^{-2} + \ldots$. Since $t^{-1}\mathcal{O}/\mathcal{O} \cong k$, the Kloosterman in this case may be identified with

$$\int_{\substack{x_-, x_+ \in t^{-1}\mathcal{O} \\ xy=1}} \psi(x_+ + x_-)\,dx_-dx_+ = \sum_{\substack{x_-, x_+ \in k \\ xy=1}} \psi(x_+ + x_-)$$

which is just the classical Kloosterman that we introduced before.

Proof sketch. The proof is analogous to the argument we gave in the classical case, Example 1.8.

Definition 3.8. We define

$$K_a = \{u_-, u_+ \in U(F)/U(\mathcal{O}) \mid {}^tu_-au_+ \in \mathfrak{g}(\mathcal{O})\}.$$

Note that $U(F)/U(\mathcal{O})$ is an infinite dimensional affine space. It is a fact that $\dim_k K_a = r(a)$.

Let $l\colon K_a \to \mathbb{G}_a$ by the usual map (summing superdiagonal and subdiagonal entries). Then by the Grothendieck-Lefschetz formula, we have

$$\mathrm{Kl}(a) = \sum_{x \in K_a(k)} \psi(l(x)) = \mathrm{tr}(\sigma, R\Gamma_c^i(K_a \otimes_k \overline{k}, l^*\mathcal{L}_\psi)).$$

As before, we have an involution $\tau(u_-, u_+) = (u_+, u_-)$ and we claim that τ acts on $R\Gamma_c$ as $(-1)^{r(a)}$.

□

3.3 Algebro-geometric setup

Now, we can regard a as fixing a diagram

$$\mathbb{D} \to \mathfrak{g}//U \times U$$

where $a(t) = (e_1, \ldots, e_n) \in \mathcal{O}^{\oplus n}$ (with not all e_i zero). Then we claim that we may interpret

$$K_a := U(\mathcal{O})\backslash\{x \in \mathfrak{g}(\mathcal{O}) \mid f(x) = a \in \mathcal{E}(\mathcal{O})\}/U(\mathcal{O})$$

as the set of lifts

$$\begin{array}{ccc} & & [\mathfrak{g}/U \times U] \\ & \overset{x}{\nearrow} & \downarrow \\ \mathbb{D} & \xrightarrow[a]{} & \mathcal{E} = \mathfrak{g}//U \times U \end{array}$$

Why? Giving such a lift is the same as giving a $U \times U$ torsor over $\mathbb{D}$ and mapping it to $\mathfrak{g}$ equivariantly with respect to $U \times U$. But over the formal disk, any torsor is trivial, so the map is determined by a map $\mathbb{D} \to \mathfrak{g}$. This is ambiguous up to the choice of trivialization, hence we mod out by the $G \times G$ action.

Globalization. Now we're going to set up the algebro-geometric family to which we can try to apply the perverse continuation principle. Let $l\colon K_a \to \mathbb{G}_a$ be as before, but replace $\mathbb{D}$ with a smooth proper curve C/k. In the most naïve formulation, we would be considering diagrams

$$\begin{array}{ccc} & & [\mathfrak{g}/U \times U] \\ & \nearrow & \downarrow \\ C & \xrightarrow{a} & \mathcal{E} = \mathfrak{g}//U \times U \end{array}$$

Here we have a problem that the only maps from C to the affine space $\mathcal{E}$ are constant. So we need to twist the situation. Observe that on $\mathfrak{g}$ we have a $\mathbb{G}_m$ action by scaling, which descends to an action on $\mathcal{E}$. This action is $(a_1, \ldots, a_n) \mapsto (ta_1, t^2a_2, \ldots)$. So we consider instead diagrams

$$\begin{array}{ccc} & & [\mathfrak{g}/U \times U \times \mathbb{G}_m] \\ & \nearrow & \downarrow \\ C & \xrightarrow{a} & \mathcal{E}/\mathbb{G}_m \end{array}$$

Let $\mathcal{L}$ be the line bundle on C induced by the projection map $C \to B\mathbb{G}_m = [\mathrm{pt}/\mathbb{G}_m]$.

Definition 3.9. Let $\mathcal{K}_a$ be the set of diagrams

$$\begin{array}{ccc} & & [\mathfrak{g}/U \times U \times \mathbb{G}_m] \\ & \overset{x}{\nearrow} & \downarrow \\ C & \xrightarrow{a} & [\mathcal{E}/\mathbb{G}_m] \\ & \underset{\mathcal{L}}{\searrow} & \downarrow \\ & & B\mathbb{G}_m \end{array}$$

Fixing the line bundle $\mathcal{L}$ on C, giving a map $C \to B\mathbb{G}_m$, consider the fibration

$$\begin{array}{c} \mathcal{K} := \{C \xrightarrow{x} [\mathfrak{g}/U \times U \times \mathbb{G}_m]\} \\ \downarrow \\ \mathcal{A} := \left\{ C \xrightarrow{a} [\mathcal{E}/\mathbb{G}_m] \mid {\scriptstyle \begin{array}{c} a(t)=(e_1,\dots,e_n) \\ e_i \in H^0(C,\mathcal{L}^{\otimes i}) \\ e_i \text{ not all } 0 \end{array}} \right\} \end{array}$$

with the fiber over $a \in \mathcal{A}$ being $\mathcal{K}_a$.

Claim. $\mathcal{K}_a$ is *affine*.

