DE RHAM IN CHARACTERISTIC p AND THE CARTIER ISOMORPHISM

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I. Notation and motivation

Our goal is to state and prove Cartier's theorem following §7 of Katz. Let $\pi : X \to S$ be smooth, $X \xrightarrow{F} X^{(p)} \xrightarrow{W} X$ the factorization of Frobenius, $\pi^{(p)} : X^{(p)} \to S$ the base change. Also let $Y \hookrightarrow X$ a normal crossings divisor relative to S, $Y^{(p)} \hookrightarrow X^{(p)}$ the base change along W.

The constructions of §1 specialize to the case of $(\mathcal{E}, \nabla) = (\mathcal{O}_{X/S}, d)$ to recover "ordinary de Rham cohomology"

$$H^q_{dR}(X/S) := H^q_{dR}(X/S, (\mathcal{O}_X, d)) = \mathbf{R}^q\pi_\star(\Omega^\bullet_{X/S}).$$

and

$$H^q_{dR}(X/S(\log Y)) := H^q_{dR}(X/S(\log Y), (\mathcal{O}_X, d(\log Y))) = \mathbf{R}^q\pi_\star(\Omega^\bullet_{X/S}(\log Y)).$$

Given any smooth morphism $f : S \to T$, since $(\mathcal{O}_X, d_{X/S}) = (\mathcal{O}_X, d_{X/T}|_{\mathrm{Der}(X/S)})$, we have seen that the de Rham cohomology comes with a canonical integrable T-connection.

Proposition 1 (Spectral sequence in de Rham cohomology)**.** *There are spectral sequences*

$$E_2^{p,q} = R^p\pi_\star^{(p)}(\mathcal{H}^q(F_\star\Omega^\bullet_{X/S})) \implies H^{p+q}_{dR}(X/S)$$
$$E_2^{p,q} = R^p\pi^{(p)}(\mathcal{H}^q(F_\star\Omega^\bullet_{X/S}(\log Y))) \implies H^{p+q}_{dR}(X/S(\log Y)).$$

Proof. In general, there are two spectral sequences for hyper-derived functors. We recall their construction for completeness. Let $\pi : X \to S$ any map. For an arbitrary complex $\Omega^\bullet$ on X, choose a flasque resolution of (e.g. by iterating the "sheaf of discontinuous sections" construction), i.e. a double complex $0 \to \Omega^\bullet \to \mathcal{A}^{\bullet,\bullet}$ such that $0 \to \Omega^p \to \mathcal{A}^{p,\bullet}$. Then the hyper-derived pushforward is by definition

$$\mathbf{R}^{p+q}\pi_\star(\Omega^\bullet) = R^{p+q}\pi_\star(\mathrm{Tot}(\mathcal{A}^{\bullet,\bullet})) = \mathcal{H}^{p+q}(\mathrm{Tot}(\pi_\star\mathcal{A}^{\bullet,\bullet})).$$

There are two spectral sequences associated to the double complex $\pi_\star\mathcal{A}^{\bullet,\bullet}$ depending on whether we start by taking vertical or horizontal cohomology, and both converge to cohomology of the total complex. If we begin with vertical cohomology, then our spectral sequence has E_1 page

$${}_I\mathrm{E}_1^{p,q} = R^q\pi_\star(\Omega^p) \implies \mathbf{R}^{p+q}\pi_\star(\Omega^\bullet).$$

This is the *first spectral sequence*, which is in our case the standard *Hodge-to-de-Rham*, or *Frölicher*, spectral sequence. One part of the Hodge decomposition is that over $\mathbb{C}$ this degenerates at the E_1-page.

On the other hand, if we begin computing cohomology horizontally first, then the E_1-page is (flipping indices so that our spectral sequences behave the same way)

$$E_1^{p,q} := \mathcal{H}^q(\pi_\star\mathcal{A}^{\bullet,p}) = \pi_\star\mathcal{H}^q(\mathcal{A}^{\bullet,p}),$$

the latter equality because pushforward is exact on complexes of flasque objects. Then, noting that the p-th column $\mathcal{H}^p(\mathcal{A}^{\bullet,q})$ is a flasque resolution of $\mathcal{H}^p(\Omega^\bullet)$, our E_1-page is the pushforward of this resolution and so we have the *second spectral sequence*, with E_2-page

$${}_{II}\mathrm{E}_2^{p,q} = R^p\pi_\star(\mathcal{H}^q(\Omega^\bullet)) \implies \mathbf{R}^{p+q}\pi_\star(\Omega^\bullet).$$

These two spectral sequences are conjugate in some abstract sense, in particular giving rise to "conjugate filtrations" of the de Rham cohomology.

