Singular Lagrangians

David Nadler
Notes by Tony Feng

1 Lagrangian singularities

1.1 Setup

Let me start with the question, *what is a Lagrangian singularity*? I want to give a sense of what they are and where they arise.

Let (M, ω) be a symplectic manifold. Everything I'll talk about today is local, so it is sufficient to think of the local picture $(M, \omega) = (\mathbb{R}^{2n} \cong T^*\mathbb{R}^n, \sum dx_i d\xi_i)$. It is useful to assume that $\omega = d\alpha$ is exact, as is certainly true locally. Here $\alpha = -\sum \xi_i dx_i$.

Let $L \subset M$ be a closed subvarieties, pure of dimension n, which is *isotropic* (the symplectic form restricts to 0 on L). If you're not comfortable with this notion when M is singulary, you can think of L as being stratified by symplectic manifolds such that ω restricts to 0 on each strata.

We want to assume that L is also exact, meaning that the cohomology class of form $\omega|_L$ is trivial, i.e. there exists $f : L \to \mathbb{R}$ such that $df = \alpha|_L$.

1.2 Examples

1. Ribbon graphs. Here the Lagrangian L is a graph.
2. Symplectic resolutions. These include resolutions of ADE singularities, e.g. A_n is $X = \{x^2 + y^2 + z^{1+n} = 0\}$. A natural source of singular Lagrangians are the exceptional divisors. More generally, you can take the pre-image of a Lagrangian subvariety.
3. Cotangent bundles ("sheaf theory"). Let X be a base manifold with stratification $\mathcal{S} = \{X_\alpha\}$. Then set $M = T^*X$ and $L = T^*_{\mathcal{S}}X = \coprod_\alpha T^*_{X_\alpha}X$. We can also take a uiion of irreducible components inside here.
4. Landau-Ginzburg models ("mirror symmetry"). Let $M = \mathbb{C}^3$ and consider the superpotential $w = xyz : M \to \mathbb{C}$. The singular Lagrangian is

   $$L = \{(x, y, z) \mid w(x, y, z) \in \mathbb{R}_{\geq 0}, \quad |x| = |y| = |z|\}.$$

   This is a cone over a two-torus.

1.3 Guiding analogies

In the rest of the talk I want to tell you how to take complicated Lagrangian singularities and "reduce" them to combinatorial Lagrangian singularities.

General	Combinatorial
singular variety X	smooth variety $\widetilde{X}$ with divisor with normal crossings
smooth $f\colon M \to \mathbb{R}$	Morse function $\widetilde{f}\colon M \to \mathbb{R}$

In these two examples, we have somehow "reduced" the singularities to coordinate functions (in the first case linear, in the second case quadratic).

General	Combinatorial
singular variety X	smooth variety $\widetilde{X}$ with divisor with normal crossings
smooth $f\colon M \to \mathbb{R}$	Morse function $\widetilde{f}\colon M \to \mathbb{R}$
singular Lagrangian $L \subset M$	singular Lagrangian $\widetilde{L} \subset M$ with *arboreal singularities*

The basic technique of the first kind is blowup. The basic technique of the second is deformation. The basic technique of the third kind is a combination of these.

Theorem 1.1. *Any Lagrangian singularity $L \subset M$ admits a* non-characteristic deformation *to a nearby Lagrangian $\widetilde{L} \subset M$ with arboreal singularities.*

When you do deformations, you can often end up changing the invariants (e.g. Fukaya categories) that you're interested in. Roughly, "non-characteristic deformation" means a deformation that doesn't change these interesting invariants.

Remark 1.2. The theorem is in fact an algorithm.

2 Arboreal singularities

Now we are addressing the question, what are arboreal singularities? The reason they are called "arboreal" is that the input data is a tree T. From this we associate a topological space L_T such that

- $\dim L_T = |T| - 1$,
- it's a local model of a singularity.