Proof. Choose some $a\colon C \to [\mathcal{E}/\mathbb{G}_m]$, i.e $e_1, \dots, e_n$ where $e_i \in H^\bullet(C, \mathcal{L}^{\otimes i})$ are not all 0. We want to know the possible lifts

$$\begin{array}{ccc} & & [\mathfrak{g}/U \times U \times \mathbb{G}_m] \\ & \overset{x}{\nearrow} & \downarrow \\ C & \xrightarrow{a} & [\mathcal{E}/\mathbb{G}_m] \end{array}$$

Well, obviously $x|_{a^{-1}([\mathcal{E}^\circ/\mathbb{G}_m])}$ is determined by a because $[\mathfrak{g}/U \times U \times \mathbb{G}_m] \to [\mathcal{E}/\mathbb{G}_m]$ is an isomorphism over $[\mathcal{E}^\circ/\mathbb{G}_m]$. Now, this means that the map is determined on

$$a^{-1}([\mathcal{E}^\circ/\mathbb{G}_m]) = C' = C - \bigcup_i \mathrm{Div}(e_i),$$

so the fiber $\mathcal{K}_a$ is the product of the space of lifts at the support of C:

$$\mathcal{K}_a = \prod_{v \in \sum_{i=1}^n \mathrm{Div}(e_i)} \mathcal{K}_{a,v}.$$

In particular, we see that if $\sum \mathrm{Div}(e_i)$ is multiplicity free then $\mathcal{K}_a = \mathbb{G}_m^{r(a)}$. ♠♠♠ TONY: [uhh, why?] □

Let $l \colon \mathcal{K} \to \mathbb{G}_a$ be the map from before. Choose a rational 1-form ω. Denote by $\mathrm{Div}(\omega)$ the divisor of ω, i.e. the divisor of zeros minus the divisor of poles. Define an open subset $\mathcal{A}' = \{\sum \mathrm{Div}(e_i) \cap \mathrm{Div}(\omega) = \emptyset\} \subset \mathcal{A}$ and defined $\mathcal{K}' \subset \mathcal{K}$ as the pre-image:

$$\begin{array}{ccc} \mathcal{K}' & \hookrightarrow & \mathcal{K} \\ \downarrow & & \downarrow \\ \mathcal{A}' & \hookrightarrow & \mathcal{A} \end{array}$$

Then we have the diagram

$$\begin{array}{ccccc} & & & & \mathbb{G}_a \\ & \nearrow^{l} & & \nearrow & \\ \mathcal{K}' & \longrightarrow & \mathcal{K} & & \\ {\scriptstyle p}\downarrow & & \downarrow & & \\ \mathcal{A}' & \longrightarrow & \mathcal{A}' & & \end{array}$$

Conjecture 3.10. *Let ψ be a character of $\mathbb{G}_a$ and $\mathcal{L}_\psi$ the associated Artin-Schreier sheaf. Then the complex $K := p_! l^* \mathcal{L}_\psi$ is a perverse sheaf, which is equal to $j_{!*} j^* \mathcal{K}$ for all open embeddings $j \colon U \hookrightarrow \mathcal{A}'$.*

In particular, if $U \subset \mathcal{A}'$ is an open subset on which $\sum \mathrm{Div}\, e_i$ is multiplicity free, then this implies that $K|_U$ is a local system of rank $2^{r(a)}$ on which τ acts by $(-1)^{r(a)}$.

The conjecture is wide open, but we know how to prove this in a special case. Let $C = \mathbb{P}^1$ and $\mathcal{L} = \mathcal{O}(1)$. Let $\omega = dt$. Then

$$\mathcal{A} = \bigoplus_{i=1}^{n} H^0(\mathbb{P}^1, \mathcal{O}(i))$$

so since $\mathcal{A}'$ allow no zeros at ∞,

$$\mathcal{A}' = \bigoplus_{i=1}^{n} (\text{polynomials in } t \text{ of degree } i).$$

In this case the maps $\mathbb{P}^1 \to [\mathfrak{g}/U \times U \times \mathbb{G}_m]$ are parametrized by

$$\begin{aligned} \mathcal{K} &= U \backslash \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}, \mathcal{O}(1))/U \\ &= U \backslash \{x_0 + x_1 t \mid x_0, x_1 \in \mathfrak{g}\}/U \end{aligned}$$

and so

$$\begin{aligned} \mathcal{K}' &= U \backslash \{x_0 + x_1 t \mid x_0 \in \mathfrak{g}, x_1 \in \mathfrak{g} - 0\}/U \\ &= \{(x, a) \in \mathfrak{g} \times A\} \end{aligned}$$

Now you calculate the function $l \colon \mathcal{K}' \to \mathbb{G}_a$. This is a serious computation, but you can do it in this case by a tricky induction and Fourier transforms.

4 The Jacquet-Rallis Fundamental Lemma

4.1 Overview

The plan is to explain Zhiwei Yun's proof of the Jacquet-Rallis Fundamental Lemma. We will give a slightly different presentation, emphasizing how it follows the blueprint we outlined last time. So let's briefly recall this blueprint.

We fixed a stack $[X/G]$, with admits a map to the invariant quotient $X//G$. Consider the induced map of "formal arc spaces" $L(X/G) \to L(X//G)$, where $L(X) = \mathrm{Hom}(\mathbb{D}, X)$. The right hand side is well-defined, but the left hand side confusing. Understanding this map basically encapsulates problems in harmonic analysis.

That was the local picture. The global picture is to replace $\mathbb{D}$ with a global curve, so we consider the induced map

$$\mathrm{Maps}(C, [X/G]) \to \mathrm{Maps}(C, X//G)$$

(with appropriate twists to make it non-trivial). Then you apply the perverse continuation principle in the global setting.

There is a product formula that relates the local and global settings. That lets you deduce information on the local side from the global side. This is the blueprint for the arguments so far, but in the future I think it will be important to understand the local picture.

4.2 Setup

Let $X = \mathfrak{g} = \mathfrak{gl}_n$, with the action of $H = \mathrm{GL}(n-1)$ induced by its embedding in $G = \mathrm{GL}(n)$, which acts via the adjoint representation.

$$H \hookrightarrow \begin{pmatrix} \boxed{H} & \\ & 1 \end{pmatrix} \subset G.$$

The first order of business is to understand the map $[\mathfrak{g}/H] \to \mathfrak{g}//H$. In this form the map is hard to digest.

