[NT, Gauss–Manin]

1. The generalisation ~~of~~ of PDE's like $\frac{d}{dz}f = \frac{\alpha}{z}f$, $\frac{d^2}{dz^2}f + a(z)\frac{d}{dz}f + bf = 0$ is "q.coh. sheaf with a connection".

"Def". A linear homogeneous PDE on a smooth scheme X over a field k is the data of $(\mathcal{E}, \nabla)$:
- $\mathcal{E}$ … a q.coh. sheaf
- ∇ … a connection.

1.1 Recall:
A connection ∇ on $\mathcal{E}$ is a k-linear map $\nabla : \mathcal{E} \to \Omega^1_{X/k} \otimes \mathcal{E}$, satisfying $\nabla(fe) = df \otimes e + f \cdot \nabla e$. Equivalently, it's (abuse notation) $\nabla : \mathrm{Der}_k(X) \to \mathrm{End}_k(\mathcal{E})$. $v \mapsto \nabla_v e := \langle v, \nabla e \rangle$.
Note this is $\mathcal{O}_X$-linear.
∇ induces $\Omega^i_{X/k} \otimes \mathcal{E} \to \Omega^{i+1}_{X/k} \otimes \mathcal{E}$: $w^i \otimes e \mapsto (dw^i) \otimes e + (-1)^i w^i \wedge \nabla e$
∇ is flat if $\nabla^2 : \mathcal{E} \to \Omega^2_{X/k} \otimes \mathcal{E}$ is 0. This implies $\Omega^\bullet_{X/k} \otimes \mathcal{E}$ is a complex. (de Rham complex of $(\mathcal{E}, \nabla)$).

1.2 · If X is a curve, local coordinate z, $\mathcal{E}$ local $\mathcal{O}_X$-basis $\{e_1, \cdots, e_n\}$, then ∇ can be expressed as: $e_i \mapsto w_i{}^j e_j$, $w_i{}^j = \cancel{\Gamma^j_{ik}\, dz}\ \alpha_i{}^j dz$, Equation for horizontal sections: $\nabla(f^i e_i) = 0$:
$df^i \otimes e_i + f^i w_i{}^j e_j \rightsquigarrow \cancel{df^i + \Gamma^i_j}\ \frac{df^i}{dz} + \alpha_j{}^i f^j = 0$ for each i.

例 $X = \mathbb{C}^\times$, $(\mathcal{E}, \nabla) = (\mathcal{O}_X, d - g \cdot dz)$, $g \in \Gamma(X, \mathcal{O}_X)$. Then the corresp. PDE is: $\frac{df}{dz} = gf$.

or just trivial rank n bundle

例 $X =$ curve, local coord. z, $(\mathcal{E}, \nabla) = \left(\mathrm{Jet}_{n-1}(\mathcal{O}_X), (\omega_i{}^j) = \begin{pmatrix} 0 & -1 & & 0 \\ 0 & \ddots & \ddots & \\ & & 0 & -1 \\ a_n(z) & \cdots & & a_1(z) \end{pmatrix} dz\right)$

choose basis s.t. $\mathcal{O}_X \to J_{n-1}(\mathcal{O}_X)$
$f \mapsto (f, \frac{d}{dz}f, \cdots, \frac{d^{n-1}}{dz^{n-1}}f)$.

Then $\nabla_{\frac{d}{dz}} e = 0$
$(f_1, \cdots, f_n)$

②

Becomes $\begin{cases} \frac{df_1}{dz} = f_2 \\ \vdots \\ \frac{df_{n-1}}{dz} = f_n \\ \frac{df_n}{dz} + a_1 f_{n-1} + \cdots + a_n f_1 = 0 \end{cases}$ $\Longleftrightarrow \frac{d^n f_1}{dz^n} + a_1 \frac{d^{n-1} f_1}{dz^{n-1}} + \cdots + a_n f_1 = 0.$

2. In view of Riemann-Hilbert, we want the solutions to our PDE (or, horizontal sections of $(\mathcal{E}, \nabla)$) to be "locally constant". In other words, locally, solutions don't depend on "path of integration". This means we should restrict to flat connections.

[Recall : curvature $R := \nabla^2 : \mathcal{E} \to \Omega^2_{X/k} \otimes \mathcal{E}$.
flat $:= R = 0 \Leftrightarrow [\nabla_{v_1}, \nabla_{v_2}] = \nabla_{[v_1, v_2]}$ (i.e. "discrepancy only results from 'path itself'".)

For flat $(\mathcal{E}, \nabla)$, $\Omega^\bullet_{X/k} \otimes \mathcal{E}$ is a complex.
~~Def~~. $H^i_{dR}(X/k, \underbrace{(\mathcal{E}, \nabla)}_{\text{flat}}) := H^i(\Gamma(X, \Omega^\bullet_{X/k} \otimes \mathcal{E}))$.

These coho. groups contain information about X, when probed by $(\mathcal{E}, \nabla)$, on its corresponding PDE.

~~Ex~~ $X = \{ y^2 = x(x-1)(x-\lambda) \} \subseteq \mathbb{A}^2$, $k = \mathbb{C}$, $\lambda \neq 0, 1$.
$(\mathcal{E}, \nabla) = (\mathcal{O}_X, d)$.

We know $\Omega^{\bullet, an}_{X/k} \otimes \mathcal{E} \cong \Omega^{\bullet, an}_{X/k}$ is an injective resolution of $\mathbb{C}$.

So $H^{i, an}_{dR} \cong H^i_{sing}(X, \mathbb{C}) = \begin{cases} \mathbb{C}, & i = 0 \\ \mathbb{C} \oplus \mathbb{C}, & i = 1 \\ 0, & \text{otherwise} \end{cases}$ ← Let's verify these and find explicit basis.

