Turrittin's Thm

F field of char 0 K field of functions in one variable $/F$ W fin-dim vs.$/K$ $n = \dim_K W$

(W, ∇) object of $MC(K/F)$ p place of K/F h uniformizing parameter at p

Thm. (Turrittin) Let (W, ∇) be an object of $MC(K/F)$. TFAE:

(1) (W, ∇) does not have a regular singular pt at p

(2) $\forall a \in \mathbb{N}$ divisible by $n!$, $\exists$ basis $\vec{f}$ of $W \otimes_K K(h^{1/a})$ in terms of which the connection is expressed as $\nabla(t\frac{d}{dt})\vec{f} = B\vec{f}$ (put $t = h^{1/a}$, $t\frac{d}{dt} = ah\frac{d}{dh}$) s.t. $\exists \nu \in \mathbb{N}$, $B = t^{-\nu}B_{-\nu}$, $B_{-\nu} \in M_n(\mathcal{O}_{p^{1/a}})$ and the image of $B_{-\nu}$ in $M_n(k(p^{1/a}))$ is not nilpotent.

(3) (W, ∇) does not satisfy (J) at p.

Example: $(x\frac{d}{dx})^3 f - \frac{1}{x}f = 0$

$$\nabla\left(x\frac{d}{dx}\right)\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{x} & 0 & 0 \end{pmatrix}}_{B}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

irregular singularity at 0

$$B_{-\nu} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & x \\ 1 & 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bmod (x)$$

no matter what change of basis, always end up with $t^{-\nu}B_{-\nu}$ $B_{-\nu}$ nilpotent in $M_n(k(p))$

Pf of Thm

(3) $\Rightarrow$ (1) last time

(2) $\Rightarrow$ (3) Use basis $\vec{f}$ to test (J)

Write $(\nabla(t\frac{d}{dt}))^j \vec{f} = B_j\vec{f}$, we have $B_{j+1} = t\frac{d}{dt}B_j + B_jB$

$B_1 = B = t^{-\nu}B_{-\nu}$ $\quad \mathrm{ord}_{p^{1/a}}(B_1) = -\nu < 0$

If $\mathrm{ord}_{p^{1/a}}(B_j) = -\nu j$ and $t^{\nu j}B_j \equiv (B_{-\nu})^j \bmod p^{1/a}$

then $t^{\nu(j+1)}B_{j+1} = t^{\nu}\underbrace{t^{\nu j}(t\frac{d}{dt})B_j}_{\in M_n(\mathcal{O}_{p^{1/a}})} + t^{\nu j}B_j t^{\nu}B \in M_n(\mathcal{O}_{p^{1/a}})$

$$t^{\nu(j+1)}B_{j+1} \equiv (t^{\nu j}B_j)(t^{\nu}B) \equiv (B_{-\nu})^{j+1} \bmod p^{1/a}$$

so $\mathrm{ord}_{p^{1/a}}(B_{j+1}) = -\nu(j+1)$. By induction, $\mathrm{ord}_{p^{1/a}}(B_j) = -\nu j$ $\forall j \in \mathbb{N}$.

$\Rightarrow$ inverse image of (W, ∇) in $MC(K(t)/F)$ does not satisfy (J)

(1) $\Rightarrow$ (2) cyclic case last time

Induct on $n = \dim W$. Suppose (W, ∇) has a nontrivial subobject (V, ∇')

SES $\quad 0 \to (V, \nabla') \to (W, \nabla) \to (U, \nabla'') \to 0 \qquad n_1 = \dim_k(V) < n$

$n_2 = \dim_k(U) < n$

(V, ∇') or (U, ∇'') does not have a regular singular pt at p

ind. hyp $\Rightarrow \exists$ basis $\begin{pmatrix} \vec{e} \\ \vec{g} \end{pmatrix}$ of $W \otimes_k K(t)$, $\vec{e}$ basis of $V \otimes_k K(t)$

$\vec{g}$ projects to basis of $U \otimes_k K(t)$

$$\nabla\left(t\frac{d}{dt}\right)\begin{pmatrix} \vec{e} \\ \vec{g} \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\begin{pmatrix} \vec{e} \\ \vec{g} \end{pmatrix} \quad \text{s.t.} \quad \begin{aligned} A &= t^{-v_1} A_{-v_1}, \quad -v_1 = \mathrm{ord}_{p^{1/a}}(A), \\ C &= t^{-v_2} C_{-v_2}, \quad -v_2 = \mathrm{ord}_{p^{1/a}}(C) \end{aligned}$$

