Connections in char p, I

So far, mostly looked at connections ($\Leftrightarrow$ ODEs) in char 0.
Now, study connections in char p.

Ex $\quad \frac{df}{dx} = 0, \qquad f \in k[x]$

In char 0, solutions are constant $\in k$.
In char p, have many more solutions, since $\frac{d}{dx}(x^p) = 0$.
$\longrightarrow$ sdns $=$ {polynomials in x^p}

I Review of curvature

Recall that a connection $\nabla : \mathcal{E} \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$
has curvature zero if

$$\underbrace{\mathcal{E} \xrightarrow{\nabla} \Omega^1_{X/S} \otimes \mathcal{E} \xrightarrow{\nabla} \Omega^2 \otimes \mathcal{E}}_{=0}$$

(The curvature is $\nabla \circ \nabla \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \Omega^2 \otimes \mathcal{E})$.)

Equivalently, $\nabla \Leftrightarrow \mathrm{Der}(X/S) \longrightarrow \mathrm{End}_S(\mathcal{E})$

$$D \longmapsto \nabla_D$$

What is ∇_D?

$$D \in \Gamma(X, T_{X/S}) = \mathrm{Hom}(\Omega^1_{X/S}, \mathcal{O}_X)$$

$$\mathcal{E} \xrightarrow{\nabla} \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{D} \mathcal{E}$$

Recall that $T_{X/S}$ has a Lie bracket,

since $[D_1, D_2]$ is a derivation.

Curvature $0 \Leftrightarrow \underbrace{\nabla_{[D_1,D_2]}}_{\text{Lie bracket of derivations}} = \underbrace{[\nabla_{D_1}, \nabla_{D_2}]}_{\text{Lie bracket (commutator) in } \mathrm{End}_{\mathcal{O}_S}(\mathcal{E})}$

$\Leftrightarrow D \mapsto \nabla_D$ is a Lie alg. homomorphism.

II. p-(restricted) Lie algebras

Lie algebra $\mathfrak{g}$ / field k

= vec space $/k$ + Lie bracket, ...

modeled on structure of

$$\mathrm{Lie}(G) = T_e \underset{\text{algebraic group }/k}{G}$$

$$\begin{tikzcd}
G \arrow[dr, dashed, "F_{G/k}"'] \arrow[drr, bend left, "Fr_G"] \arrow[ddr, bend right] & & \\
& G^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & G \arrow[d] \\
& \operatorname{Spec} k \arrow[r, "Fr_k"'] & \operatorname{Spec} k
\end{tikzcd}$$

If $G = \operatorname{Spec} k[\{x_i\}]/\{f_j\}$

then $G^{(p)} = \operatorname{Spec} k[\{x_i\}]/\{f_j^p\}$

$F_{G/k}(x_i) = x_i^p$.

If $k = \mathbb{F}_p$, then $G^{(p)} = G$, $F_{G/k}$ is an endomorphism of G

Ex $G = GL_n$, $F(a_{ij}) = a_{ij}^p$.

$F \curvearrowright G$ induces extra structure on $\mathfrak{g} = \mathrm{Lie}(G)$

$X \mapsto X^{[p]}$

Ex $\mathfrak{g} = \mathfrak{gl}_n$, $(x_{ij})_{ij}^{[p]} = (x_{ij}^p)$.

This structure is axiomatized under the name p-(restricted) Lie algebra.

Ex A = assoc. alg. $/\mathbb{F}_p$. Then $a^{[p]} := a^p$ defines p-restricted Lie alg structure on A.

$$(a+b)^p = a^p + b^p + \sum_i \underbrace{s_i(a,b)}_{\text{universal expressions in Lie bracket}}$$

eg, $p=2$, $(a+b)^2 = a^2 + b^2 + \underbrace{ab+ba}_{[a,b]}$

$p=3$ $(a+b)^3 = a^3 + b^3 + a^2b + aba + ba^2 + ab^2 + bab + b^2a$

$= a^3 + b^3 + [a,[a,b]] + [b,[b,a]]$

$a^2b - aba - aba + ba^2$

Axioms

- $(\lambda x)^{[p]} = \lambda^p x^{[p]}$
- $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum s_i(x,y)$
- $\mathrm{ad}(x^{[p]}) = (\mathrm{ad}\, x)^p$

III p-curvature

<u>Miracle</u>: if $D \in \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$ in char p, then

$$D^p \in \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$$

$$D^p(fg) = D^p(f)\, g + f\, D^p(g) + \sum_i \binom{p}{i} (D^i f)(D^{p-i} g)$$

$\Rightarrow$ p-Lie alg structure on $T_{X/S}$.

