p-curvature & Exponent of Nilpotence

Recall the set-up & some defs from previous talk:

Let T be a scheme of char $p>0$, $f: S \to T$ smooth,
$(\mathcal{E}, \nabla) \in MIC(S/T)$, i.e. $\nabla: \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_S} \Omega^1_{S/T}$, ∇ integrable.

$$\leftrightarrow \nabla: Der(S/T) \to End_T(\mathcal{E})$$

$$\nabla(D): \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_S} \Omega^1_{S/T} \xrightarrow{id \otimes D} \mathcal{E}$$

The p-curvature of ∇ is a map $\psi: Der(S/T) \to End_S(\mathcal{E})$,
$\psi(D) = \nabla(D)^p - \nabla(D^p)$ with properties:

- $\psi(D_1 + D_2) = \psi(D_1) + \psi(D_2)$
- $\psi(gD) = g^p \psi(D)$, $g \in \mathcal{O}_S$
- $\nabla(D)$, $\nabla(D^p)$, $\psi(D)$ all commute
- $\psi(D_1)$, $\psi(D_2)$ commute.

Corollary 5.5 Let $f: S \to T$ smooth, T char $p>0$, $(\mathcal{E}, \nabla) \in MIC(S/T)$, $n \in \mathbb{Z}_{\geq 1}$. TFAE:

1) $\exists$ a filtration of $(\mathcal{E}, \nabla)$ of length n,
$\mathcal{E} = F^0 \supseteq F^1 \supseteq \dots \supseteq F^n = \{0\}$, $\nabla_i : F^i \to F^i \otimes_{\mathcal{O}_S} \Omega^1_{S/T}$, $\nabla_i = \nabla|_{F^i}$,
whose associated graded object has p-curvature 0,
$gr^i = F^i/F^{i+1}$, $\nabla_{gr^i} = \bar{\nabla}_i : F^i/F^{i+1} \to F^i/F^{i+1} \otimes_{\mathcal{O}_S} \Omega^1_{S/T}$.
$\psi_{gr^i}: Der(S/T) \to End_S(F^i/F^{i+1})$, $D \mapsto \nabla_{gr^i}(D)^p - \nabla_{gr^i}(D^p) = 0$
2) For any $D_1, \dots, D_n \in Der(S/T)$, $\psi(D_1) \cdots \psi(D_n) = 0$
3) $\exists$ a covering of S by affine opens U w/ coordinates $u_1, \dots, u_r$ & $\Omega^1_{U/T}$ free on $du_1, \dots, du_r$ s.t. $\forall (w_1, \dots, w_r) \in \mathbb{Z}^r$ w/ $\sum_i w_i = n$, $\left(\nabla\left(\frac{\partial}{\partial u_1}\right)\right)^{pw_1} \cdots \left(\nabla\left(\frac{\partial}{\partial u_r}\right)\right)^{pw_r} = 0$

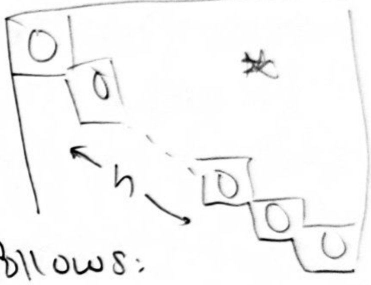
<u>Def</u> $(\mathcal{E}, \nabla)$ is <u>nilpotent of exponent $\leq n$</u> when one of the equivalent conditions of 5.5 holds.

$(\mathcal{E}, \nabla)$ is <u>nilpotent</u> if it is nilp of exp $\leq n$ for some n.

- $\mathrm{Nilp}(S/T)$ = full subcategory of $\mathrm{MIC}(S/T)$ of nilpotent $(\mathcal{E}, \nabla)$
- $\mathrm{Nilp}^n(S/T)$ = " of nilpotent of exp $\leq n$ $(\mathcal{E}, \nabla)$
- $\mathrm{Nilp}^1(S/T)$ " of p-curvature 0 $(\mathcal{E}, \nabla)$

<u>Proof of corollary 5.5</u>

1) $\Rightarrow$ 2) We have a filtration $\mathcal{E} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^n = \{0\}$ whose graded object has p-curvature 0, so upon choosing a basis for $\mathcal{E}$ by extending the basis through each step of the filtration, for any D, $\psi(D)$ acts via



So $\psi(D_1) \cdots \psi(D_n) = 0$ for any $D_1, \ldots, D_n \in \mathrm{Der}(S/T)$.