Let V be an n-dimensional vector space, and for $v \in V$ denote by $v^\vee \in V^\vee$ the dual element such that $\langle v^\vee, v \rangle = 1$. Then we can view

$$H = \{g \in G \mid gv = v, gv^\vee = v^\vee\}.$$

This implies that the quotient stack $BH = [\mathrm{pt}/H]$ should be the classifying space for triples

$$\{(V, v, v^\vee) \mid \dim V = n, \langle v, v^\vee \rangle = 1\}.$$

The group G acts transitively on $\{(v, v^\vee) \mid \langle v, v^\vee \rangle = 1\}$, and $[\mathfrak{g}/H]$ is the classifying space for quadruples

$$[\mathfrak{g}/H] = \left\{(V, x, v, v^\vee) \mid \substack{\dim V = n \\ x \in \mathrm{End}(V) \\ \langle v, v^\vee \rangle = 1}\right\}$$

The condition $\langle v, v^\vee \rangle = 1$ turns out to be annoying to work with, so it's useful to put in a family. That is the motivation for defining

$$\mathcal{Y} = \{(V, x, v, v^\vee) \mid \text{same as before, except without } \langle v, v^\vee \rangle = 1\}.$$

Now we have a map $\mathcal{Y} \to \mathbb{G}_a$ sending $(V, x, v, v^\vee) \mapsto \langle v^\vee, v \rangle$. (The fact that $\mathbb{G}_a$ is a group scheme is not important; we could just have well written $\mathbb{A}^1$.) Then obviously the fiber over 1 is $\mathcal{Y}_1 = [\mathfrak{g}/H]$.

> The key to understanding $\mathcal{Y}$ concretely is to understand what "numbers" one can make from a quadruple $(V, x, v, v^\vee)$.

So what numbers can we make? Well, we certainly have the characteristic coefficients $a_i(V, x) = \mathrm{tr}(\bigwedge^i x)$. They define a map

$$\mathcal{Y} \to \mathfrak{a} := \mathrm{Spec}\ k[a_1, \ldots, a_n].$$

There is a rank n vector bundle $\mathfrak{r}$ over $\mathfrak{a}$, with fiber over $a = (a_1, \ldots, a_n)$ being

$$\mathfrak{r}_a = k[x]/(x^n - a_1 x^{n-1} + \ldots + (-1)^n \mathfrak{a}_n).$$

This is a trivial vector bundle with basis given by the classes $1, x, \ldots, x^{n-1}$. Note that $\mathfrak{r}$ has an $O_{\mathfrak{a}}$-algebra structure.

Now consider the dual vector bundle $\mathfrak{b} \to \mathfrak{a}$, with fiber $\mathfrak{b}_a = \mathrm{Hom}(\mathfrak{r}_a, k)$. Then $\mathfrak{b}$ is an $\mathfrak{r}$-module. For any $a \in \mathfrak{a}$ and $b \in \mathfrak{b}_a$, we have a map

$$\gamma_{a,b} \colon \mathfrak{r}_a \to \mathfrak{b}_a$$

sending $r \mapsto r \cdot b$. This map can be put into a family

$$\gamma \colon \mathfrak{r} \times_{\mathfrak{a}} \mathfrak{b} \to \mathfrak{b} \times_{\mathfrak{a}} \mathfrak{b},$$

which constitutes a map of trivial vector bundles over $\mathfrak{b}$. In particular, we have $\det \gamma \in k[\mathfrak{b}]$.

Now we construct a morphism $\mathcal{Y} \to \mathfrak{b}$. Concretely this means that given a quadruple $(V, x, v, v^\vee)$ we want to produce a pair (a, b) such that $a \in \mathfrak{a}, b \in \mathfrak{b}_a$.

1. Obviously we should set $a = (a_i = \mathrm{tr} \wedge^i x)$.
2. Now, by the Cayley-Hamilton Theorem we know that V is an $\mathfrak{r}_a = k[x]/(x^n - a_1 x^{n-1} + \ldots + (-1)^n a_n)$-module, with x acting by x.
3. A vector $v \in V$ induces a map $c_v \colon \mathfrak{r}_a \to V$ sending $r \mapsto r \cdot v$. Dualizing, we get $c_v^\vee \colon V^\vee \to \mathfrak{b}_a$. Evaluating this on $v^\vee$, we obtain $b \in \mathfrak{b}_a$.

So we have produced a map $(V, x, v, v^\vee) \mapsto (a, b)$.

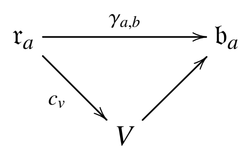
Exercise 4.1. In terms of the coordinates $(a_1, \ldots, a_n, b_0, \ldots, b_{n-1})$ on $\mathfrak{b}$, show that the map looks like $b_i(V, x, v, v^\vee) = \langle v^\vee, x^i v\rangle$.

In particular, the map $\mathcal{Y}_1 \to \mathfrak{b}$ is precisely the fiber over $b_0 = 1$.

Lemma 4.2. *Let* $(V, x, v, v^\vee) \mapsto (a, b)$. *Then the map*

$$\mathfrak{r}_a \xrightarrow{\gamma_{a,b}} \mathfrak{b}_a$$

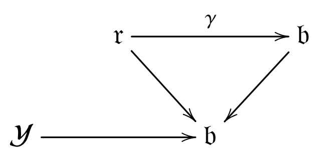
factors through



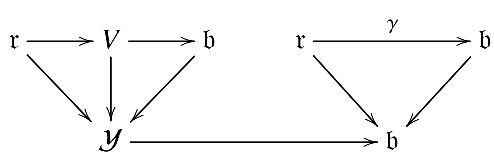
where as before $c_v(r) = r \cdot a$.

Remark 4.3. Although this Lemma is basic, it took me a long time to reaize!

