

10/19

Goal. 1) Define global nilpotence. 2) Bound the local nilpotence. 3) Local to global.

Main Theorem. T : global affine variety. $S \rightarrow T$: smooth, S : connected.
 X/S : sm. proper and $Y: \text{NCD}(S) \hookrightarrow X$.

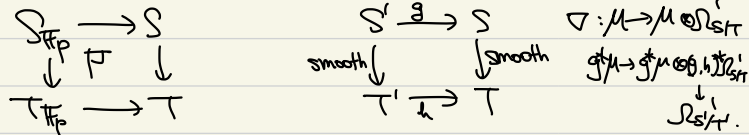
Let $h(i) := \{0 \leq p \leq i : H_{\text{étale}}^{p, i-p}(X/S(\log Y)) \neq 0\}$.

Then, $H_{\text{étale}}^i(X/S(\log Y))$ is globally nilpotent of level $h(i)$.

§ Global nilpotence

Def. A global affine variety is $\text{Spec } R$ for R : int. dom $\neq \mathbb{Z}$ with $\text{char}(\text{Frac } R) = 0$.

Consider $S \rightarrow T$ sm and primes not invertible on S .



Given $(M, \nabla) \in \text{MIC}(S/T) \rightarrow (M_{\mathbb{F}_p}, \nabla_{\mathbb{F}_p}) \in \text{MIC}(S_{\mathbb{F}_p}/T_{\mathbb{F}_p})$ via pullback.

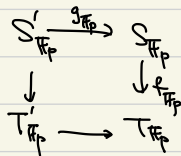
M : locally free.

Def. (M, ∇) is globally nilpotent of level ν on S/T if $(M_{\mathbb{F}_p}, \nabla_{\mathbb{F}_p}) \in \text{Nilp}^\nu(S_{\mathbb{F}_p}/T_{\mathbb{F}_p})$ (convention $\nu = \infty \Rightarrow \text{Nilp}$).

Consider the following diagram: $S' \xrightarrow{g} S$, $f': S' \rightarrow T'$, $T' \xrightarrow{h} T$, $f: S \rightarrow T$. f, f' : sm. T, T' : global aff. var.

Prop. Let $(M, \nabla) \in \text{MIC}(S/T)$ with M : fin. locally free on S . Then, if $(M, \nabla) \in \text{Nilp}^\nu(S/T)$ then $(g, h)^*(M, \nabla) \in \text{Nilp}^\nu(S'/T')$.
 $= (g^* M, (g, h)^* \nabla)$

Al. Rose's talk.



$S_{\mathbb{F}_p} = \emptyset \Rightarrow$ initial element so it's fine.

§ Local nilpotence level

conjugate spectral sequence.

Recall $X \xrightarrow{F} X^{(p)}$ gives rise to $R^p \pi_* (\mathcal{H}^i(F_* \Omega_{X/S})) \Rightarrow R^q \pi_* \Omega_{X/S}$.

$$\begin{array}{ccc} X & \xrightarrow{F} & X^{(p)} \\ \pi \downarrow & & \downarrow \pi^{(p)} \\ S & & S \end{array}$$

Connor proved Cartier isomorphism $C^{-1}: \Omega_{X^{(p)}/S}^i \xrightarrow{\sim} \mathcal{H}^i(F_* \Omega_{X/S})$ ($\mathcal{O}_{X^{(p)}}$ -modules).

The conj ss becomes $R^p \pi_* (\Omega_{X^{(p)}/S}^i) \Rightarrow R^q \pi_* (\Omega_{X/S}^i)$.

Theorem. Assume either

- the formation of $R^p \pi_* (\Omega_{X/S}^i)$ commutes with arb. base change.
- $F_{obs}: S \rightarrow S$ is flat ($\Leftrightarrow S$: regular by Kunz theorem).

Then, the canonical integrable T-connection on $H_{\text{ét}}^n(X/S)$ is nilpotent.

Pf. $X \xrightarrow{F} X^{(p)} \xrightarrow{W} X \Rightarrow F_{obs}^* R^p \pi_* (\Omega_{X/S}^i) \xrightarrow{\sim} R^p \pi_* (W^* \Omega_{X/S}^i)$

$$\begin{array}{ccccc} X & \xrightarrow{F} & X^{(p)} & \xrightarrow{W} & X \\ \pi \searrow & & \downarrow \pi^{(p)} & & \downarrow \pi \\ & & S & \xrightarrow{F_{obs}} & S \end{array}$$

$$E_{2, \text{can}}^{p, i} = F_{obs}^* R^p \pi_* (\Omega_{X/S}^i) \Rightarrow R^q \pi_* (\Omega_{X/S}^i)$$

$$F_{obs} = W_{S/T}^* F_{S/T} \rightsquigarrow F_{S/T}^* (W_{S/T}^* R^p \pi_* (\Omega_{X/S}^i))$$

: p-curvature 0 connection (Cartier Descent, Tomiyata).

Corollary. For $i \geq 0$ int. let $h(i) := \# \{ p \leq i : E_2^{p, i} \text{ is non-zero} \}$.

Then for any smooth morphism $f: S \rightarrow T$,

$$H_{\text{ét}}^i(X/S) \in \text{Nil}_p^{h(i)}(S/T).$$

§ Local to global

Katz's base change. S : noetherian scheme. $\pi: X \rightarrow S$ proper. K^\bullet : complex of ab. sheaves on S

s.t. 1) $K^i = 0$ for $i < 0$ & $i \gg 0$.

2) K^\bullet : coherent \mathcal{O}_X -mod. flat/ S .

3) The diff of K^\bullet is $\pi^*(\mathcal{O}_S)$ -linear.

as long as $\dim X_S$ is

bounded above, $H^p(X_S, \mathcal{F}) = 0$

for all $p > d$ for any ab. sheaf of X .

TFAE. i) $\forall n \geq 0$, $R^n \pi_* (K^\bullet)$ is loc. free.

\Rightarrow Grauert's theorem: S : reduced

ii) $\forall g: S' \rightarrow S$, $g^* R^n \pi_* (K^\bullet) \xrightarrow{\sim} R^n \pi'_* (g^* K^\bullet) \forall n \geq 0$.

$\cdot \eta \mapsto h^p(X_\eta, \mathcal{F}_\eta) \text{ const}$

iii) $\forall g: pt \rightarrow S$.

$\forall n \geq 0$.

$\cdot R^p \pi_* \mathcal{F}$ loc. free,

$R^p \pi_* \mathcal{O}(k) \rightarrow H^p(\cdot)$

Define $H_{\text{étale}}^{p,i}(X/S) := R^p \pi_* \mathcal{O}(k)_S$.

\rightarrow base change commutes and go down one level.

(Deligne) $H_{\text{étale}}^{p,i}(X/S)$ and $H_{\text{étale}}^h(X/S)$ are locally free on S .

proof of Main Thm -

$R^i \pi_* \mathcal{O}(k)_S$, $R^n \pi_* \mathcal{O}(k)_S$ commutes with base change: StkP.

$\Rightarrow h(i)$ bounds the # of pts s.t. $H^{p,i-p}$ is nonzero.

$\Rightarrow H_{\text{étale}}^i(X/S)$ is globally nilpotent of level $\leq h(i)$.