2) $\Rightarrow$ 1) We construct a filtration of $\mathcal{E}$ as follows:

Let $F^0 = \mathcal{E}$. Let F^1 = largest subobject of F^0 ($F^1 \subsetneq F^0$) which is stable under ∇.

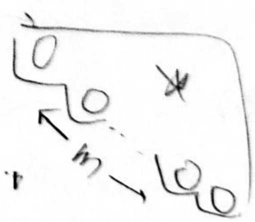
Then F^0/F^1 has no ∇-stable submodules

Then since $\psi(D)$ commutes w/ all $\nabla(D)$ & $\nabla(D)$ on F^0/F^1 has no special form bc,

$\psi(D)$ must act as scalar matrices on F^0/F^1

By 2), $\psi(D)^n = 0$, so $\psi = 0$ on F^0/F^1

continue with F^1. The filtration ends bc $\mathcal{E}$ has finite rank. $\mathcal{E} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^m = \{0\}$. Choose a basis as before to get a matrix of the form.



If $m > n$, we can remove subspaces of $\mathcal{E}$ in the filtration corresponding to where we are forced to have zeros in the matrix due to condition $\psi^n = 0$.

③

2) $\Rightarrow$ 3) Choose a covering of S by open affines U w/ coordinates $u_1, \dots, u_r$ s.t. $\Omega^1_{U/T}$ is free on $du_1, \dots, du_r$

$$\psi\left(\frac{\partial}{\partial u_i}\right) = \nabla\left(\frac{\partial}{\partial u_i}\right)^p - \nabla\left(\frac{\partial}{\partial u_i}^p\right) = \nabla\left(\frac{\partial}{\partial u_i}\right)^p$$

($\frac{\partial}{\partial u_i}^p = 0$ because it involves bringing down a power of p)

Let $(w_1, \dots, w_r) \in \mathbb{Z}^r$ s.t. $\sum_i w_i = n$. Then

$$\nabla\left(\frac{\partial}{\partial u_1}\right)^{pw_1} \cdots \nabla\left(\frac{\partial}{\partial u_r}\right)^{pw_r} = \psi\left(\frac{\partial}{\partial u_1}\right)^{w_1} \cdots \psi\left(\frac{\partial}{\partial u_r}\right)^{w_r} = 0 \text{ by } 2$$

3) $\Rightarrow$ 2) Let $D_1, \dots, D_n \in Der(S/T)$ & restrict to the cover in 3.

Write $D_i = \sum_j a_{ij} \frac{\partial}{\partial u_j}$, so $\psi(D_i) = \sum_j a_{ij} \nabla(\frac{\partial}{\partial u_j})^p$,

$$\psi(D_1) \cdots \psi(D_n) = \sum c \cdot \nabla\left(\frac{\partial}{\partial u_{i_1}}\right)^p \cdots \nabla\left(\frac{\partial}{\partial u_{i_n}}\right)^p = 0. \blacksquare$$

Properties
- $Nilp(S/T)$ is an exact abelian subcategory of $MIC(S/T)$.
- $Nilp^n(S/T)$ stable under taking subobjects & quotients
- $Nilp(S/T)$ stable under taking Hom & $\otimes$ (when defined)

Proposition 5.8 $(\mathcal{E}, \nabla)$ is nilpotent iff whenever $D \in Der(S/T)$ is nilpotent, $\nabla(D)$ is nilpotent.

Proof $\Rightarrow$ Suppose D is nilpotent. We show by induction on $\nu \geq 1$ that if $D^{p^\nu} = 0$, then $\nabla(D)^{p^\nu}$ is nilpotent, and thus $\nabla(D)$ is nilpotent.

Base Case: If $D^p = 0$, then $\psi(D) = \nabla(D)^p - \nabla(D^p) = \nabla(D)^p$ is nilpotent.

Inductive Step: Suppose $D^{p^\nu} = 0$. Then $(D^p)^{p^{\nu-1}} = 0$, so $\nabla(D^p)^{p^{\nu-1}}$ is nilpotent, so $\nabla(D^p)$ is nilpotent.

