F field of char 0

K field of functions in 1 var over F. (So K is func field of proj. smooth abs. irred. curve over F)

W fin dim vec space $/K$.

K-connection on W is additive mapping

$$\nabla : W \to \Omega^1_{K/F} \otimes_K W$$

satisfying leibniz rule

$$\nabla(fw) = df \otimes w + f\nabla(w)$$

Equivalently it's a K-linear mapping

$$\nabla : \mathrm{Der}(K/F) \longrightarrow \mathrm{End}_K(W)$$

s.t.

$$\nabla(D)(fw) = D(f)w + f\nabla(D)(w)$$

Rmk: We defined the n-dim case in Rachel's talk, (W, ∇) is the vector bundle we get over generic fiber.

Morphisms btwn two such obj (W, ∇) and (W', ∇') are K-linear maps

$$\varphi(\nabla(D)(w)) = \nabla'(D')(\varphi(w))$$

Form ab. cat. $MC(K/F)$ in this way, has internal tensor.

p a place of K/F (closed point on proj smooth curve with func field K)

$\mathrm{ord}_p : K \to \mathbb{Z} \cup \{\infty\}$ associated val

$$\mathcal{O}_p = \{ f \in K \mid \mathrm{ord}_p(f) \geq 0 \}$$

$$m_p = \{ f \in K \mid \mathrm{ord}_p(f) \geq 1 \}.$$

$\mathrm{Der}_p(K/F)$ $\quad \mathcal{O}_p$-submodule s.t.

$$\mathrm{Der}_p(K/F) := \{ D \in \mathrm{Der}(K/F) \mid D(m_p) \subset m_p \}$$
$$= \{ f\, h \tfrac{\partial}{\partial h} \mid f \in \mathcal{O}_p \}$$
$$= \mathrm{Hom}_K(\Omega^1_{\mathcal{O}_p/F}(\log[p]), K)$$

h a uniformizer but actually for any non unit

$$\frac{d}{dx} = f' \cdot \frac{d}{df}$$

$$f \frac{d}{df} = \frac{f}{f'x} \times \frac{d}{dx}$$

<u>Def:</u> $(W, \nabla) \in \mathrm{ob}(MC(K/F))$ has <u>regular single pt</u> at p if $\exists\, \mathcal{O}_p$ lattice W_p of W s.t.

$$\mathrm{Der}_p(K/F)(W_p) \subset W_p$$

$\iff \exists$ basis $\vec{e}$ of W as K-space s.t.

$$\nabla\left(h \frac{d}{dh}\right) \vec{e} = B\vec{e} \qquad B \in M_n(\mathcal{O}_p)$$

any uniformizer h at p

see [DGS] for char p.

<u>Prop:</u> Suppose

$$0 \to (V, \nabla') \to (W, \nabla) \to (U, \nabla'') \to 0$$

exact in $MC(K/F)$. (W, ∇) has RSP at $p \iff (V, \nabla')$ and (W, ∇'') do.

<u>Pf:</u> Suppose (V, ∇') and (W, ∇'') have RSP at p.

Basis: $\vec{e}$, matrix A $\qquad$ Basis: $\vec{g}$ matrix B

So have K basis of W of form $\begin{pmatrix} \vec{e} \\ \vec{f} \end{pmatrix}$ ← lift of $\vec{g}$ to W.

So

$$\nabla\left(h\frac{d}{dh}\right)\begin{pmatrix}\vec{e}\\ \vec{f}\end{pmatrix} = \begin{pmatrix}A & 0\\ B & C\end{pmatrix}\begin{pmatrix}\vec{e}\\ \vec{f}\end{pmatrix}$$

$$\begin{matrix}\cap ?\\ M_n(\mathcal{O}_p)\end{matrix}$$

Claim:

$$\nabla\left(h\frac{\partial}{\partial h}\right)\begin{pmatrix}\vec{e}\\ h^v\vec{f}\end{pmatrix} = \begin{pmatrix}A & 0\\ h^vB & C+r\end{pmatrix}\begin{pmatrix}e\\ h^v\end{pmatrix}$$

Pf: Leibniz rule.

$$\nabla h\frac{d}{dh}(h^v\vec{f}) = h\frac{\partial}{\partial h}h^v\vec{f} + h^v\nabla\left(h\frac{d}{dh}\right)\vec{f} =$$

$$v h^v\vec{f} + h^v(Be + C\vec{f}) =$$

$$h^vB\vec{e} + (C+r)h^v\vec{f} \qquad \square \text{ Claim}$$

Conversely, suppose (W,∇) has reg sing pt at p and W_p an $\mathcal{O}_p$ lattice s.t.

$$\mathrm{Der}_p(K/F)(W_p) \subset W_p.$$

Consider $V\cap W_p$ and $U\cap(\text{Image of } W_p)$. These satisfy the criteria b/c horiz. morph def.