Now we leave generality behind. In our specific case, the second spectral sequence applied to $\pi : X \to S$ and $\Omega^\bullet$ gives

$$E_2^{p,q} = R^p(\pi^{(p)} \circ F)_\star(\mathcal{H}^q(\Omega^\bullet_{X/S})) \implies H^{p+q}_{dR}(X/S).$$

Note that $F_\star$ is a homeomorphism on topological spaces. A simple diagram chase on stalks shows that pushforward along homeomorphisms is exact. This implies by definition that

$$R^p(\pi^{(p)} \circ F)_\star = (R^p\pi_\star^{(p)}) \circ F_\star \quad \text{and} \quad F_\star \mathcal{H}^q(\Omega^\bullet) = \mathcal{H}^q(F_\star\Omega^\bullet). \tag{1}$$

The former can be checked on injective resolutions, the latter is definitional. Combining these, our spectral sequence becomes

$$E_2^{p,q} = R^p\pi_\star^{(p)}(\mathcal{H}^q(F_\star\Omega^\bullet_{X/S})) \implies H^{p+q}_{dR}(X/S).$$

The same argument verbatim applied to $\Omega^\bullet(\log Y)$ gives the second spectral sequence in the proposition.

Someday, someone should explain to me how any of this works in derived categories. □

These spectral sequences are only useful if $\mathcal{H}^q(F_\star\Omega^\bullet_{X/S})$ and $\mathcal{H}^q(F_\star\Omega^\bullet_{X/S}(\log Y))$ have a simple description. This is the content of Cartier's isomorphism.

II. Ordinary Cartier

Theorem 2. *There is a unique isomorphism of $\mathcal{O}_{X^{(p)}/S}$-modules*

$$C^{-1} : \Omega^i_{X^{(p)}/S} \xrightarrow{\sim} \mathcal{H}^i(F_\star\Omega^\bullet_{X/S})$$

which satisfies

- $C^{-1}(1) = 1$,
- $C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau)$, *and*
- $C^{-1}(d(W^{-1}(f)) = [f^{p-1}df]$.

Example 3. Let S a characteristic p scheme (e.g. $S = \operatorname{Spec} k$ for k a characteristic p field) and $X = \mathbb{A}^n_S$. Then $X^{(p)} = \mathbb{A}^n_S$ also, and $F : X \to X^{(p)}$ corresponds to the $\mathcal{O}_S$-map $\mathcal{O}_S[x_1, \dots, x_n] \to \mathcal{O}_S[y_1, \dots, y_n]$ given by

$$x_k \mapsto y_k^p.$$

(We distinguish the coordinates on X and $X^{(p)}$ for clarity). In this case, $\Omega^i_{X/S}$ is the free $\mathcal{O}_X$-module generated by the $dy_{i_1} \wedge \cdots \wedge dy_{i_k}$. Pushing this forward along F gives a module with the same underlying set and the x_i acting through y_i^p. This is still free over $\mathcal{O}_{X^{(p)}}$, but with larger basis

$$F_\star\Omega^j_{X/S} = \mathcal{O}_{X^{(p)}}\langle y_1^{w_1} \cdots y_n^{w_n} dy_{\alpha_1} \wedge \cdots \wedge dy_{\alpha_j} : \alpha_i \neq \alpha_k \forall i \neq k,\ \ 0 \leq w_i \leq (p-1)\forall i\rangle.$$

The exterior derivative maps are still given by total derivatives (with respect to the y_i).

