Goal. 1) Define global nilpotence. 2) Bound the local nilpotence. 3) Local to global.

Main Theorem. T: global affine variety. $S \to T$: smooth, S: connected.
X/S: sm. proper and Y: NCD$/S \hookrightarrow X$.
Let $h(i) := \{0 \le p \le i : H^{p,i-p}_{\mathrm{Hodge}}(X/S(\log Y)) \text{ is nonzero}\}$.
Then, $H^i_{dR}(X/S(\log Y))$ is globally nilpotent of level $h(i)$.

§ Global nilpotence

Def. A global affine variety is $\mathrm{Spec}\, R$ for R: int.dom f.g$/\mathbb{Z}$ with $\mathrm{char}(\mathrm{Frac} R) = 0$.

Consider $S \to T$ sm and primes not invertible on S.

$$\begin{tikzcd} S_{\mathbb{F}_p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d] \\ T_{\mathbb{F}_p} \arrow[r] & T \end{tikzcd} \qquad \begin{tikzcd} S' \arrow[r, "g"] \arrow[d, "\text{smooth}"'] & S \arrow[d, "\text{smooth}"] \\ T' \arrow[r, "h"] & T \end{tikzcd}$$

$\nabla : M \to M \otimes \Omega^1_{S/T}$
$g^*M \to g^*M \otimes (g,h)^*\Omega^1_{S/T}$
$\downarrow$
$\Omega^1_{S'/T'}$.

Given $(M, \nabla) \in MIC(S/T) \longrightarrow (M_{\mathbb{F}_p}, \nabla_{\mathbb{F}_p}) \in MIC(S_{\mathbb{F}_p}/T_{\mathbb{F}_p})$ via pullback.
M: locally free.

Def. (M, ∇) is globally nilpotent of level ν on S/T if
$(M_{\mathbb{F}_p}, \nabla_{\mathbb{F}_p}) \in Nilp^{\nu}(S_{\mathbb{F}_p}/T_{\mathbb{F}_p})$ (Convention $\nu = \infty \Rightarrow Nilp$).

Consider the following diagram:
$$\begin{tikzcd} S' \arrow[r, "g"] \arrow[d, "f'"'] \arrow[dr, phantom, "\circlearrowright"] & S \arrow[d, "f"] \\ T' \arrow[r, "h"] & T \end{tikzcd}$$
f, f': sm. T, T': global aff. var.

Prop. Let $(M, \nabla) \in MIC(S/T)$ with M: fin. locally free on S. Then,
if $(M, \nabla) \in Nilp^{\nu}(S/T)$ then $(g,h)^*(M, \nabla) \in Nilp^{\nu}(S'/T')$.
$= (g^*M, (g,h)^*\nabla)$

Pf. Rose's talk.

$$\begin{tikzcd} S'_{\mathbb{F}_p} \arrow[r, "g_{\mathbb{F}_p}"] \arrow[d] & S_{\mathbb{F}_p} \arrow[d, "f_{\mathbb{F}_p}"] \\ T'_{\mathbb{F}_p} \arrow[r] & T_{\mathbb{F}_p} \end{tikzcd}$$

$S_{\mathbb{F}_p} = \emptyset \Rightarrow$ initial element so it's fine.

§ Local nilpotence level

conjugate spectral sequence.

Recall
$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr, "\pi"'] & X^{(p)} \arrow[d, "\pi^{(p)}"] \\ & S \end{tikzcd}$$
gives rise to $R^p\pi_*^{(p)}(\mathcal{H}^q(F_*\Omega^\bullet_{X/S})) \Rightarrow R^n\pi_*\Omega^\bullet_{X/S}$.

Connor proved Cartier isomorphism $C^{-1}: \Omega^i_{X^{(p)}/S} \xrightarrow{\sim} \mathcal{H}^i(F_*\Omega^\bullet_{X/S})$ ($\mathcal{O}_{X^{(p)}}$-modules).