Then $\psi(D) = \nabla(D)^p - \nabla(D^p)$, so $\nabla(D)^p = \psi(D) + \nabla(D^p)$

Since $\psi(D)$ and $\nabla(D^p)$ are nilpotent & commute, $\nabla(D)^p$ is nilpotent, so $\nabla(D)^{p^\nu}$ & $\nabla(D)$ are nilpotent.

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($\Leftarrow$) Choose a covering of S by open affines U étale / $\mathbb{A}^r_T$ w/ coordinates $u_1, \dots, u_r$ s.t. $\Omega^1_{U/T}$ free on $du_1, \dots, du_r$.

We have each $\left(\frac{\partial}{\partial u_i}\right)^p = 0$ in $\mathrm{Der}(U/T)$, so $\nabla\left(\frac{\partial}{\partial u_i}\right)$ is nilpotent

Let $n_u \in \mathbb{Z}$ s.t. $\nabla\left(\frac{\partial}{\partial u_i}\right)^{p^{n_u}} = 0 \;\; \forall i$ in $\mathrm{End}_T(\mathcal{E})$

Take $n = \sup_{\{u\}} n_u$. Then $(\mathcal{E}, \nabla)$ is nilp of exp $\leq n^2$ (?)

<u>Theorem 5.9</u> Let $f: S \to T$, $f': S' \to T'$ be smooth morphism s.t the diagram commutes, T has char p.

$$\begin{array}{ccc} S' & \xrightarrow{g} & S \\ \downarrow f' & & \downarrow f \\ T' & \xrightarrow{h} & T \end{array}$$

Then under $(g,h)^*: MIC(S/T) \to MIC(S'/T')$,
$\forall n \geq 1$, $(g,h)^*(\mathrm{Nilp}^n(S/T)) \subseteq \mathrm{Nilp}^n(S'/T')$.

<u>Proof</u> by induction. Suppose theorem holds for $\nu = 1, \dots, n-1$.

Let $(\mathcal{E}, \nabla) \in \mathrm{Nilp}^n(S/T)$. Then $\exists$ a filtration

$$\mathcal{E} = F^0 \supseteq F^1 \supseteq \dots \supseteq F^n = \{0\}$$

The subspace $\mathcal{E}' = F^{n-1} \subseteq \mathcal{E}$ is $F^{n-1}/F^n = gr_{n-1}$, so has p-curvature 0.

Also, $\mathcal{E}'' = \mathcal{E}/\mathcal{E}'$ has filtration $\mathcal{E}/\mathcal{E}' = F_0' \supseteq F^1/\mathcal{E}' = F_1' \supseteq \dots \supseteq F_{n-1}' = \{0\}$,

& each graded object $F_i'/F_{i+1}' = F^i/\mathcal{E}' \Big/ F^{i+1}/\mathcal{E}' \simeq F^i/F^{i+1}$

has p-curvature 0, so $(\mathcal{E}/\mathcal{E}', \bar{\nabla}) \in \mathrm{Nilp}^{n-1}(S/T)$.

Thus, we have a s.e.s. in $MIC(S/T)$,

$0 \to (\mathcal{E}', \nabla') \to (\mathcal{E}, \nabla) \to (\mathcal{E}'', \nabla'') \to 0$, with

$(\mathcal{E}', \nabla') \in \mathrm{Nilp}^1(S/T)$ $\quad (\mathcal{E}, \nabla) \in \mathrm{Nilp}^n(S/T)$, $\quad (\mathcal{E}'', \nabla'') \in \mathrm{Nilp}^{n-1}(S/T)$

Pulling back to $MIC(S'/T')$ gives

$(g,h)^*(\mathcal{E}', \nabla') \to (g,h)^*(\mathcal{E}, \nabla) \to (g,h)^*(\mathcal{E}'', \nabla'') \to 0$

By induction, $(g,h)^*(\mathcal{E}', \nabla') \in \mathrm{Nilp}^1(S'/T')$,
$(g,h)^*(\mathcal{E}'', \nabla'') \in \mathrm{Nilp}^{n-1}(S'/T')$.

Thus $(g,h)^*(\mathcal{E}, \nabla) \in \mathrm{Nilp}^n(S'/T')$

4. Show that the orthogonal trajectories of the family of hyperbolas $xy = k$ are the hyperbola given by $x^2 - y^2 = k$.

we can see this as follows:

Write $V_1 \xrightarrow[\alpha]{} V_2 \xrightarrow[\beta]{} V_3 \longrightarrow 0$.