$\left.\begin{array}{ll} v_1 > 0, \ A_{-v_1} \text{ has non-nilpotent image in} & M_{n_1}(K(p^{1/a})) \\ v_2 > 0, \ A_{-v_2} & M_{n_2}(K(p^{1/a})) \end{array}\right\}$ at least one holds

Replace $\begin{pmatrix} \vec{e} \\ \vec{g} \end{pmatrix}$ by $\begin{pmatrix} \vec{e} \\ t^N \vec{g} \end{pmatrix}$ for N large

$$\nabla\left(t\frac{d}{dt}\right)\begin{pmatrix} \vec{e} \\ t^N\vec{g} \end{pmatrix} = \begin{pmatrix} A & 0 \\ t^N B & C + N I_{n_2} \end{pmatrix}\begin{pmatrix} \vec{e} \\ t^N\vec{g} \end{pmatrix}$$

This connection matrix has a pole of order $v = \max\{v_1, v_2\} > 0$

and $t^v \begin{pmatrix} A & 0 \\ t^N B & C + N I_{n_2} \end{pmatrix} \equiv \begin{pmatrix} t^{\max\{v,\tau\}} A & \\ & t^{\max\{v,\tau\}} C \end{pmatrix}$ mod $p^{1/a}$ not nilpotent

rmks. (W, ∇) has a regular singular pt at p

$\Leftrightarrow \forall w \in W \quad m = \dim_k \mathrm{Span}\{w, \nabla(h\frac{d}{dh})(w), \cdots\} \qquad \vec{e} = \begin{pmatrix} w \\ \nabla(h\frac{d}{dh})w \\ \vdots \\ \nabla(h\frac{d}{dh})^{m-1} w \end{pmatrix}$

$$\nabla\left(h\frac{d}{dh}\right)\vec{e} = \begin{pmatrix} 0 & I_{m-1} \\ a_0 & a_1 \cdots a_m \end{pmatrix}\vec{e} \qquad \mathrm{ord}(a_i) \geq 0 \ \forall i$$

$(W, \nabla) = \oplus$ cyclic $\qquad (W, \nabla)$ has regular singular pt at $p \Leftrightarrow$ each cyclic piece does

When $\mathrm{char}(F) \nmid n!$, the proof should still work

inverse image
$$\begin{array}{cccc} p' & L & (W\otimes_k L, \nabla_L) \\ | & | & \\ p & k & (W,\nabla) \end{array}$$
L/K function fields in one variable$/F$

p' lies over p

Prop. (W,∇) has a regular singular pt at p iff the inverse image $(W\otimes_k L, \nabla_L)$ of (W,∇) in $MC(L/F)$ does at p'.

Pf. $Der_{p'}(L/F) \simeq Der_p(K/F)\otimes_{\mathcal{O}_p}\mathcal{O}_{p'}$

If W_p is an $\mathcal{O}_p$-lattice in W, stable under $Der_p(K/F)$,

then $W_p\otimes_{\mathcal{O}_p}\mathcal{O}_{p'}$ is an $\mathcal{O}_{p'}$-lattice in $W\otimes_k L$, stable under $Der_{p'}(L/F)$

If $(W\otimes_k L)_{p'}$ is an $\mathcal{O}_{p'}$-lattice in $W\otimes_k L$, stable under $Der_{p'}(L/F)$

then $W\cap(W\otimes_k L)_{p'}$ is an $\mathcal{O}_p$-lattice in W, stable under $Der_p(K/F)$

direct image
$$\begin{array}{ccc} p_1,\cdots,p_r & K & (W,\nabla) \\ \backslash\ / & | & \\ p'' & M & \end{array}$$
K/M function fields in one variable$/F$

$p_1,\cdots,p_r$ all the places of K which lie over p''

Prop. The direct image $(W \text{ as } M\text{-vs}, \nabla|_{Der(M/F)})$ of (W,∇) in $MC(M/F)$ has a regular singular pt at p'' iff (W,∇) does at each p_i lying over p''.

Pf. $Der_{p_i}(K/F) \simeq Der_{p''}(M/F)\otimes_{\mathcal{O}_{p''}}\mathcal{O}_{p_i}$, $i=1,\cdots,r$

If W_{p_i} is an $\mathcal{O}_{p_i}$-lattice in W, stable under $Der_{p_i}(K/F)$,

then $\bigoplus_{i=1}^{r} W_{p_i}$ is an $\mathcal{O}_{p''}$-lattice in W, stable under $Der_{p''}(M/F)$.