<u>Defn</u> ∇ has <u>p-curvature zero</u> if

$D \mapsto \nabla_D$ is p-Lie alg. homomorphism

$$T_{X/S} \to \mathrm{End}_{\mathcal{O}_S}(\mathcal{E})$$

i.e. if ∇ flat and $\boxed{\nabla_{D^p} = (\nabla_D)^p}$

Suppose $(\mathcal{E}, \nabla) =$ module with flat connection (MIC)

Then define $\psi(D) = \nabla_{D^{[p]}} - (\nabla_D)^p \in \mathrm{End}_S(\mathcal{E})$.

This defines the action of $\mathrm{Sym}^{\bullet}_X(T_{X/S})$ on $\mathcal{E}$.

The map $D \mapsto \psi(D)$ is called the

p-curvature.

Ex $X = \mathbb{G}_m = \mathrm{Spec}\, \mathbb{F}_p[t^{\pm}]$

$$\downarrow$$

$S = \mathrm{Spec}\, \mathbb{F}_p$

$\mathcal{E} = \mathcal{O}_X$, $\nabla = d + \underset{\substack{\cap \\ \mathbb{F}_p}}{A} \frac{dt}{t}$

Let $D = t\partial_t \Rightarrow D^p = D$

$$\nabla_D(f) = \langle D, df + A\,dt \rangle = \langle t\partial_t, df + A\frac{dt}{t} f\rangle$$

$$= t\frac{\partial f}{\partial t} + Af$$

$$\Rightarrow \psi(D) = \nabla_{D^{[p]}} - (\nabla_D)^p = \nabla_D - (\nabla_D)^p$$

Let's apply to $f = 1$ (constant)

$\Rightarrow \psi(D)\, 1 = A - A^p$ vanishes $\Leftrightarrow A \in \mathbb{F}_p$.

<u>p-linearity</u>: $\psi(D_1 + D_2) = \psi(D_1) + \psi(D_2)$

$$\psi(gD) = g^p \psi(D)$$

<u>Thm</u> $\psi(D) \in \mathrm{End}_S(\mathcal{E})^{\nabla=0} \subset \mathrm{End}_S(\mathcal{E})$

"p-curvature is valued in flat endomorphisms"

i.e. $\psi(D)$ commutes with $\nabla_{D'}$ $\forall D'$!

<u>Pf</u> May check locally $\Rightarrow$ assume S affine,

$$X \xrightarrow{\text{ét}} \mathbb{A}^n \longrightarrow S$$

$$\Rightarrow \Omega^1_{X/S} = \mathcal{O}_X \langle dx_1, \dots, dx_n \rangle$$

$$T_{X/S} = \mathcal{O}_X \langle \partial_{x_1}, \dots, \partial_{x_n} \rangle$$

Suppose $D = \sum_i a_i \frac{\partial}{\partial x_i} \Rightarrow \psi(D) = \sum_i a_i^p \, \psi\left(\frac{\partial}{\partial x_i}\right)$

$D' = \sum_i b_i \frac{\partial}{\partial x_i}$

$\Rightarrow [D, D'] = \sum_{i,j} a_i^p b_j \left[\psi\left(\frac{\partial}{\partial x_i}\right), \nabla_{\partial/\partial x_j}\right]$

$$= \sum_{i,j} a_i^p b_j^p \cancel{[(\nabla_{\partial/\partial x_i})^p, \nabla_{\partial/\partial x_j}]}$$

0

Since $\frac{\partial}{\partial x_i}$ commutes with $\frac{\partial}{\partial x_j}$ and ∇ is flat,

$$[\nabla_{\partial/\partial x_i}, \nabla_{\partial/\partial x_j}] = 0 \qquad \square$$

IV D-modules

Sheaf of differential operators on smooth X

is $D_X :=$ $\mathcal{O}_X$-algebra gen by vector fields

$$= \mathcal{O}_X \langle T_X \rangle \Big/ \begin{array}{l} D_1 D_2 - D_2 D_1 = [D_1, D_2] \\ Df - fD = D(f) \end{array}$$

D-module = sheaf of D_X-modules.