In order to compute $\mathcal{H}^i(F_\star\Omega^\bullet_{X/S})$, we can factor out the $\mathcal{O}_{X^{(p)}}$-linearity via

$$F_\star\Omega^j_{X/S} \cong \mathcal{O}_{X^{(p)}} \otimes_{\mathbb{F}_p} K^j(n),$$

where $K^j(n)$ is the $\mathbb{F}_p$-vector space with the same basis as above. The differentials descend to the $K^j(n)$ because the exterior derivative is $\mathcal{O}_S[y_1^p, \dots, y_n^p]$-linear. Since everything is faithfully flat over a field, we get

$$\mathcal{H}^i(F_\star\Omega^\bullet_{X/S}) \cong \mathcal{O}_{X^{(p)}} \otimes_{\mathbb{F}_p} \mathcal{H}^i(K^\bullet(n)).$$

Consider the case of $\mathbb{A}^1$, i.e. the complex $K^\bullet(1)$:

$$0 \longrightarrow \mathbb{F}_p\langle 1, \dots, y^{p-1}\rangle \longrightarrow \mathbb{F}_p\langle dy, ydy, \dots, y^{p-1}dy\rangle \longrightarrow 0\ .$$

The exterior derivative lets us compute

$$\mathcal{H}^i(K^\bullet(1)) = \begin{cases} \mathbb{F}_p \cdot [1] & i = 0 \\ \mathbb{F}_p \cdot [y^{p-1}dy] & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Comparing this to $\Omega^\bullet_{X^{(p)}/S} = \mathcal{O}_{X^{(p)}}\langle 1, dx\rangle$, we can define the isomorphism by $1 \mapsto 1, dx \mapsto y^{p-1}dy$. This will be our model in general.

For a general $\mathbb{A}^n_S$, notice that as complexes

$$K^\bullet(n) = K^\bullet(1)^{\otimes n}$$

(where the tensor products are over $\mathbb{F}_p$). So by Kunneth,

$$\mathcal{H}^0(K^\bullet(n)) = \mathbb{F}_p, \qquad \mathcal{H}^1(K^\bullet(n)) = \mathbb{F}_p\langle y_1^{p-1}dy_1, \ldots, y_n^{p-1}dy_n\rangle,$$

and

$$\mathcal{H}^i(K^\bullet(n)) = \bigwedge^i \mathcal{H}^1(K^\bullet(n)) = \mathbb{F}_p\langle (y_{k_1}\cdots y_{k_i})^{p-1}dy_{k_1}\wedge\cdots\wedge dy_{k_i}\rangle.$$

So we have

$$\mathcal{H}^i(F_\star\Omega^\bullet_{X/S}) = \mathcal{O}_{X^{(p)}}\langle (y_{k_1}\cdots y_{k_i})^{p-1}dy_{k_1}\wedge\cdots\wedge dy_{k_i}\rangle$$

On the other hand,

$$\Omega^i_{X^{(p)}/S} = \mathcal{O}_{X^{(p)}}\langle dx_{k_1}\wedge\cdots\wedge dx_{k_i}\rangle.$$

So the map in degree 1 given by $dx_k \mapsto y_k^{p-1}dy_k \forall k$ extends multiplicatively to the desired isomorphism. This isomorphism, of course, satisfies the conditions of the theorem.

This example also constitutes the essential part of the proof.

Proof. Let $X, S, X^{(p)}, \pi$, etc., as in the statement of the theorem. The multiplicativity condition implies that C^{-1} is determined by its degree-one part (the degree-zero part is determined by linearity and the constant term condition). Since $\Omega^i_{X^{(p)}/S} = \bigwedge^i \Omega^1_{X^{(p)}/S}$, we can extend any $\mathcal{O}_{X^{(p)}/S}$-map on the degree-1 piece. So it suffices to construct the map in degree 1.

An $\mathcal{O}_{X^{(p)}}$-map $C^{-1} : \Omega^1_{X^{(p)}/S} \to \mathcal{H}^1(F_\star\Omega^\bullet_{X/S})$ is, by the universal property of the sheaf of differentials, the same as a $(\pi^{(p)})^{-1}(\mathcal{O}_S)$-linear derivation $\mathcal{O}_{X^{(p)}} \to \mathcal{H}^1(F_\star\Omega^\bullet_{X/S})$. Since $\mathcal{O}_{X^{(p)}} = \mathcal{O}_X \otimes_{\pi^{-1}(\mathcal{O}_S)} \pi^{-1}(\mathcal{O}_S)$ ($\pi^{-1}(\mathcal{O}_S)$ acting on itself by absolute Frobenius), this is the same data as a bi-"linear" map:

$$\delta : \mathcal{O}_X \times \pi^{-1}(\mathcal{O}_S) \to \mathcal{H}^1(F_\star\Omega^\bullet_{X/S})$$

which is bi-additive and satisfies

- $\delta(fs, s') = \delta(f, s^p s')$,
- $\delta(gf, s) = g^p\delta(f, s) + f^p\delta(g, s)$, and
- $\delta(f, 1) = [f^{p-1}df]$ (corresponding to the third bullet in the theorem).