Let $v \in V_2$. Then for $\bar{v} \in V_3$, $\psi^{n-1}(\bar{v}) = \overline{\psi^{n-1}(v)} = 0$, so

$\psi^{n-1}(v) \in \ker(\beta) = \operatorname{im}(\alpha)$.

We have $V_1/_U = \operatorname{im}(\alpha)$ for some U, so since $V_1 \in \mathrm{Nilp}^1$,

$\operatorname{im}(\alpha) \in \mathrm{Nilp}^1$. Thus $\psi(\psi^{n-1}(v)) = \psi^n(v) = 0$, so

$V_2 \in \mathrm{Nilp}^n$.

Now we prove the base case, $(g,h)^*(\mathrm{Nilp}^1(S/T)) \subseteq \mathrm{Nilp}^1(S'/T')$

Case 1 $\quad S' = S \times_T T'$, $g = pr_1$, $f' = pr_2$, $(\mathcal{E}, \nabla) \in \mathrm{Nilp}'(S/T)$.

$$\begin{tikzcd} S' \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & S \arrow[d, "f"] \\ T' \arrow[r, "h"'] & T \end{tikzcd}$$

Suppose S is affine & étale over $\mathbb{A}^r_T$, so

$\Omega^1_{S/T}$ is free on $ds_1, \dots, ds_r$

Thus S' étale over $\mathbb{A}^r_{T'}$ & $\Omega^1_{S'/T'}$ free

on $ds_1', \dots, ds_r'$, $s_i' = g^*(s_i)$

$$\begin{tikzcd} S' \arrow[r] \arrow[d, "\text{étale}"'] \arrow[dr, phantom, "\perp"] & S \arrow[d, "\text{étale}"] \\ \mathbb{A}^r_{T'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & \mathbb{A}^r_T \arrow[d] \\ T' \arrow[r] & T \end{tikzcd}$$

$\nabla(\frac{\partial}{\partial s_i'}) \in \mathrm{End}_{T'}(g^*(\mathcal{E}))$ is induced from

$\nabla(\frac{\partial}{\partial s_i}) \in \mathrm{End}_T(\mathcal{E})$ via extension of scalars

$\mathcal{O}_T \to \mathcal{O}_{T'}$, so $\nabla(\frac{\partial}{\partial s_i})^p = 0 \ \forall i \Rightarrow \nabla(\frac{\partial}{\partial s_i'})^p = 0 \ \forall i$.

Case 2 $\quad T' = T$, $h = id$, $(\mathcal{E}, \nabla) \in \mathrm{Nilp}'(S/T)$.

$$\begin{tikzcd} S' \arrow[r, "g"] \arrow[d, "f \circ g"'] & S \arrow[d, "f"] \\ T \arrow[r, equal, "id"'] & T \end{tikzcd}$$

Recall from Tony's talk:

We have an equivalence of categories:

$\left(\begin{matrix} \text{q.coh} \\ \text{sheaves} \\ \text{on } S^{(p)} \end{matrix}\right) \xrightarrow{\sim} \left(\mathrm{Nilp}^1(S/T) \right.$

$\mathcal{E}^{\nabla} \longleftarrow (\mathcal{E}, \nabla)$

$\mathcal{F} \longrightarrow (F^*(\mathcal{F}), \nabla_{can})$,

where $\mathcal{F} \simeq (F^*\mathcal{F})^{\nabla_{can}}$

$$\begin{tikzcd} S' \arrow[r, "F'"] \arrow[d, "g"'] & S'^{(p)} \arrow[r] \arrow[d, "g'^{(p)}"] & S' \arrow[d, "g"] \\ S \arrow[r, "F"'] & S^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & S \arrow[d, "f"] \\ & T \arrow[r, "F_T"'] & T \end{tikzcd}$$

Thus, $(\mathcal{E}, \nabla) \simeq (F^*(\mathcal{F}), \nabla_{can})$ for some $\mathcal{F} \simeq \mathcal{E}^{\nabla}$,

naturally a q.coh $S^{(p)}$-module. Then

$(g, id)^*(\mathcal{E}, \nabla) = (g, id)^*(F^*(\mathcal{F}), \nabla_{can}) = (g^*F^*(\mathcal{F}), \nabla_{can}) =$