A module + flat connection $(\mathcal{E}, \nabla) \rightsquigarrow$ D-module with underlying $\mathcal{O}_X$-module $\mathcal{E}$

$$D \cdot v = \nabla_D (v)$$

$$D_1 D_2 v = \nabla_{D_1}(\nabla_{D_2}(v)) \quad \text{etc.}$$

Have a relative version for smooth $\pi: X \to S$

$D_{X/S}$, etc.

Analogy

Lie alg $\mathfrak{g}$

Vector fields $T_{X/S}$

Universal enveloping $U\mathfrak{g}$

Differential operators $D_{X/S}$

Since $\mathrm{ad}(x^{[p]}) = \mathrm{ad}(x)^p$

$x^{[p]} - x^p \in Z(U(\mathfrak{g}))$

This defines a homomorphism

$\mathrm{Sym}\,\mathfrak{g}^{(p)} \to Z(U(\mathfrak{g}))$

whose image is called the p-center.

Consider $T_{X/S} \to D_{X/S}$

$D \longmapsto D^{[p]} - \underbrace{D D \cdots}_{p \text{ times}}$

This defines a map

$\mathrm{Sym}^{\cdot}_{X}(T_{X/S}) \xrightarrow{Z} D_{X/S}$.

$\Rightarrow$ equips any $D_{X/S}$-module $\mathcal{E}$ with the structure of a quasicoherent sheaf $v(\mathcal{E})$ on

$\mathrm{Spec}_X\, \mathrm{Sym}^{\cdot}_X(T_{X/S}) = T^*_{X/S}$.

Fact The map $\mathrm{Sym}^\bullet_X(T_{X/S}) \xrightarrow{z} D_{X/S}$ is p-linear / $\mathcal{O}_X$

meaning (i) $z(D_1+D_2) = z(D_1)+z(D_2)$

(ii) $z(gD) = g^p z(D)$

$\Rightarrow$ Can be regarded as an alg homomorphism

$$\mathrm{Sym}^\bullet_{X^{(p)}} T_{X^{(p)}/S} \xrightarrow{z^{(p)}} (F_{X/S})_* D_{X/S} \qquad X \xrightarrow{F_{X/S}} X^{(p)}$$

Why (i)?

$\to$ $z(D_1+D_2) = (D_1+D_2)^{[p]} - (D_1+D_2)^p$

$= D_1^{[p]} + D_2^{[p]} + \sum s_i[D_1,D_2] - D_1^p - D_2^p - \sum s_i(D_1,D_2)$

Why (ii)?

$\to$ More complicated.

Thm [BMR?]

$$\mathrm{Sym}^\bullet_{X^{(p)}} T_{X^{(p)}/S} \xrightarrow{z^{(p)}} (F_{X/S})_* D_{X/S}$$

defines an isomorphism onto the image of $(F_{X/S})_* D_{X/S}$, and $(F_{X/S})_* D_{X/S}$ is an Azumaya algebra over it, of rank $(\dim X)^2$

Ex $X = \mathbb{A}^1_{\mathbb{F}_p}$, $D_{X/S} = \mathbb{F}_p[t, \partial_t]$

$$\mathrm{Sym}^\bullet_{X^{(1)}} T_{X^{(1)}/S} = \mathbb{F}_p[u, \xi] \longrightarrow D_{X/S}$$

$$u \longmapsto t^p$$

$$\xi \longmapsto \partial_t^p$$

Recall: an Azumaya algebra $\mathcal{A}$ = a quasicoherent sheaf of algebras locally isomorphic to $\mathrm{End}(\mathcal{E})$

$\mathrm{QCoh}(X, \mathcal{A})$ = category of quasicoh $\mathcal{A}$-modules

A splitting $\mathcal{A} \simeq \mathrm{End}(\mathcal{E})$ induces

$$\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X, \mathcal{A})$$

$$\mathcal{F} \longmapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}$$

Ex central simple k-algebra D,

A splitting is $D \simeq \mathrm{Mat}_n(k)$.