We define our map as the one corresponding to $\delta(f, s) = [sf^{p-1}df]$. This satisfies the conditions above. It is in particular bi-additive:

$$\delta(f+g, s) - \delta(f, s) - \delta(g, s) = s((f+g)^{p-1}(df+dg) - f^{p-1}df - g^{p-1}dg) = d\left(s\cdot\frac{(f+g)^p - f^p - g^p}{p}\right),$$

so at the level of cohomology δ is biadditive, and we have constructed a global morphism. Note that this C^{-1} agrees with the map constructed in our affine space example.

We can check that C^{-1} is an isomorphism Zariski-locally. Smooth morphisms are Zariski-locally of the form étale-over-affine-space. If $X \to \mathbb{A}^n_S \to S$ is this factorization, then we choose coordinates $x_1, \ldots, x_n$ on X corresponding to the standard coordinates on $\mathbb{A}^n_S$. The étale map induces an isomorphism $\Omega^1_{X/S} = \Omega^1_{\mathbb{A}^n_S/S}$, compatible with exterior differentiation and the relative Frobenius maps. All of our arguments above for the case of $X = \mathbb{A}^n_S$ carry over mostly verbatim to this case, so we reduce to the case of affine space above, where we saw that our map is an isomorphism.[1] □

III. Cartier with log singularities

Now let $i : Y \hookrightarrow X$ the inclusion of a normal crossing divisor over S. Write $Y^{(p)} \hookrightarrow X^{(p)}$ for the pullback along relative Frobenius. The canonical composition $d(\log Y) : \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \hookrightarrow \Omega^1_{X/S}(\log Y)$ makes $(\mathcal{O}_X, d(\log Y))$ into an integrable vector bundle for X/S with log singularities along Y, so we define

$$H^q_{dR}(X/S(\log Y)) := \mathbf{R}^q\pi_\star(\Omega^\bullet_{X/S}(\log Y)).$$

We have an analogous spectral sequence

$$E_2^{p,q}(\log Y) = R^p\pi_\star^{(p)}(\mathcal{H}^q(F_\star\Omega^\bullet_{X/S}(\log Y))) \implies \mathbf{R}^{p+q}\pi_\star(\Omega^\bullet_{X/S}(\log Y)) = H^{p+q}_{dR}(X/S(\log Y)).$$

[1] Arthur corrected me on this point. There is one subtlety in knowing that pullback along the étale map is compatible with pushforward along F. This uses that F is not only a homeomorphism, a universal homeomorphism, and maybe more besides.

Theorem 4 (Cartier with log singularities)**.** *The map C^{-1} induces an isomorphism*

$$\Omega^i_{X^{(p)}/S}(\log Y^{(p)}) \cong \mathcal{H}^i(F_\star \Omega^\bullet_{X/S}(\log Y))$$

As before, the proof will boil down to the case of (an open subset of) affine space, so let's do that example first. It is a fact of algebraic geometry that in the theorem "smooth = étale over affine" one can take a normal crossings divisor to be the pullback of a union of coordinate hyperplanes, so without loss of generality we have the following.

Example 5. Let $X = \mathbb{A}^n_S$ with coordinates y_i and $Y = V(y_1 \cdots y_v)$ for some $v \leq n$. We write $X^{(p)}$ with coordinates x_i, so $Y^{(p)} = V(x_1 \cdots x_v)$. Then we have

$$\Omega^i_{X/S}(\log Y) = \mathcal{O}_{X^{(p)}} \langle \omega_{k_1} \wedge \cdots \wedge \omega_{k_i} \rangle$$

where ω_k is dy_k/y_k if $k \leq v$ and dy_k otherwise. Pushing forward gives

$$F_\star \Omega^\bullet_{X/S}(\log Y) = \mathcal{O}_{X^{(p)}} \langle x_1^{w_1} \cdots x_n^{w_n} \omega_{k_1} \wedge \cdots \wedge \omega_{k_i} : 0 \leq w_i < (p-1) \rangle = \mathcal{O}_{X^{(p)}} \otimes L^\bullet(n, v)$$

with differentials descending to $L^\bullet$, so

$$\mathcal{H}^i(F_\star \Omega^\bullet_{X/S}(\log Y)) = \mathcal{O}_{X^{(p)}} \otimes_{\mathbb{F}_p} \mathcal{H}^i(L^\bullet(n, v)).$$