Conversely, suppose $(W \text{ as } M\text{-vs}, \nabla|_{Der(M/F)})$ has a regular singular pt at p'',

τ uniformizing parameter at p''

every $w\in W$ annihilated by a monic polynomial in $\nabla(\tau\frac{d}{d\tau})$ whose coefficients are in $\mathcal{O}_{p''}$

In $MC(K/F)$, the cyclic subobject of (W,∇) gen by w is a quotient of an object with regular singular pt at p_i, so it has a regular singular pt at p_i

base change

Prop. $\bar{F}$ algebraic closure of F, $\bar{p}$ induced place of $K\bar{F}/\bar{F}$, $(W_{\bar{F}}, \nabla_{\bar{F}})$ in $MC(K\bar{F}/\bar{F})$.

Then (W,∇) has a regular singular pt at p iff $(W_{\bar{F}}, \nabla_{\bar{F}})$ does at $\bar{p}$

Pf. Use equivalence regular singular pt $\Leftrightarrow$ (J)

calculate with a K-basis of W and a parameter at p.

The Monodromy around a Regular Singular Point

F field of char 0 $\quad$ K field of functions in one variable $/F$ $\quad$ W fin-dim vs.$/K$ $\quad$ $n=\dim_K W$

(W,∇) object of $MC(K/F)$ $\quad$ p place of K/F $\quad$ h uniformizing parameter at p

p rational pt of K/F (i.e. $k(p)=F$) $\quad$ (W,∇) has a regular singular pt at p.

Thm. (Manin) Suppose (W,∇) is an object of $MC(K/F)$ which has a regular singular pt at a rational pt p of K/F. In terms of a uniformizing parameter h at p, and a basis $\vec{e}$ of an $\mathcal{O}_p$-lattice W_p of W, we express the connection as $\nabla(h\frac{d}{dh})\vec{e} = B\vec{e}$, $B \in M_n(\mathcal{O}_p)$. Suppose that the matrix $B(p) \in M_n(F)$ (the value of B at p, whose conjugacy class depends only on the lattice W_p, not on the choice of a basis of W_p or on the choice of the uniformizing parameter h) has all of its eigenvalues in F. Then

(1) The set of images of the eigenvalues of $B(p)$ in the additive gp $F/\mathbb{Z}$ is independent of the choice of the $\mathrm{Der}_p(K/F)$-stable $\mathcal{O}_p$-lattice W_p in W.

(2) Fix a set-theoretic section $\varphi: F/\mathbb{Z} \to F$ of the projection mapping $F \to F/\mathbb{Z}$

$\exists !$ $\mathcal{O}_p$-lattice W_p' of W, stable under $\mathrm{Der}_p(K/F)$, in terms of basis $\vec{e}\,'$ of which the connection is expressed as $\nabla(h\frac{d}{dh})\vec{e}\,' = C\vec{e}\,'$, $C \in M_n(\mathcal{O}_p)$

s.t. the eigenvalues of $C(p) \in M_n(F)$ are all fixed by $F \to F/\mathbb{Z} \xrightarrow{\varphi} F$

(α_i, α_j eigenvalues of $C(p)$, $\alpha_i - \alpha_j \in \mathbb{Z} \Rightarrow \alpha_i = \alpha_j$)

The matrix $C(p)$ is called the local monodromy at p.

(3) The completion $\widehat{W_p'}$ of the $\mathcal{O}_p$-lattice W_p' of W above admits a basis $\hat{e}$ in terms of which the connection is simply $\nabla(h\frac{d}{dh})\hat{e} = C(p)\hat{e}$

local horizontal sections $\quad h^{-C(p)}\,\hat{e}$

$$C(p) = \underset{\text{semisimple}}{D} + \underset{\text{nilpotent}}{N} \qquad [D,N]=0, \qquad U^{-1}DU = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},\ U \in GL_n(F)$$

$$h^{-C(p)} = h^{-D}\, h^{-N}$$

$$h^{-D} = U^{-1}\begin{pmatrix} h^{-\lambda_1} & & \\ & \ddots & \\ & & h^{-\lambda_n} \end{pmatrix} U$$

$$h^{-N} = \sum_{j=0}^{\infty} \frac{(-N)^j}{j!} (\log h)^j = \exp(-N \log h)$$

Example: Gauss hypergeometric equation $x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$