This means that if we pull back the bundles $\mathfrak{r} \times_{\mathfrak{a}} \mathfrak{b}$ and $\mathfrak{b} \times_{\mathfrak{a}} \times \mathfrak{b}$ to $\mathcal{Y}$ via the map



then we get a diagram:



4.3 Loops

The map $\mathcal{Y} \to \mathfrak{b}$ induces a morphism $\mathrm{Maps}(\mathbb{D}, \mathcal{Y}) \to \mathrm{Maps}(\mathbb{D}, \mathfrak{b})$, with the latter being pairs of tuples $(a_1, \ldots, a_n, b_0, \ldots, b_{n-1})$ such that $a_i, b_j \in \mathcal{O}$ and $\det \gamma(a, b) \in \mathcal{O} \cap F^\times$ ♠♠♠ TONY: [this determinant condition might have been incorporated into an earlier definition - I didn't catch it], where $\mathcal{O} = k[[t]]$ and $F = k((t))$. Let $N_{a,b}$ be the fiber over (a, b), i.e. the set of $y \colon \mathbb{D} \to \mathcal{Y}$ lying over (a, b).

Definition 4.4. Let b be such that $b_0 = 1$ (the $\mathfrak{g}/H$ situation). Let $y \colon \mathbb{D} \to \mathcal{Y}$ lie over (a, b). We define the *untwisted Jacquet-Rallis* integral

$$\#N_{a,b} = \int_{H(F)} 1_{\mathfrak{g}(\mathcal{O})}(h^{-1}\gamma h)\, dh$$

where $\gamma \mapsto (a, b)$.

There are also "twisted Jacquet-Rallis integrals." The Jacquet-Rallis Fundamental Lemma is about an (almost) equality between the twisted and untwisted versions.

A morphism $y\colon \mathbb{D} \to \mathcal{Y}$ is equivalent to a quadruple $y = (V, x, v, v^\vee)$ where V is a vector bundle on $\mathbb{D}$ (i.e. a rank n free $\mathcal{O}$-module), $x \in \operatorname{End}(V)$, $v \in V$, and $v^\vee \in V^\vee$. This leads to the data of (a, b) and a map of free $\mathcal{O}$-modules $\gamma_{a,b}\colon \mathfrak{r}_a \to \mathfrak{b}_a$. Observe that $\gamma_{a,b} \otimes F$ is an isomorphism, since we have demanded $\det \gamma \in F^\times$. In particular, $\gamma_{a,b}$ is injective.

Because the lemma is robust over any base, and in particular $\mathbb{D}$, we get the factorization

$$\begin{array}{ccccc} \mathfrak{r}_a & \xrightarrow{c_v} & V & \longrightarrow & \mathfrak{b}_a \\ & \searrow & \xrightarrow[\gamma_{a,b}]{} & \nearrow & \end{array}$$

over $\mathbb{D}$. This shows that $V \to \mathfrak{b}_a$ is injective. So V gives an $\mathfrak{r}_a$-submodule of $\mathfrak{b}_a$ such that $\gamma_{a,b}(\mathfrak{r}_a) \subset V \subset \mathfrak{b}_a$.

Definition 4.5. Define the space

$$\mathcal{N}_{a,b} = \{V \mid \gamma_{a,b}(\mathfrak{r}_a) \subset V \subset \mathfrak{b}_a\}.$$

In particular, this is a finite-dimensional algebraic variety over k.

We have a stratification $N_{a,b} = \bigsqcup_s N^s_{a,b}$ where $s = \dim(\mathfrak{b}_a/V)$. For example, if $\gamma_{a,b}$ is an isomorphism then there is only one choice for V. If $\gamma_{a,b}$ has large cokernel, then there are many choices. For studying this, it's better to switch to a global situation.

It is a fact that there is a nice open subset $\mathfrak{b}^{\mathrm{reg}} \subset \mathfrak{b}$ such that the map restricted to it is an isomorphism:

$$\begin{array}{ccc} \mathcal{Y}^{\mathrm{reg}} & \longrightarrow & \mathcal{Y} \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ \mathfrak{b}^{\mathrm{reg}} & \longrightarrow & \mathfrak{b} \end{array}$$

This can be proven again by a Cayley-Hamilton argument, using the factorization lemma.

Now suppose that we have a map $C \to \mathfrak{b}$, where C is some curve, and we are considering lifts

$$\begin{array}{ccc} & & \mathcal{Y} \\ & \nearrow & \downarrow \\ C & \xrightarrow[a,b]{} & \mathfrak{b} \end{array}$$

Over $C' = (a, b)^{-1}(\mathfrak{b}^{\mathrm{reg}})$ we have no choice, while on the remaining points we have some thing to be decided on a formal disk, so

$$N_{a,b} = \prod_{v \in C - C'} N_{a_v, b_v}.$$

Since a, b are not functions but sections of a line bundle, to make sense of this you should pick trivializations.

4.4 Globalization

Let $(V, x, v, v^\vee)$ be a quadruple as before. Consider a $\mathbb{G}_m^2$ action on such quadruples by $(\alpha, \beta) \cdot (V, x, v, v^\vee) = (V, \alpha x, v, \beta v^\vee)$. We could obviously put another $\mathbb{G}_m$ if we wanted, but we don't need it.

Consider the same map as before: $(V, x, v, v^\vee) \to (a, b)$. By the formula, we can see that $(\alpha, \beta) \cdot (a_i) = \alpha^i a_i$ and $(\alpha, \beta) \cdot (b_j) = \beta \alpha^j b_j$ makes this equivariant with respect to our chosen torus action. (This comes from the formula $b_j = \langle v^\vee, x^j v \rangle$.)