We have a tensor product decomposition

$$L^\bullet(n, v) = L^\bullet(1, 1)^{\otimes v} \otimes K^\bullet(1)^{\otimes (n-v)},$$

so again we reduce by Kunneth to $L^\bullet(1, 1)$, where

$$\mathcal{H}^i(L^\bullet(1,1)) = \begin{cases} \mathbb{F}_p \cdot 1 & i = 0 \\ \mathbb{F}_p \cdot dy/y & i = 1 \\ 0 & \text{otherwise} \end{cases}.$$

So on $\mathbb{A}^n$ we can take

$$\begin{aligned} C^{-1} : \Omega^1_{X^{(p)}/S}(\log Y^{(p)}) &\longrightarrow \mathcal{H}^1(F_\star \Omega^\bullet_{X/S}(\log Y)) \\ \frac{dx_k}{x_k} &\longmapsto \frac{dy_k}{y_k} && 0 \leq k \leq v \\ dx_k &\longmapsto y_k^{p-1} dy_k && v < k \leq n \end{aligned}$$

and extend by multiplicativity.

Proof of Theorem. We will deduce this from the regular Cartier isomorphism applied to $(X - Y)/S$. Note that we have natural inclusions

$$\Omega^i_{X^{(p)}/S}(\log Y^{(p)}) \subseteq \Omega^i_{(X^{(p)} - Y^{(p)})/S} = \Omega^i_{(X-Y)^{(p)}/S}.$$

On the other hand,

$$F_\star \Omega^i_{X/S}(\log Y) \subseteq F_\star \Omega^i_{(X-Y)/S},$$

so we get $\mathcal{O}_{X^{(p)}}$-morphisms of cohomology sheaves

$$h^i : \mathcal{H}^i(F_\star \Omega^\bullet_{X/S}(\log Y)) \to \mathcal{H}^i(F_\star \Omega^\bullet_{(X-Y)/S})$$

(not *a priori* injective). We wish for a factorization

$$\begin{array}{ccc} \Omega^i_{(X-Y)^{(p)}/S} & \xrightarrow{C^{-1}} & \mathcal{H}^i(F_\star \Omega^\bullet_{(X-Y)/S}) \\ \uparrow & & \uparrow \\ \Omega^i_{X^{(p)}/S}(\log Y^{(p)}) & \overset{?}{\dashrightarrow} & \mathcal{H}^i(F_\star \Omega^\bullet_{X/S}(\log Y)) \end{array}$$

First we show the right map is injective. This can be checked locally, so we reduce to the case of affine space. There we can take advantage of our explicit bases:

$$\begin{array}{ccc} \mathcal{H}^i(F_\star\Omega^\bullet_{(X-Y)/S}) & = & \mathcal{O}_{(X-Y)^{(p)}} \otimes_{\mathbb{F}_p} \mathcal{H}^i(K^\bullet(n)) \\ \uparrow & & \uparrow \\ \mathcal{H}^i(F_\star\Omega^\bullet_{X/S}(\log Y)) & = & \mathcal{O}_{X^{(p)}} \otimes_{\mathbb{F}_p} \mathcal{H}^i(L^\bullet(n,v)) \end{array}$$

The left vertical map is restriction, so the right map is given in degree 1 by

$$\begin{aligned} 1 \otimes \left[\frac{dy_k}{y_k}\right] &\longmapsto \left[\frac{y_k^{p-1}dy_k}{y_k^p}\right] \longmapsto \frac{1}{x_k} \otimes \left[y_k^{p-1}dy_k\right] && 0 \leq k \leq v \\ 1 \otimes [y_k^{p-1}dy_k] &\longmapsto 1 \otimes [y_k^{p-1}dy_k] && v < k \leq n, \end{aligned}$$

which is injective as the image of the basis is linearly independent. Since wedge power is exact, this implies the same in all higher degrees.