$\square$ Prop

Def: (W,∇) cyclic if $\exists\, w\in W$ s.t. for some (thus any) $D\in\mathrm{Der}(K/F)$

nonzero

$$w,\ \nabla(D)(w),\ \nabla(D)^2(w)\ \ldots$$

span W. (For fixed w, span is ind of choice of $D\in\mathrm{Der}(K|F)$)

Rmk: We care about cyclic ones b/c they correspond to our DEs.

Cor: $(W, \nabla) \in ob(MC(K/F))$. (W, ∇) has RSP at p iff every cyclic subobject of it does.

Pf: $(\Rightarrow)$ by exact sequence

$(\Leftarrow)$ (W,∇) is quotient of direct sum of finitely many cyclic subobj. $\square$

Problem: For RSP, need to find a particular basis. Want a condition independent of basis.

Def: $(W,\nabla) \in Ob(MC(K/F))$, $W_p \subset W$ $\mathcal{O}_p$ lattice, (W,∇) satisfies "Jurkat's estimate" (J) at p for W_p if $\exists\, \mu \in \mathbb{Z}$ s.t. $\forall j \geq 1$ $(\in \mathbb{Z})$ $\forall D_1, \dots, D_j \in Der_p(K/F)$

$$\nabla(D_1) \cdot \nabla(D_2) \cdots \nabla(D_j)(W_p) \subset h^{\mu}(W_p).$$

Equivalently, pick $\mathcal{O}_p$ basis D_0 of $Der_p(K/F)$.

Induction and Leibniz rule give us

$$\nabla(D_1) \cdots \nabla(D_j) = \sum_{v=0}^{j} a_v (\nabla(D_0))^v$$

so $(J) \iff (\nabla(D_0))^j (W_p) \subset h^{\mu}(W_p) \quad \forall j \geq 1$

$\iff$ $\mathcal{O}_p$ basis $\vec{e}$ of W_p

$$(\nabla(D_0))^j \vec{e} = B_j \vec{e} \quad \text{s.t. } ord_p(B_j) \geq \mu \quad \forall j \geq 1.$$

$\vec{e}$ of W satisfies (J).

Prop(*): (W, ∇) satisfies (J) at p for one base, satisfies for all base

Ex: (from Rachel's talk)

$$y'' + \frac{2t-1}{t(t-1)} y' + \frac{\frac{1}{4}}{t(t-1)} y = 0 \qquad K = \mathbb{C}(t) \quad F = \mathbb{C}$$

Recall that 0 was regular singular point.

$$\nabla\left(t\frac{\partial}{\partial t}\right)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\frac{1}{4}t}{t-1} & 1 - \frac{2t-1}{t-1} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - t\frac{\partial}{\partial t}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

satisfies RSP condition. But same connection can be expressed as

$$\nabla\left(t\frac{\partial}{\partial t}\right)\begin{pmatrix} f_1 \\ t^{-2}f_2 \end{pmatrix} = \begin{pmatrix} 0 & t^2 \\ \underbrace{\frac{-\frac{1}{4}}{t(t-1)}}_{\text{not holomorphic}} & -2 - \frac{t}{t-1} \end{pmatrix}\begin{pmatrix} f_1 \\ t^{-2}f_2 \end{pmatrix} - t\frac{\partial}{\partial t}\begin{pmatrix} f_1 \\ t^{-2}f_2 \end{pmatrix}$$

Satisfies (J) for $\mu = -2$. Proof will show why.

Prop: If (W, ∇) has RSP at p, satisfies (J) at p.

Pf! RSP gives us

$$\nabla\left(h\frac{\partial}{\partial h}\right)\vec{e} = B\vec{e} \qquad B \in M_n(\mathcal{O}_p)$$

Leibniz rule:

$$B_{j+1} = h\frac{\partial}{\partial h}(B_j) + B_j B. \qquad \begin{array}{l} \text{ord}(B_j) \geq 0 \\ \text{pick } \mu = 0 \end{array}$$

Converse is true too for cyclic objects!