For the global setup we need to fix some global data: let C be a smooth projective curve over k, and D, E divisors on C of large degree. Let

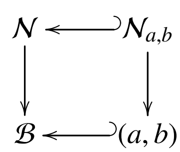
$$\mathcal{B} = \mathrm{Maps}(C \to \mathfrak{b} \wedge^{\mathbb{G}_m \times \mathbb{G}_m} \mathcal{O}(D) \times \mathcal{O}(E)) \text{ such that } \det \gamma(a, b) \neq 0.$$

The space $\mathrm{Maps}(C \to \mathfrak{b} \wedge^{\mathbb{G}_m \times \mathbb{G}_m} \mathcal{O}(D) \times \mathcal{O}(E))$ is just some large affine space, namely $\bigoplus_{i=1}^n H^0(C, \mathcal{O}(iD)) \oplus \bigoplus_{j=0}^{n-1} H^0(C, \mathcal{O}(jD + E))$. But then we impose the condition that over (a, b) we have $\det \gamma(a, b) \neq 0$ in defining $\mathcal{B}$.

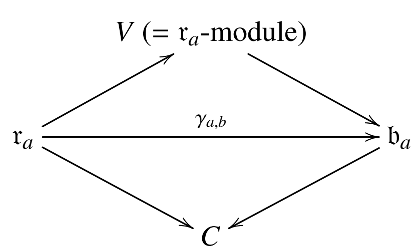
Now let $\mathcal{N} = \mathrm{Maps}(C, \mathcal{Y} \bigwedge^{\mathbb{G}_m \times \mathbb{G}_m} \mathcal{O}(D) \times \mathcal{O}(E))$, which which parametrizes the set of $(V, x, v, v^\vee)$ where

- V is a vector bundle over C of rank n,
- $x \colon V \to V(D)$,
- $v \colon \mathcal{O} \to V$, and
- $v^\vee \colon V^\vee \to V^\vee(E)$.

Consider the fibration $\mathcal{N} \to \mathcal{B}$. Fix some (a, b) on the base, and let's try to understand the fibers.



In these terms, $\mathcal{N}_{a,b}$ is the set of diagrams



The choice of V is equivalent to a choice of finite quotients of $\mathfrak{b}_a / \mathfrak{a}_a$. Again, we have a stratification by the dimension of the quotient: let $\mathcal{N}_{a,b}^i$ be the set of V as above, such that $\dim(\mathfrak{b}_a / V) = i$.

Theorem 4.6 (Yun)**.** *If* $\deg D, \deg E \gg i$ *then* $\mathcal{N}^i_{a,b}$ *is smooth, and* $\mathcal{N}^i_{a,b} \to \mathcal{B}$ *is proper and small.*

Proof sketch. One can prove this using the theory of Hilbert schemes. Over C, we have a projective bundle $S := \mathbb{P}(O \oplus O(D))$. For every a, b we get a spectral curve Y_a over C. Then $\mathfrak{r}_a, \mathfrak{b}_a$ become (free) O_{Y_a}-modules, so the quotient $\mathfrak{b}_a/V$ can be interpreted as the ideal sheaf defining a point in $\mathrm{Hilb}_?(S)$. This construction defines a map $q\colon \mathcal{N}^i_{a,b} \to \mathrm{Hilb}_i(S)$, sending a quadruple $(V, x, v, v^\vee)$, lying over (a, b), to some $Z \in \mathrm{Hilb}_i(S)$ via the constructions we have described. You can check that

$$q^{-1}(Z) = \left\{ a \in \underbrace{H^0(S, \mathcal{E}_1)}_{A}, b \in \underbrace{H^0(S, \mathcal{E}_2)}_{B} \colon a|_Z = b|_Z = 0 \right\}.$$

Since these are very ample bundles, and $O_Z = i$, the conditions $a|_Z = 0$ and $b|_Z = 0$ are each codimension i linear conditions. So we see that the fibers $q^{-1}(Z)$ are *linear* spaces, of constant dimension $\dim A + \dim B - 2i$. This proves that $\mathcal{N}$ is smooth.

Now what about the smallness? From the discussion above we see that

$$\mathcal{N} \times_{\mathcal{B}} \mathcal{N} = \{Z_1, Z_2 \in \mathrm{Hilb}_i(S); a \in A, b \in B \colon a|_{Z_1} = a|_{Z_2} = b|_{Z_1} = b|_{Z_2} = 0\}.$$

Then $q \times q$ defines a map from $\mathcal{N} \times_{\mathcal{B}} \mathcal{N}$ to $Q := \mathrm{Hilb} \times \mathrm{Hilb}$ sending $(Z_1, Z_2, a, b) \mapsto Z_1, Z_2$.

Since Q is a product of two Hilbert schemes, we get a stratification of Q into strata $Q^j = \{Z_1, Z_2\}$ such that the length of $Z_1 \cup Z_2$ is j, for some $i \le j \le 2i$. Then by the same argument, the fiber $(q \times q)^{-1}(Q^j) = \dim A + \dim B - 2j$, since the conditions can be written as $a|_{Z_1 \cup Z_2} = 0$, etc.

So the only thing to prove now is that $\dim Q^j = j$. I think that you can prove this using that $q^{-1}(?) \to ?$ is small, but to be honest I don't quite remember. □

5 The Langlands-Shelstad Fundamental Lemma

5.1 Algebro-geometric setup

Finally, we discuss the Langlands-Shelstad Fundamental Lemma and the non-standard Fundamental Lemma of Waldspurger. The Langlands-Shelstad Fundamental Lemma has important applications to endoscopy theory. Waldspurger's non-standard Fundamenta Lemma is an important technical tool.

Since it is not really impossible to set up endoscopy theory in the amount of time that's left, we're actually going to talk about Waldspurger's Fundamental Lemma, which requires relatively little setup. (In the problem session, you will be exposed to the simplest case of endoscopy.)

Let G be a reductive group over k and $\mathfrak{g} = \mathrm{Lie}(G)$, which admits an adjoint action of G. We consider this time the map

$$[\mathfrak{g}/G] \to \mathfrak{g}//G = \mathrm{Spec}\ k[\mathfrak{g}]^G.$$

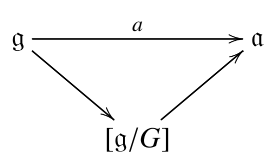
It was known classically that $k[\mathfrak{g}]^G = k[a_1, \ldots, a_n]$. The elements a_i cannot be chosen canonically, but the degrees $d_i := \deg a_i$ are canonical.