V Cartier descent

$$\begin{array}{c} \mathcal{O}_X \\ \uparrow Fr^*_{X/S} \end{array}$$

Recall that $d(f^p)=0$

$\Rightarrow$ for $(\mathcal{E},\nabla)$ on X, $\mathcal{E}^\nabla$ is a module over $\mathcal{O}_{X^{(p)}}$

Conversely, given a $\mathcal{E}' \in QCoh(X^{(p)})$,

$Fr^*\mathcal{E}' = \mathcal{E}' \otimes_{\mathcal{O}_{X^{(p)}}} \mathcal{O}_X$ has a canonical connection

$\nabla = 0 \otimes d$

Thm This induces equivalence of categories

$$MIC(X, \psi=0) \xrightarrow{\sim} QCoh(X^{(p)})$$

Sketch 1. Explicit formulas. Suppose $\dim X = 1$.

$$(\mathcal{E}^{\nabla=0}) \otimes_{\mathcal{O}_{X^{(p)}}} \mathcal{O}_X \overset{?}{\xrightarrow{\sim}} \mathcal{E}.$$

Need $\mathcal{E}$ with $\Psi = 0$ to have "enough" flat sections. Try $P : \mathcal{E} \to \mathcal{E}^{\nabla}$

$$P(f) = f - t\nabla_{\partial_t}(f) + \frac{t^2}{2!}\nabla_{\partial_t}^2(f) \cdots$$

(extract constant term)

$\Psi = 0$ used to see that $\nabla_{\partial_t}^p(f) = 0$.

Then write (Taylor expansion)

$$f = P(f) + tP(\nabla_{\partial_t} f) + t^2 P(\nabla_{\partial_t}^2 f) + \cdots$$

again a finite sum by $\Psi = 0$.

<u>Pf using microlocal geometry</u>

$\mathcal{E} \rightsquigarrow \nu(\mathcal{E}) \in QCoh(T^*X^{(p)})$

The action of $\Psi(D)$ defines the

action of $\mathrm{Sym}^{\bullet}_{X^{(p)}} T^{*}X^{(p)}$.

$\psi = 0 \Rightarrow \nu(\mathcal{E})$ supported on the 0-section

$$\begin{tikzcd} D_{X/S} \circlearrowright \mathcal{E} \arrow[d] & F_{X/S*}\mathcal{E} \arrow[d] & \nu(\mathcal{E}) \circlearrowleft (F_{X/S})_* D_{X/S} \arrow[d] \\ X \arrow[r, "F_{X/S}" ', "\text{affine}"] & X^{(p)} \arrow[r, hook, "0\text{-section}" '] & T^{*}X^{(p)} \end{tikzcd}$$

$\mathcal{E} \circlearrowleft D_{X/S}$ structure is captured by $F_{X/S*}\mathcal{E} \circlearrowleft F_{X/S*}D_{X/S}$

Take $\mathcal{E}_0 = (\mathcal{O}_X, d)$

$\Rightarrow \nu(\mathcal{E}_0) \circlearrowleft F_{X/S*} D_{X/S}$

rank = (dim X)

provides splitting of $(F_{X/S})_* D_{X/S} \Big|_{X^{(p)}}$

$= \mathrm{End}(\nu(\mathcal{E}_0))$

$(F_{X/S})_* \mathcal{E}$

Hence $\nu(\mathcal{E}) \overset{\shortparallel}{=} \nu(\mathcal{E}_0) \underset{\mathcal{O}_{X^{(p)}}}{\otimes} W$

$$= \left((F_{X/S})_* \mathcal{O}_X \right) \underset{\mathcal{O}_{X^{(p)}}}{\otimes} W$$

projection formula $= (F_{X/S})_* \left(F_{X/S}^* (W) \right)$

$\Rightarrow \quad \mathcal{E} \simeq F_{X/S}^* (W).$