$(F'^*(g'^{(p)*}\mathcal{F}), \nabla_{can})$. Since $g'^{(p)*}\mathcal{F}$ is a q-coh $S'^{(p)}$-module,

this is $\simeq$ some $(\mathcal{E}', \nabla') \in \mathrm{Nilp}^1(S'/T')$

Finally, note that putting case 1 & case 2 together gives us the base case:

$$\begin{tikzcd} S' \arrow[rr, bend left, "g"] \arrow[r, "\phi^S"] \arrow[d, "f'"'] & S\times_T T' \arrow[r, "pr_2"] \arrow[dl, "pr_1"] & S \arrow[d, "f"] \\ T' \arrow[rr, "h"'] & & T \end{tikzcd}$$

$(\mathcal{E},\nabla) \in Nilp^1(S/T)$

$\Rightarrow (\mathcal{E}',\nabla') = (pr_2, h)^*(\mathcal{E},\nabla) \in Nilp^1(S\times_T T'/T')$

$\Rightarrow (\mathcal{E}'',\nabla'') = (\phi, id)^*(\mathcal{E}',\nabla') \in Nilp^1(S'/T')$. ■

Theorem 5.10 Let $\pi: X\to S$, $f: S\to T$ smooth, T has char p. Let $n=$ relative dimension of X/S, suppose S affine, $(\mathcal{E},\nabla)\in Nilp^\nu(X/T_{(\log F)})$. Consider the spectral sequence

$$E_1^{p,q} = C^p(\{U_\alpha\}, \mathcal{H}^q_{dR}(X/S_{(\log F)}, (\mathcal{E},\nabla))) \Rightarrow H^{p+q}_{dR}(X/S_{(\log F)}, (\mathcal{E},\nabla))$$

associated to a covering $\{U_\alpha\}$ of X by open subsets étale / $\mathbb{A}^n_S$.

1) $E_r^{p,q} \in Nilp^\nu(S/T)$ $\forall p,q,r$
2) $\forall i\geq 0$ set $\tau(i) = \#\{p\in\mathbb{Z} : E_\infty^{p,i-p}\neq 0\}$
   Then $H^i_{dR}(X/S_{(\log F)}, (\mathcal{E},\nabla)) \in Nilp^{\nu\cdot\tau(i)}(S/T)$
3) $\tau(i)\leq i+1$, $\tau(i)\leq 2n-i+1$.

Proof How does the spectral sequence work?

• $C^p(\{U_\alpha\}, \mathcal{H}^q_{dR}(X/S,(\mathcal{E},\nabla))) = \prod_{i_0<\cdots<i_p} H^q_{dR}(U_{i_0\cdots i_p}/S, (\mathcal{E},\nabla)|_{U_{i_0\cdots i_p}})$

E_1: to get E_2 page, we take cohomology along vertical arrows

$$\begin{tikzcd} \vdots & q & \\ \prod_{\alpha,\beta} H^0_{dR}(U_{\alpha\beta}/S) \arrow[u] \arrow[r, "\nabla"'] & \prod_{\alpha,\beta} H^1_{dR}(U_{\alpha\beta}/S) \arrow[u] \arrow[r, "\nabla"'] & \cdots \\ \prod_\alpha H^0_{dR}(U_\alpha/S) \arrow[u, "\partial"] \arrow[r, "\nabla"'] & \prod_\alpha H^1_{dR}(U_\alpha/S) \arrow[u] \arrow[r, "\nabla"'] & \prod_\alpha H^2_{dR}(U_\alpha/S) \arrow[r] & {} \end{tikzcd}$$

4. Show that the orthogonal trajectories of the family of hyperbolas $xy=k$ are the hyperbola given by $x^2-y^2=k$.

Proof of 1) Since each $E_r^{p,q}$ is given by a quotient of a subobject of $E_{r-1}^{p,q}$, $E_{r-1}^{p,q} \in \mathrm{Nilp}^\vee(S/T)$ $\forall p,q$

$\Rightarrow E_r^{p,q} \in \mathrm{Nilp}^\vee(S/T)$ $\forall p,q$, $\forall r$.

So we show $E_1^{p,q} = \prod_{i_0 < \cdots < i_p} H^q_{dR}(U_{i_0 \cdots i_p}/S, (\mathcal{E},\nabla)|_{U_{i_0\cdots i_p}}) \in \mathrm{Nilp}^\vee(S/T)$

Since each U_α étale $/\mathbb{A}^n_S$, $\cap U_\alpha$ étale $/\mathbb{A}^n_S$.