Example 5.1. If $\mathfrak{g} = \mathfrak{gl}_n$, then $a_i(x) = \mathrm{tr}(\wedge^i x)$ and $\deg a_i = i$ (if ch $k > 2n$). Another choice is $a_i(x) = \mathrm{tr}(x^i)$.

The map

$$a \colon \mathfrak{g} \to \mathfrak{a} = \mathrm{Spec}\ k[a_1, \ldots, a_n]$$

factors through the stack quotient

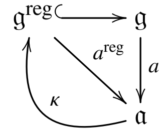


The fundamental difference between this case and the two that we have considered so far is that there is a *big disparity* between the stack quotient $[\mathfrak{g}/G]$ and $\mathfrak{a}$, since the stabilizers of G on $\mathfrak{g}$ are big. In particular, it will certainly not be true that $[\mathfrak{g}/G] \to \mathfrak{a}$ is an isomorphism over a dense open subset, and we'll have to deal with this later.

Definition 5.2. Let $I \to \mathfrak{g}$ be the *stabilizer group scheme*, whose fibers are the stabilizers:

$$I_x = \{g \in G \mid \mathrm{ad}(g)x = x\}.$$

This is a bad group scheme because it is not flat over $\mathfrak{g}$. However, if we restrict ourselves to the regular locus $\mathfrak{g}^{\mathrm{reg}} = \{x \in \mathfrak{g} \mid \dim I_x = n\}$ then $I^{\mathrm{reg}} \to \mathfrak{g}^{\mathrm{reg}}$ *is* a smooth, commutative group scheme. Another way to say this is that there is action a section κ of $\mathfrak{a}$ (due to Konstant) which lands in the regular set.



Facts:

- $\mathfrak{a}^{\text{reg}}$ is smooth,
- the fibers of $\mathfrak{a}^{\text{reg}}$ are G-orbits.
- I^{reg} descends to $\mathfrak{a}$. You could just pull back via κ, but it can be said more invariantly.

  Lemma 5.3. *There exists a smooth group scheme $J \to \mathfrak{a}$ (unique up to unique isomorphism) equipped with a G-equivariant isoomorphism*

  $$(a^* J)|_{\mathfrak{g}^{\text{reg}}} \cong I|_{\mathfrak{g}^{\text{reg}}}$$

  Moreover, this extends to $a^* J \to I$.

  Example 5.4. Let $G = \text{GL}(n)$ and $a \in \mathfrak{a}$. Let $x \in \mathfrak{g}^{\text{reg}}$ and $a = a(x)$. Then we have $J_a = (k[x]/x^n - a_1 x^{n-1} + \ldots)^\times$, whose pullback maps to I_x by $x \mapsto x$ (the point is that this is precisely the ring of the centralizer of a regular $x \in \mathfrak{g}$).

Now, we have the diagram of stacks over $\mathfrak{a}$:

$$\begin{tikzcd} B_{\mathfrak{a}}J \arrow[r, "\text{Kostant}", "\cong"'] \arrow[rrd, bend right] & {[\mathfrak{g}^{\text{reg}}/G]} \arrow[r] \arrow[rd] & {[\mathfrak{g}/G]} \arrow[d] \\ & & \mathfrak{a} \end{tikzcd}$$

where $B_{\mathfrak{a}}J$ is the relative classifying stack of $J \to \mathfrak{a}$ (whose fiber B_a over a is $B(J_a)$).

A more canonical way to say this is that BJ acts simply transitively on $[\mathfrak{g}^{\text{reg}}/G]$, and the action extends to $[\mathfrak{g}/G]$.

In the previous two Fundamental Lemmas, there was always an open subset of $\mathfrak{a}$ on which the map from the stack quotient was an isomorphism. We don't have that here. To restore this happy situation, we consider the quotient by BJ: $[\mathfrak{g}/G]/BJ$. Then we claim that over the regular semisimple locus, this is an isomorphism.

$$\begin{tikzcd} {[\mathfrak{g}^{rss}/G]/BJ} \arrow[r, "\cong"] \arrow[d] & {[\mathfrak{g}/G]/BJ} \arrow[d, "a"] \\ \mathfrak{a}^{rs} \arrow[r] & \mathfrak{a} \end{tikzcd}$$

(If $a \in \mathfrak{a}^{rss}$ then $a^{-1}(a) \subset \mathfrak{g}^{\text{reg}}$. ♠♠♠ TONY: [Yes, Ngô actually wrote that.])

Example 5.5. Consider GL_n. Intuitively, the map $\mathfrak{g}/G$ sends matrices to their characteristic polynomials. When you restrict to the regular semi-simple locus, then everything with the same characteristic polynomial is conjugate.

5.2 Analytic setup

Let $F = k((t))$, $\mathcal{O} = k[[t]]$, $\varphi \in C_c^\infty(\mathfrak{g}(F))$, and $\gamma \in \mathfrak{g}(F)$. We are interested in an orbital integral

$$O_\gamma(\varphi) = \int_{G(F)/G_\gamma(F)} \varphi(\mathrm{ad}(g)^{-1}\gamma)\frac{dg}{dg_\gamma}.$$

Here dg is a Haar measure on $G(F)$, normalized so that the volume of $G(\mathcal{O})$ is 1, and dg_γ is a Haar measure on $G_\gamma(F)$. But how do we normalize it? My impression is that the conventional choice so far is not that good. I'll describe a canonical choice that I prefer.