Since the right map is injective, if a factorization of the Cartier isomorphism exists, it is unique, and so we can check that the map factors *locally.* This reduces again to affine space, where we can check it on the basis of ω_k. For each $k \leq v$:

$$\begin{array}{ccc} \frac{1}{x_k}dx_k & \longmapsto & \frac{1}{x_k}[y_k^{p-1}dy_k] \\ \uparrow & & \\ \frac{dx_k}{x_k} & & \end{array}$$

which of course factors through the bottom right as desired. For $k > v$, we have $dx_k \mapsto [y_k^{p-1}dy_k]$, which is in the image also. So locally the Cartier isomorphism factors through the logarithmic differentials. By uniqueness of the factorizations, these glue to give a global restricted map

$$C^{-1}(\log Y) : \Omega^i_{X^{(p)}/S}(\log Y^{(p)}) \to \mathcal{H}^i(F_\star\Omega^\bullet_{X/S}(\log Y)).$$

Note that this map is locally an isomorphism: in the étale-over-affine case, we saw above that the map bijects our $\mathcal{O}_{X^{(p)}}$-bases. So $C^{-1}(\log Y)$ is an isomorphism. □

IV. Application

We have seen that

Corollary 6. *The spectral sequences for (log) de Rham cohomology reduce to*

$$E_2^{p,q} = R^p\pi_\star^{(p)}(\Omega^q_{X^{(p)}/S}) \implies H^{p+q}_{dR}(X/S)$$
$$E_2^{p,q}(\log Y) = R^p\pi_\star^{(p)}(\Omega^q_{X^{(p)}/S}(\log Y^{(p)})) \implies H^{p+q}_{dR}(X/S(\log Y)).$$

Assume now that either

(1) $F_{abs} : S \to S$ is flat, as in the case of S smooth over a field, or
(2) the formation of the $R^p\pi_\star(\Omega^q_{X/S})$ and $R^p\pi_\star(\Omega^q_{X/S}(\log Y))$ commute with *arbitrary* base change of X/S.

Either of these hypotheses ensures the cohomology and base change theorem holds for the Cartesian diagram

$$\begin{array}{ccc} X^{(p)} & \xrightarrow{W} & X \\ \downarrow{\scriptstyle \pi^{(p)}} & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{F_{abs}} & S \end{array}$$

(see for example, Stacks [02KH] for the flat case) and so the natural map

$$F^\star_{abs}R^p\pi_\star(\Omega^q_{X/S}) \to R^p\pi_\star^{(p)}(W^\star(\Omega^q_{X/S})) = R^p\pi_\star^{(p)}(\Omega^q_{X^{(p)}/S})$$

is an isomorphism. Our final version of the spectral sequence is

$$E_2^{p,q} = F_{abs}^{\star} R^p \pi_{\star}(\Omega_{X/S}^q) \implies H_{dR}^{p+q}(X/S),$$

likewise with log differentials.

We already had a spectral sequence that looked like this, but it began on the E_1 page, without the $F_{abs}^{\star}$. So why do we care? The first reason is that it gives an alternate construction of the Gauss-Manin connection on these $E_2^{p,q}$ as the canonical integrable connection coming from the pullback. More importantly, recall the following result from Tony's talk.

Theorem 7. *If a coherent sheaf with integrable connection is of the form $F_{abs}^{\star}(\mathcal{F}, \nabla)$, then it has p-curvature zero.*

So the E_2-page of our spectral sequence consists only of elements of $\mathrm{MIC}(X/S)$ (with their Gauss-Manin connections) which have p-curvature zero. Taking kernels and cokernels preserves this property, of course, so the E_∞ page also consists entirely of p-curvature zero elements of MIC. Since $H_{dR}^n(X/S)$ has a filtration by the $E_\infty^{p,n-p}$ and p-curvature is additive in extensions (Rose's talk), we have shown

Corollary 8. *The number of nonzero $E_2^{p,q}$ with $p + q = n$ is an upper bound on the exponent of nilpotence of $H_{dR}^n(X/S)$ with Gauss-Manin connection.*

This is at least as good a bound as $\leq (n + 1)$ and will often be better!