<u>Thm (Fuchs, Turrittin, Lutz)</u>: Let (W, ∇) be cyclic object of $MC(K/F)$
$w \in W$ cyclic vector, p a place of K/F, h uniformizer at p,
$\dim_K(W) = n$. TFAE

(1) (W, ∇) does not have RSP at p

(2)

$$\vec{e} = \begin{pmatrix} w \\ \nabla(h\frac{d}{dh})w \\ \vdots \\ (\nabla(h\frac{d}{dh}))^{n-1} w \end{pmatrix} \quad \text{base}$$

Connection is

$$\nabla(h\tfrac{d}{dh})\vec{e} = \begin{pmatrix} 0 & & I_{n-1} & \\ \vdots & & & \\ \vdots & & & \\ f_0 & f_1 & \cdots & f_{n-1} \end{pmatrix} \vec{e}$$

$\operatorname{ord}_p(f_i) < 0$ for some f_i

(5) (W, ∇) does not satisfy (J) at p.

<u>Proof of Prop (*)</u>: $\vec{e}$ basis of W, $\mu \in \mathbb{Z}$ s.t. $j \geq 1$

$$(\nabla(h\tfrac{h}{dh}))^j \vec{e} = B_j \vec{e} \qquad \operatorname{ord}(B_j) \geq \mu$$

$\vec{f}$ another basis

$$\vec{f} = A\vec{e} \qquad \vec{e} = A^{-1}\vec{f} \qquad A \in GL_n(K).$$

Define C_j by

$$(\nabla(h\tfrac{h}{dh}))^j \vec{f} = C_j \vec{f}$$

Calculate C_j in terms of B_i

$$\left(\nabla\left(h\frac{\partial}{\partial h}\right)\right)^j \vec{f} = \left(\nabla\left(h\frac{\partial}{\partial h}\right)\right)^j \left(A \cdot \vec{e}\right)$$

$$= \sum_{i=0}^{j} \binom{j}{i} \left(\left(h\frac{\partial}{\partial h}\right)^{j-i} (A)\right) \cdot B_i e$$

Leibniz rule (induction)

$$C_j = \sum_{i=0}^{j} \binom{j}{i} \left(\left(h\frac{\partial}{\partial h}\right)^{j-i} (A)\right) \cdot B_i A^{-1}$$

Note that

$$\mathrm{ord}_p\left(h\frac{\partial f}{\partial h}\right) \geq \mathrm{ord}_p(f) \qquad \forall f \in K$$

So

$$\mathrm{ord}_p(C_j) \geq \min_{0 \leq i \leq j} \left(\mathrm{ord}_p(A) + \mathrm{ord}(B_i) + \mathrm{ord}_p(A^{-1})\right)$$

$$\geq \mu + \mathrm{ord}_p(A) + \mathrm{ord}_p(A^{-1})$$

$\square$

So in our example $\mathrm{ord}_p(A) = -2$ $\quad \mathrm{ord}_p(A^{-1}) = 0$ $\quad$ hence $\mu = -2$ works.

Let $a > 0$ $(\in \mathbb{Z})$. $K(h^{1/a})/K$ there is unique prime $\mathfrak{p}^{1/a}$ that extends $\mathfrak{p}$. $h^{1/a}$ uniformizer.

<u>Prop:</u> (W, ∇) obj of $MC(K/F)$. (W, ∇) satisfies (J) at $\mathfrak{p}$ iff inverse image in $MC(K(h^{1/a})/F)$ satisfies (J) at $\mathfrak{p}^{1/a}$.

<u>Pf:</u> Using K-base $\vec{e}$ of W $h\frac{\partial}{\partial h}$ as $\mathcal{O}_{\mathfrak{p}}$ base for both (W, ∇) and inverse image, we get the same sequence of matrices. $\square$

In statement of thm add two more conditions

(3) $\forall$ multiples a of $n!$, inverse image of (W, ∇) in $MC(K(h^{1/a})/F)$ admits basis $\vec{f}$ s.t. writing $t = h^{1/a}$

$$\nabla\left(t\frac{\partial}{\partial t}\right)\vec{f} = B\vec{f}$$

s.t. for some $r \geq 1$ integer

$$B = t^{-r}B_{-r} \qquad B_{-r} \in M_n(\mathcal{O}_{\mathfrak{p}^{1/a}})$$

and image of B_{-r} in $M_n(k(\mathfrak{p}))$ is not nilpotent.