Let $\varphi = 1_{\mathfrak{g}(\mathcal{O})}$. Suppose $\gamma \mapsto a \in \mathfrak{a}(\mathcal{O})$ such that $a \in \mathfrak{a}^{rs}(F)$. Then we can consider pulling back $J \to \mathfrak{a}$ via a:

$$\begin{array}{ccc} & & J \\ & & \downarrow \\ \mathrm{Spec}\ \mathcal{O} & \xrightarrow{a} & \mathfrak{a}. \end{array}$$

The generic fiber of this pullback coincides with I_γ:

$$\begin{array}{ccc} J_a = a^*J & \longleftarrow & a^*J_F = I_\gamma \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ \mathcal{O} & \longleftarrow & \mathrm{Spec}\ F \end{array}$$

So from a we have produced a compact open subgroup $J_a(\mathcal{O}) \subset I_\gamma(F)$. Inside J_a we have the connected component J_a°, and the right normalization should be $\mathrm{vol}(J_a^\circ(\mathcal{O}), dg_\gamma) = 1$.

Remark 5.6. What's the difference between $J_a^\circ(\mathcal{O})$ and $I_\gamma(\mathcal{O})$? They're the same when γ is regular, but in general they're different, and we have $J_a(\mathcal{O}) \subset I_\gamma(\mathcal{O})$.

So why is this normalization better? Consider the map

$$\begin{array}{c} \mathfrak{g}(F) \\ \downarrow \\ \mathfrak{a}(F) \end{array}$$

Suppose we want to integrate φ along the fiber of some $a \in \mathfrak{a}^n(F)$. If $x, x' \in \mathfrak{g}(F)$ both lie over a, then there exists $g \in G(\overline{F})$ such that $\mathrm{ad}(g)x = x'$. But x, x' may not be conjugate over $G(F)$. In fact, the set $\Xi_a := \{x \in \mathfrak{g}(F), a(x) = a\}/G(F)$ has a simply-transitive action of $\ker(H^1(F, J_a) \to H^1(F, G))$, which is a finite set.

Definition 5.7. Define the *stable orbital integral*

$$\mathrm{SO}_a(\varphi) = \sum_{x \in \Xi_a} O_x(\varphi)$$

which corresponds geometrically to integrating along the fiber with the good choice of measure.

The harmonic analysis problem is to understand $a \mapsto \mathrm{SO}_a(\varphi)$, where $\varphi = 1_{\mathfrak{g}(O)}$. Let me say something brief about endoscopy. By doing Fourier analysis on the finite group $\ker(H^1(F, J_a) \to H^1(F, G))$, you can go back and forth between orbital integrals and "κ-orbital integrals," which we haven't defined. The Langlands-Shelstad Fundamental Lemma is actually about the κ-orbital integrals.

Definition 5.8. We say that two root systems are *isogenous* if there is an identification of their underlying real vector spaces such that their roots lie on the same line.

Example 5.9. If G, G' are dual semi-simple groups, e.g. $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(2n)$, then their root systems are isogeneous.

Theorem 5.10. *Let G, G' be semisimple groups with isogeneous root systems. Then under the identification*

$$\begin{array}{ccc} \mathfrak{g} & & \mathfrak{g}' \\ \downarrow & & \downarrow \\ \mathfrak{a} & = & \mathfrak{a}' \end{array}$$

for all $a \in \mathfrak{a}^{rs}(O) = (\mathfrak{a}')^{rs}(O)$, we have

$$\mathrm{SO}_a(1_{\mathfrak{g}(O)}) = \mathrm{SO}_a(1_{\mathfrak{g}'(O)}).$$

5.3 Local geometric picture

Consider maps

$$\begin{array}{ccc} & & [\mathfrak{g}/G] \\ & \overset{x}{\nearrow} & \downarrow \\ \mathbb{D} & \xrightarrow[a]{} & \mathfrak{a} \end{array}$$

For $a \in \mathfrak{a}(O) \cap \mathfrak{a}^{rs}(F)$, the space $\mathrm{Maps}(\mathbb{D}, [\mathfrak{g}/G])$ (all maps considered over $a \in \mathfrak{a}$) is $(-\infty)$-dimensional, since the stabilizer of a point is infinite-dimensional. That's bad, so we want some way to make it nicer.

Given some γ, say $\gamma = \kappa(a)$, let

$$M_a = \{g \in G(F)/G(O) \mid \mathrm{ad}(g)^{-1}\gamma \in \mathfrak{g}(O)\}$$

This is an affine Springer fiber, a finite-dimensional algebraic variety.

Recall that we have a map $a^*J \to I$. We have an action of $J_a(F)/J_a(O)$ on M_a. There is an open orbit $M_a^{\mathrm{reg}} \subset M_a$ on which the action is simply-transitive.

Theorem 5.11. *We have* $\dim M_a = \dim J_a(F)/I_a(O)$.

We claim that the stable orbital integral of a is equal to the "mass" of the stack $[M_a/P_a]$, where $P_a = I_a(F)/J_a(O)$:

$$\mathrm{SO}_a = \#[M_a/P_a]$$

This stack has a single honest point (corresponding to the open orbit), and a bunch of stacky points.

Now go back to considering maps

$$\begin{array}{ccc} & & [\mathfrak{g}/G] \\ & \overset{x}{\nearrow} & \downarrow \\ \mathbb{D} & \xrightarrow{a} & \mathfrak{a} \end{array}$$

The space $\mathrm{Maps}(\mathbb{D}, [\mathfrak{g}/G])$ has an action of BJ_a, which is a stack of dimension $-\infty$. Thus the quotient stack $\mathrm{Maps}(\mathbb{D}, [\mathfrak{g}/G])/BJ_a$ has a chance to actually by finite dimensional, and it turns out that it is precisely $[M_a/P_a]$.

5.4 Global moduli space

Now fix a curve C and D a divisor of large degree. We have a map $\mathfrak{g} \to \mathfrak{a}$, and there is a $\mathbb{G}_m$-action on $\mathfrak{g}$ by scaling. This is equivariant if we descend it to $\mathfrak{a}$ by $t \cdot (a_1, \ldots, a_n) = (t^{d_1} a_1, \ldots, t^{d_n} a_n)$.