$a=b=\frac{1}{2}$, $c=1$ $\qquad x(1-x)y''+(1-2x)y'-\frac{1}{4}y=0$

regular singular pts at $x=0,1,\infty$

At $x=0$, $h=x$ $\qquad (h\frac{d}{dh})^2 y + \frac{h}{h-1}(h\frac{d}{dh}y)+\frac{\frac{1}{4}h}{h-1}y=0$

$$\nabla(h\tfrac{d}{dh})\begin{pmatrix}e_1\\e_2\end{pmatrix} = \underbrace{\begin{pmatrix}0&1\\ \frac{\frac{1}{4}h}{1-h} & \frac{h}{1-h}\end{pmatrix}}_{C}\begin{pmatrix}e_1\\e_2\end{pmatrix}$$

$$C(p)=C|_{h=0}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$h^{-C(p)} = \begin{pmatrix}1&-\log x\\0&1\end{pmatrix} \longrightarrow \exp(-2\pi i C(p))\,h^{-C(p)} = \begin{pmatrix}1&-2\pi i\\0&1\end{pmatrix}h^{-C(p)}$$

$$\begin{matrix}\hat{e}_1-(\log x)\hat{e}_2\\ \hat{e}_2\end{matrix} \xrightarrow[\text{turn around p counterclockwise once}]{F\hookrightarrow\mathbb{C}} \begin{matrix}\hat{e}_1-(2\pi i+\log x)\hat{e}_2\\ \hat{e}_2\end{matrix}$$

At $x=1$, $h=x-1$ $\qquad (h\frac{d}{dh})^2 y + \frac{h}{1+h}(h\frac{d}{dh}y)-\frac{\frac{1}{4}h}{1+h}y=0$

$$\nabla(h\tfrac{d}{dh})\begin{pmatrix}e_1\\e_2\end{pmatrix} = \underbrace{\begin{pmatrix}0&1\\ -\frac{\frac{1}{4}h}{1+h} & -\frac{h}{1+h}\end{pmatrix}}_{C}\begin{pmatrix}e_1\\e_2\end{pmatrix}$$

$$C(p)=C|_{h=0}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$h^{-C(p)} = \begin{pmatrix}1&-\log(x-1)\\0&1\end{pmatrix}$$

At $x=\infty$, $h=\frac{1}{x}$ $\qquad (h\frac{d}{dh})^2 y + \frac{1}{h-1}(h\frac{d}{dh}y)-\frac{\frac{1}{4}}{h-1}y=0$

$$\nabla(h\tfrac{d}{dh})\Big|_{h=0}\begin{pmatrix}e_1\\e_2\end{pmatrix} = \begin{pmatrix}0&1\\ \frac{\frac{1}{4}}{h-1} & \frac{1}{1-h}\end{pmatrix}\Bigg|_{h=0}\begin{pmatrix}e_1\\e_2\end{pmatrix} = \underbrace{\begin{pmatrix}0&1\\-\frac{1}{4}&1\end{pmatrix}}_{\begin{pmatrix}2&0\\1&2\end{pmatrix}\begin{pmatrix}\frac{1}{2}&1\\0&\frac{1}{2}\end{pmatrix}\begin{pmatrix}2&0\\1&2\end{pmatrix}^{-1}}\begin{pmatrix}e_1\\e_2\end{pmatrix}$$

$$\nabla(h\tfrac{d}{dh})\begin{pmatrix}\hat{e}_1\\\hat{e}_2\end{pmatrix} = \underbrace{\begin{pmatrix}\frac{1}{2}&1\\0&\frac{1}{2}\end{pmatrix}}_{C(p)}\begin{pmatrix}\hat{e}_1\\\hat{e}_2\end{pmatrix}$$

$$h^{-C(p)} = \exp(-\tfrac{1}{2}\log h)\begin{pmatrix}1&-\log h\\0&1\end{pmatrix}$$

$$\longrightarrow \exp(-2\pi i\, C(p))\,h^{-C(p)} = \exp(-2\pi i\cdot\tfrac{1}{2})\exp\left(-2\pi i\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)h^{-C(p)}$$
$$= \begin{pmatrix}-1&\\&-1\end{pmatrix}\begin{pmatrix}1&-2\pi i\\0&1\end{pmatrix}h^{-C(p)}$$