To describe the construction, start out with

$$\coprod_{\alpha \in T} \mathbb{R}^{T-\{\alpha\}}.$$

Think of this as functions on the tree, omitting one vertex. Then we do some gluing. For each edge (α, β), we're going to glue $R^{T-\{\alpha\}}$ and $R^{T-\{\beta\}}$. We have $R^{T-\{\alpha\}} = R^{\{\beta\}} \times R^{T'}$ and $R^{\{\alpha\}} \times R^{T'}$. We're going to describe how to glue $R^{\{\beta\}}$ and R^{α}, and the rest of the factors are carried along. The gluing is between the rays $x_\beta \geq 0$ and $x_\alpha \geq 0$.

Remark 2.1. There is another, completely combinatorial definition. There is a poset P_T determined from T, whose "order complex" is L_T. The poset is the poset of correspondences of trees

$$T \hookleftarrow S \twoheadrightarrow Q$$

where $T \hookleftarrow S$ is a full subtree and $S \twoheadrightarrow Q$ is an edge collapsing.

Example 2.2.

1. If $T = \text{pt}$, then $L_T = \text{pt}$ (also $P_T = \text{pt}$).
2. If T is the A_2 quiver (two-element tree), then the arboreal singularity is a trivalent tree.
3. More generally, if T is the A_n tree then L_T is a tropical hyperplane.

For a long time I thought that the A_n were all that are necessary, and I still haven't disproved this, but now it seems that L_T for $T \sim D_4$ is fundamentally different.

For $n = 1$, we have only A_1. it is a well-known theorem that any graph on a surface can be deformed to trivalent... almost. There are also degenerate arboreal singularities indexed by "leafy trees" obtained by identifying some leaves of a tree.

I won't really discuss how you can put these in as Lagrangians. You can do this at the price of picking a root vertex.

3 Non-characteristic deformations

In the last part of the talk, I'll answer the question "what is a non-characteristic deformation?"

The notion of deformation is probably familiar, and the term "non-characteristic" comes from PDE, where it refers to codirections in the cotangent bundle along which you can propagate solutions. It was then appropriated in sheaf theory to mean codirections in which the sheaf "doesn't change."

Informal definition. I want to explain the "geometric" meaning of non-characteristic. Think of a Lagrangian $\mathcal{L} \subset N$ where N is a Lagrangian contact manifold. You can think of $\mathcal{L}$ as a singular subvariety that lives in the plane to first order. For reasonable Lagrangians, $\mathcal{L}$ will be displaced by Reeb flow for all small $\epsilon > 0$. That's because $\mathcal{L}$ lives in an orthogonal plane (to first order) to the Reeb flow.

So if you want to blow up or something, you need to make sure that you don't introduce any very small Reeb chords between the Lagrangian and itself. Informally, a deformation $\mathcal{L}_t$ is *non-characteristic* if there is such an $\epsilon > 0$ *uniformly in t*.

Now we give the "official version." To a singular Lagrangian $L \subset M$ one can assign a "quantum category" $C_L(M)$. There are different ways to get your hands on what this is:

- (algebra) modules over a deformation quantization supported on $\mathcal{L}$
- (topology) microlocal sheaves on L

- (analysis) Fukaya category of branes near L.

In my experience the easiest to work with is microlocal sheaves. A non-characteristic deformation is one in which this category doesn't change (in the strong sense of forming a local system).

Definition 3.1. $L_t \subset M$ is non-characteristic if $C_{L_t}(M)$ is independent of t.

This allows for calculations in practice: you can define the categories completely combinatorially. A key step is:

Theorem 3.2. *If $L_T \subset M$ is an arboreal singularity associated to T, then $C_{L_T}(M) \cong$ $\mathbf{Mod}(T)$ (with T thought of as the quiver with respect to some directing of arrows).*

Example 3.3. Going back to an earlier example, if $M = \mathbb{C}^3$ with the superpotential $w = xyz$, then L is a cone on T^2 and

$$C_L(M) \cong \mathrm{Coh}(\mathbb{P}^1 \setminus \{0, 1, \infty\}).$$