Let $\mathcal{A}$ be the space of maps $C \to \mathfrak{a} \wedge^{\mathbb{G}_m} \mathcal{O}(D)$, which is just

$$\mathcal{A} = \bigoplus_{i=1}^{n} H^0(C, \mathcal{O}(d_i D)).$$

Composing with $[\mathfrak{g}/G] \to \mathfrak{a}$ induces a map

$$\begin{array}{ll} \mathcal{M} & = \mathrm{Maps}(C, [\mathfrak{g}/G] \wedge^{\mathbb{G}_m} \mathcal{O}(D)) \\ \downarrow & \\ \mathcal{A} & = \mathrm{Maps}(C, \mathfrak{a} \wedge^{\mathbb{G}_m} \mathcal{O}(D)) \end{array}$$

This is the *Hitchin fibration*. If you unwind the definitions, you'll find that

$$\mathcal{M} = \left\{ (V, \varphi) \mid {}^{V\ G\text{-torsor over } C}_{\varphi \in H^0(C, \mathrm{ad}(V)(D))} \right\}.$$

Remark 5.12. Hitchin originally defined this with D being the canonical divisor.

We have

$$\begin{array}{ccc} & & J \\ & & \downarrow \\ C & \xrightarrow{a} & \mathfrak{a} \wedge^{\mathbb{G}_m} \mathcal{O}(D) \end{array}$$

Let $C' \subset C$ be the pre-image of the open subset $\mathfrak{a}^{rs} \wedge^{\mathbb{G}_m} \mathcal{O}(D)$. Then $J_a = a^* J$ is a smooth group scheme over C'.

Let P_a be the space of torsors for J_a, which fit into a principal J_a-bundle $\mathcal{P}$ over C called the *Picard stack*. ♠♠♠ TONY: [compare this with earlier definition?]

We derive the universal map $a^*J \to I$ to get an action of $\mathcal{P}_a$ on $\mathcal{M}_a$ (this is nothing but Cayley-Hamilton formulated in another language). Indeed, consider φ in the data of $\mathcal{M}$. You can twist by an $\mathrm{End}(J_a)$-torsor.

We claim that

$$[\mathcal{M}_a/\mathcal{P}_a] = \prod_{v \in C - C'} [\mathcal{M}_{a,v}/\mathcal{P}_{a,v}].$$

In particular, if $a^*(\mathfrak{a} - \mathfrak{a}^{rs})$ is multiplicity-free then $[\mathcal{M}_{a,v}/\mathcal{P}_{a,v}] = 1$ and we find that $\mathcal{P}_a$ acts simply transitively on $\mathcal{M}_a$, so $\mathcal{M}_a$ is an abelian variety. If G is semisimple, then $\pi_0(\mathcal{P}_a) = [A/M] \times N$ where A is an abelian variety, M is a finite subgroup, and N is another finite group.

The most important thing is to understand how $\pi_0(\mathcal{P}_a)$ depends on a. Since $\mathcal{P} \to \mathcal{A}$ is a smooth group scheme, $\pi_0(\mathcal{P}_a)$ can be put together into a sheaf $\pi_0(\mathcal{P})$. It's very interesting to see how this sheaf jumps. It turns out that rank $\pi_0(v_a)$ increases when a comes from a Levi subgroup (as the spectral curve will become reducible). On the other hand, the torsionof $\pi_0(\mathcal{P}_a)$ increases when a comes from an endoscopy group.

For now, let's ignore this. We have the Hitchin fibration $\mathcal{M} \to \mathcal{A}$, which is a map of Artin stacks. However, there is an open subset of $\mathcal{A}$ called $\mathcal{A}^{ell}$ ("elliptic elements"), consisting of the a such that $\pi_0(P_a)$ is finite, such that the fiber $\mathcal{M}^{ell} \to \mathcal{A}^{ell}$ is a proper morphism of *Deligne-Mumford* stacks.

$$\begin{array}{ccc} \mathcal{M} & \hookleftarrow & \mathcal{M}^{ell} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{ell}} \\ \mathcal{A} & \hookleftarrow & \mathcal{A}^{ell} \end{array}$$

Theorem 5.13. *Suppose* $\deg D \gg 0$. *Then*

$$f_*^{ell}\mathbb{Q}_\ell = \bigoplus_n {}^{\mathrm{p}}\mathcal{H}^n(f_*^{ell}\mathbb{Q}_\ell)$$

where

$${}^{\mathrm{p}}\mathcal{H}^n(-) = \bigoplus_{\alpha \in A_n} K_\alpha,$$

with K_α *a simple perverse sheaf.*

Now, $\pi_0(\mathcal{P})$ acts on the ${}^{\mathrm{p}}H^n(f_*^{ell}\mathbb{Q}_\ell)$. Let ${}^{\mathrm{p}}H^n(f_*^{ell}\mathbb{Q}_\ell)^{st}$ be the largest direct factor with the trivial action of $\pi_0(\mathcal{P})$. Then you can deduce from the general support theorem for abelian fibrations that if K is a direct factor of ${}^{\mathrm{p}}H^n(f_*^{ell}\mathbb{Q}_\ell)^{st}$, then $\mathrm{supp}\, K = A^{ell}$ (so there is no direct factor of small support). This is true for $k = \mathbb{C}, k = \mathbb{F}_q$. We hope to generalize this to all fields, but there are some technical limitations. This is what is necessary to apply the perverse continuation principle.

Now you have basically proved the nonstandard fundamental lemma. The point is that if G, G' have isogenous root systems, then comparing the maps

$$\mathcal{M} \xrightarrow{f} \mathcal{A} \xleftarrow{f'} \mathcal{M}'$$

you see that the generic fibers for f and f' are isogenous abelian schemes. This implies that $(f_* \mathbb{Q}_\ell)^{st} \cong (f'_* \mathbb{Q}_\ell)^{st}$.

Remark 5.14. For the Langlands-Shelstad fundamental lemma, you look no longer at the stable factor but at factors for characters of $\pi_0(\mathcal{P})$.