The conj ss becomes $R^p\pi_*^{(p)}(\Omega^q_{X^{(p)}/S}) \Rightarrow R^n\pi_*(\Omega^\bullet_{X/S})$.

Theorem. Assume either

- the formation of $R^p\pi_*(\Omega^q_{X/S})$ commutes with arb. base change.
- $F_{abs}: S \to S$ is flat ($\Leftrightarrow S$: regular by Kunz theorem).

Then, the canonical integrable T-connection on $H^n_{dR}(X/S)$ is nilpotent.

pf.
$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr, "\pi"'] & X^{(p)} \arrow[r, "W"] \arrow[d, "\pi^{(p)}"] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\pi"] \\ & S \arrow[r, "F_{abs}"'] & S \end{tikzcd}$$
$\Rightarrow F_{abs}^* R^p\pi_*(\Omega^q_{X/S}) \xrightarrow{\sim} R^p\pi_*^{(p)}(W^*\Omega^q_{X/S})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cong \Omega^q_{X^{(p)}/S}$

$$E_{2,\mathrm{con}}^{p,q} = F_{abs}^* R^p\pi_*(\Omega^q_{X/S}) \Rightarrow R^n\pi_*(\Omega^\bullet_{X/S}).$$

$F_{abs} = W_{S/T} \circ F_{S/T} \longrightarrow F_{S/T}^*(W_{S/T}^* R^p\pi_*(\Omega^q_{X/S}))$

: p-curvature 0 connection (Cartier Descent, Tony's talk.)

Corollary. For $i \geq 0$ int. let $h(i) := \#\{p \leq i : E_2^{p,i-p} \text{ is non-zero}\}$.

Then for any smooth morphism $f: S \to T$,
$$H^i_{dR}(X/S) \in \mathrm{Nilp}^{h(i)}(S/T).$$

§ Local to global

Katz's base change. S: noetherian scheme. $\pi : X \to S$ proper. $K^\bullet$: complex of ab. sheaves on S

s.t. 1) $K^i = 0$ for $i<0$ & $i \gg 0$.

2) K^i: coherent $\mathcal{O}_X$-mod. flat/S.

3) The diff of $K^\bullet$ is $\pi^{-1}(\mathcal{O}_S)$-linear.

TFAE. i) $\forall n \geq 0$, $R^n\pi_*(K^\bullet)$ is loc. free.

ii) $\forall g : S' \to S$, $g^* R^n\pi_*(K^\bullet) \xrightarrow{\sim} R^n\pi'_*(g^*K^\bullet)$ $\forall n \geq 0$.

iii) $\forall g : pt \to S$. " $\forall n \geq 0$.

Define $H^{p,q}_{Hodge}(X/S) := R^q\pi_*\Omega^p_{X/S}$.

as long as $\dim X_s$ is bounded above, $H^p(X_s, \mathcal{F}) = 0$ for all $p > d$ for any ab. sheaf of X.

$\Rightarrow$ Grauert's theorem : S: reduced

· $y \mapsto h^p(X_y, \mathcal{F}_y)$ const

· $R^pf_*\mathcal{F}$ loc. free,

$R^pf_*\mathcal{F} \otimes k(y) \xrightarrow{\sim} H^p(\ ,\ )$

$\Rightarrow$ base change comm. and go down one level.

(Deligne) $H^{p,q}_{Hodge}(X/S)$ and $H^n_{dR}(X/S)$ are locally free on S.

proof of Main Thm.

$R^q\pi_*\Omega^p_{X/S}$, $R^n\pi_*\Omega^\bullet_{X/S}$ commute with base change : $S_{\mathbb{F}_p}$.

$\Rightarrow$ $h(i)$ bounds the # of p's s.t. $H^{p,i-p}$ is nonzero.

$\Rightarrow$ $H^i_{dR}(X/S)$ is globally nilpotent of level $\leq h(i)$.