Rmk. The conjugacy class of N is independent of the choice of $\varphi: F/\mathbb{Z} \to F$

Choose the basis $\hat{e}$ of W_p so that the connection is expressed as $\nabla(h\frac{d}{dh})\hat{e} = C\hat{e}$ with $C \in M_n(F)$

s.t. $C = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix}$ $\quad J_i = \begin{pmatrix} \lambda_i & 1 & \\ & \ddots & 1 \\ & & \lambda_i \end{pmatrix} \in M_{n_i}(F)$

Put $v_i = \varphi(\lambda_i) - \lambda_i$ $\quad \hat{e}' = \begin{pmatrix} h^{v_1} I_{n_1} & & \\ & \ddots & \\ & & h^{v_r} I_{n_r} \end{pmatrix} \hat{e}$

$\nabla(h\frac{d}{dh})\hat{e}' = \begin{pmatrix} J_1 + v_1 I_{n_1} & & \\ & \ddots & \\ & & J_r + v_r I_{n_r} \end{pmatrix} \hat{e}'$ $\quad \hat{e}'$ basis of the unique lattice specified by φ

Recall. local monodromy $C(p) = D + N$

quasi-unipotent with exponent of nilpotence $\leq \nu$ if all eigenvalues of D in $\mathbb{Q}$ and $N^\nu = 0$

If p any place of K/F (not necessarily rational), we say that the local monodromy at p is quasi-unipotent with exponent of nilpotence $\leq \nu$ if this becomes true after the change of base $F \to \bar{F}$ at the induced place $\bar{p}$ of $K\bar{F}/\bar{F}$.

inverse image
$$\begin{array}{ccc} p' & & L \\ | & & | \\ p & & K \end{array}$$
L/K function fields in one variable$/F$

p' lies over p

Prop. The inverse image $(W \otimes_K L, \nabla_L)$ of (W, ∇) in $MC(L/F)$, which has a regular singular pt at p', has quasi-unipotent local monodromy at p' of exponent of nilpotence $\leq \nu$ iff (W, ∇) does at p.

Pf. base change $F \to \bar{F}$ $\quad$ reduce to $F = \bar{F}$

$\nabla(h\frac{d}{dh})\hat{e} = C(p)\hat{e}$, $C(p) \in M_n(F)$ $\quad \hat{e}$ basis of $\hat{W}$

$\nabla(\varepsilon(p'/p)\, h\frac{d}{dh})\hat{e} = \varepsilon(p'/p)\, C(p)\hat{e}$ $\quad \hat{e}$ "same" basis $\widehat{W \otimes_K L}$ $\quad \varepsilon(p'/p)$ ram. index

direct image
$$\begin{array}{ccc} p_1, \cdots, p_r & & K \\ \diagdown \ \diagup & & | \\ p'' & & M \end{array}$$
K/M function fields in one variable$/F$

$p_1, \cdots, p_r$ all the places of K which lie over p''

Prop. Let (W, ∇) have regular singular pts at each $p_1, \cdots, p_r$. The direct image $(W$ as M-vs, $\nabla|_{Der(M/F)})$ of (W, ∇) in $MC(M/F)$ has quasi-unipotent local monodromy at p'' of exponent of nilpotence $\leq \nu$ iff (W, ∇) does at each place p_i of K lying over p''.

Pf. base-change $F \to \bar{F}$ $\quad$ reduce to $F = \bar{F}$

τ uniformizing parameter at p''

For each p_i lying over p'', choose a lattice W_{p_i} $\quad \hat{e}_i$ basis of completion $\hat{W}_{p_i}$

$\nabla(\varepsilon(p_i/p'')\, \tau\frac{d}{d\tau})\hat{e}_i = C(p_i)\hat{e}_i$, $C(p_i) \in M_n(F)$

$\mathcal{O}_{p''}$-lattice $\bigoplus_i W_{p_i}$ in W as M-vs $\quad$ basis $\tau^{a/\varepsilon(p_i/p'')}\hat{e}_i$, $a = 0, 1, \cdots, \varepsilon(p_i/p'') - 1$, $i = 1, \cdots, r$

On each block $\nabla(\tau\frac{d}{d\tau})(\tau^{a/\varepsilon(p_i/p'')}\hat{e}_i) = \frac{1}{\varepsilon(p_i/p'')}(aI_n + C(p_i))(\tau^{a/\varepsilon(p_i/p'')}\hat{e}_i)$