We show for $(\mathcal{E},\nabla) \in \mathrm{Nilp}^\vee(S/T)$, X étale $/\mathbb{A}^n_S$, $H^i_{dR}(X/S, (\mathcal{E},\nabla)) \in \mathrm{Nilp}^\vee(S/T)$.

Pf Let $\Omega^1_{X/S}$ be free on $dx_1, \dots, dx_n$, & for each $D \in \mathrm{Der}(S/T)$, let $D_0 \in \mathrm{Der}(X/T)$ be the unique extension which kills $dx_1, \dots, dx_n$

Then $\nabla_{GM} : \mathrm{Der}(S/T) \to \mathrm{End}_T(\Omega^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$

$$\nabla_{GM}(D)((dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \otimes e) = (dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \otimes \nabla(D_0)(e)$$

Then $\nabla_{GM}(D)^p((dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \otimes e) = (dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \otimes \nabla(D_0)^p(e)$

Now, given $D_1, \dots, D_n \in \mathrm{Der}(S/T)$, take $(D_1)_0, \dots, (D_n)_0 \in \mathrm{Der}(X/T)$,

$$\psi(D_1) \cdots \psi(D_n) = \nabla_{GM}(D_1)^p \cdots \nabla_{GM}(D_n)^p$$

$$= (dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \otimes \nabla((D_1)_0)^p \cdots \nabla((D_n)_0)^p(e)$$

$$= (dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \otimes 0$$

$$= 0$$ ✓

$E_1^{p,q} \Rightarrow H^{p+q}_{dR}(X/S, (\mathcal{E}, \nabla))$ means that

$E_\infty^{p,q} = gr^p_{tor} H^{p+q}_{dR}(X/S, (\mathcal{E}, \nabla))$, and

$\exists$ a p-filtration on $E_\infty^{p,q}$

$$H^i_{dR}(X/S, (\mathcal{E}, \nabla)) = E_\infty^{0,i} \oplus E_\infty^{1,i-1} \oplus \cdots \oplus E_\infty^{i,0}$$

$$E_\infty^{0,i} = gr^0 H^i = F^0/F^1$$
$$E_\infty^{1,i-1} = gr^1 H^i = F^1/F^2$$
$$E_\infty^{2,i-2} = gr^2 H^i = F^2/F^3$$
$$\vdots$$
$$E_\infty^{i,0} = gr^i H^i = F^i/0 = E_\infty^{i,0}$$

filtration: $F^0 = H^i \supseteq F^1 = E_\infty^{1,i-1} \oplus \cdots \oplus E_\infty^{i,0}$

$\supseteq F^2 = E_\infty^{2,i-2} \oplus \cdots \oplus E_\infty^{i,0}$

$\supseteq \cdots \supseteq F^i = E_\infty^{i,0}$

Then we have a filtration on $H^i_{dR}(X/S, (\mathcal{E}, \nabla))$ of length $T(i)$, with each component in $\mathrm{Nilp}^u(S/T)$.

Clearly, $T(i) \le i+1$. Since $E_\infty^{p,q} = 0$ whenever $p > n$,

$T(i) \le 2n - i + 1$

To show $E_\infty^{p,q} = 0$ for all $p > n$, it suffices to show $E_2^{p,q} = 0$ for all $p > n$ (omitted).

Now add the logarithmic singularities

Let $\pi : X \to S$ smooth, $i : Y \hookrightarrow X$ the inclusion of a smooth divisor with normal crossings relative to S, $f : S \to T$ smooth, T with char p.

$$\mathrm{Nilp}^{\nu}(X/T(\log Y)) = \text{full sub cat of } MIC(X/T(\log Y))$$
nilpotent of exp $\leq \nu$

p-curvature of $(\mathcal{E}, \nabla)$: $\quad \psi : \mathrm{Der}_T(X/T) \to \mathrm{End}_T(\mathcal{E})$

$$\psi(D) = (\nabla(D))^p - \nabla(D^p)$$

Discussion 23: Separable Equations 3

4. Show that the orthogonal trajectories of the family of hyperbolas $xy = k$ are the hyperbola given by $x^2 - y^2 = k$.