4) $\forall$ mult a of $n!$, inv image of (W, ∇) in $MC(K(h^{1/a})/F)$ does not satisfy (J) at $\mathfrak{p}^{1/a}$.

Pf of Thm: (1) $\Rightarrow$ (2) Def

(2) $\Rightarrow$ (3) Change base $K \to K(t)$ $t^a = h$ we have:

$$\frac{1}{a}\nabla\left(t\frac{d}{dt}\right)\vec{e} = \underbrace{\begin{pmatrix} 0 & & I_{n-1} & \\ \vdots & & & \\ 0 & & & \\ f_1 & \cdots & & f_{n-1} \end{pmatrix}}_{C}\vec{e}$$

By assumption $\mathrm{ord}_{p^{1/a}}(f_i) < 0$ at least one i and $\forall j$ $n! \mid \mathrm{ord}_{p^{1/a}}(f_j)$

$$r := \max_{0 \le j \le n-1}\left(-\mathrm{ord}_{p^{1/a}}(f_j)/n-j\right)$$

This is strictly positive. Consider basis $\vec{f}$ of $W \otimes K(t)$ given by

$$\vec{f} = \underbrace{\begin{pmatrix} 1 & & & 0 \\ & t^r & & \\ & & t^{2r} & \\ 0 & & & \ddots \\ & & & & t^{(n-1)r} \end{pmatrix}}_{A}\vec{e}$$

So

$$\nabla\left(t\frac{d}{dt}\right)\vec{f} = \nabla\left(t\frac{d}{dt}\right)(A\vec{e}) = \left(\left(t\frac{d}{dt}\right)(A)\right)\vec{e} + A \cdot \underset{\substack{\parallel \\ aC\vec{e}}}{\nabla\left(t\frac{d}{dt}\right)(\vec{e})}$$

$$= \underbrace{\left(\left(t\frac{d}{dt}\right)(A)\right)A^{-1} + AaCA^{-1}}_{B}\vec{f}.$$

$$B = \left(\begin{pmatrix} 0 & & & \\ & r & & \\ & & \ddots & \\ & & & (n-1)r \end{pmatrix}\right) + a\begin{pmatrix} 0 & & t^{-r} I_{n-1} & & \\ \vdots & & & & \\ t^{(n-1)r}f_0 & \cdots & t^{(n-i-1)r}f_i & \cdots & f_{n-1} \end{pmatrix}$$

$$\mathrm{ord}_{p^{1/a}}\left(t^{(n-j-1)v}c_j\right) = (n-j-1)v + \mathrm{ord}_{p^{1/a}}(c_j) \geq (n-j-1)v - (n-j)v = -v$$

equality for at least one j.

Write $g_j = t^{(n-j)v} c_j \quad j = 0, \dots, n-1$. $\quad g_j \in \mathcal{O}_{p^{1/a}} \ \forall j$, $g_i \notin m_{p^{1/a}} \ \exists i$.

$$B_{-r} = t^r \begin{pmatrix} 0_r & & \\ & \ddots & \\ & & c_{n-1}r \end{pmatrix} + a \begin{pmatrix} 0 & I_{n-1} \\ \vdots & \\ g_0 & \cdots \ g_{n-1} \end{pmatrix}$$

$$\equiv a \begin{pmatrix} 0 & I_{n-1} \\ \vdots & \\ 0 & \\ g_0 & \cdots \ g_{n-1} \end{pmatrix} \mod p^{1/a}$$

Not nilpotent mod $p^{1/a}$.

$$\det(T I_n - B_{-r}) \equiv T^n - \sum_{i=0}^{n-1} a^{n-i} g_i T^i \mod p^{1/a}$$

B_{-r} not nilpotent mod $p^{1/a}$ $\iff$ $g_i \notin m_{p^{1/a}}$ some i

(3) $\Rightarrow$ (4) Use $\vec{f}$

$$\left(\nabla\left(t\frac{\partial}{\partial t}\right)\right)^j \vec{f} = B_j f \qquad B_{j+1} = \left(t\frac{\partial}{\partial t}\right) B_j + B_j B$$

we have

$$B_j = t^{-rj} B_{-rj} \qquad B_{-rj} \in M_n(\mathcal{O}_{p^{1/a}})$$

$$B_{-rj} \equiv (B_{-r})^j \mod p^{1/a}$$

so $\mathrm{ord}_{p^{1/a}}(B_j) = -rj$.