We also know $H^i_{dR} \cong H^{i, an}_{dR}$ [Grothendieck, 1965]

$\Omega^\bullet_{X/\mathbb{C}} : \quad 0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathbb{C}} \to \cancel{\Omega^2_{X/\mathbb{C}}} \xrightarrow{0} 0$

$H^0_{dR} =$ constant functions $\simeq \mathbb{C}$
$H^1_{dR} = \operatorname{coker}(\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathbb{C}})$.

Let $R = \mathbb{C}[x, y] / y^2 - x(x-1)(x-\lambda)$. Then $\Omega^1 = \overset{R\text{-module}}{R[dx, dy]} / 2y\,dy = P_x\,dx$
$P(x) = x(x-1)(x-\lambda)$

Consider the following element in Ω^1:

$\omega = Ay\,dx + 2B\,dy$, where $A, B \in \mathbb{C}[x]$ s.t. $AP + BP_x = 1$.

<u>Claim</u>: $\{\omega, x\omega\}$ is a $\mathbb{C}$-basis for $H^1_{dR} \simeq \mathbb{C}^2$.

Sketch: note: $\begin{cases} y\omega = dx \\ dy = \frac{P_x}{2}\omega \end{cases}$. So ~~any~~ all element in Ω^1 can be written as $(C + Dy)\omega$, $C, D \in \mathbb{C}[x,y]$. All y^2 terms can be absorbed, so may ~~assume~~ take $C, D \in \mathbb{C}[x]$, then this representation is unique.

exercise: $Dy\omega$ is exact, $C\omega$ can be uniquely written as exact part $+ a\omega + bx\omega$, $a, b \in \mathbb{C}$.

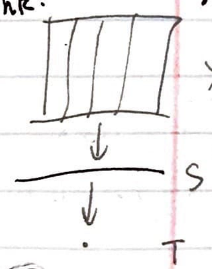
3. Continue with general theory. From now on, work with general base: $\begin{array}{c} X \\ \downarrow \\ S \end{array}$ smooth map of schemes.

To such a pair associate $MC(X/S)$: category of q.coh. modules on X with connections. morphisms are $\varphi: (\mathcal{E}_1, \nabla_1) \to (\mathcal{E}_2, \nabla_2)$, $\forall v \in \mathrm{Der}(X/S)$, $\varphi(\nabla_{1,v} e_1) = \nabla_{2,v}(\varphi(e_1))$.

- $MC(X/S)$ has $\otimes$, partial internal Hom, is abelian. It has full subcat. $MIC(X/S)$, $\otimes$, Hom, abelian.

- Given $\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$ $(\mathcal{E},\nabla)$, can pull back $(\mathcal{E},\nabla)$.

think:



- Given $\begin{tikzcd} X \arrow[d, "\pi"] \\ S \arrow[d] \\ T \end{tikzcd}$ $(\mathcal{E}, \nabla_{/T})$, can forget: $(\mathcal{E}, \nabla_{/T}) \mapsto (\mathcal{E}, \nabla_{/S})$

$$\begin{tikzcd} \nabla_{/S}: \mathcal{E} \arrow[r] \arrow[dr, "\nabla_{/T}"'] & \Omega^1_{X/S} \otimes \mathcal{E} \\ & \Omega^1_{X/T} \otimes \mathcal{E} \arrow[u] \end{tikzcd}$$

Recall: $\pi^*\Omega_{S/T} \to \Omega^1_{X/T} \to \Omega^1_{X/S} \to 0$ exact.

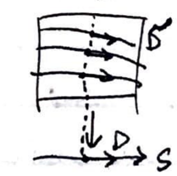
smooth $\Rightarrow$ $0 \to \underbrace{\pi^*\Omega_{S/T}}_{=\text{"horizontal"}} \to \Omega^1_{X/T} \to \underbrace{\Omega^1_{X/S}}_{=\text{"vertical"}} \to 0$ exact.

$$\begin{array}{l} X \; (\mathcal{E}, \nabla_{/S}) \cdots \text{flat} \\ \downarrow \pi \\ S \end{array} \quad \rightsquigarrow \quad \Omega^\bullet_{X/S} \otimes \mathcal{E}, \quad R^i\pi_*(\Omega^\bullet_{X/S} \otimes \mathcal{E}) =: H^i_{dR}(X/S, (\mathcal{E}, \nabla_{/S})) \quad (\text{a q.coh. sheaf on } S)$$

If π is "good", then the fibre of H^i_{dR} ~~computes~~ is the dR cohomology of $(\mathcal{E}|_{X_s}, \nabla|_{X_s})$. In general, H^i_{dR} contains information about this S-family.

4. Consider the following situation: $\begin{array}{l} X \; (\mathcal{E}, \nabla) \\ \downarrow \pi \\ S \\ \downarrow \\ T \end{array}$ if $(\mathcal{E}, \nabla)$ is relative to S, then can form $H^i_{dR}(X/S, (\mathcal{E}, \nabla_{/S}))$. But in general, we want to probe X using more than just $(\mathcal{E}, \nabla)$ relative to S, but also $(\mathcal{E}, \nabla)$ relative to T (i.e. "completely general"). Of course we can restrict: $(\mathcal{E}, \nabla_{/T}) \rightsquigarrow (\mathcal{E}, \nabla_{/S})$ and form $H^i_{dR}(X/S, (\mathcal{E}, \nabla_{/S}))$, but then horizontal information is lost.

e.g. $\begin{array}{l} X = \text{affine surface} \\ \downarrow \\ S = \mathbb{A}^1 = \operatorname{Spec} k[s] \\ \downarrow \\ T = \operatorname{Spec} k \end{array}$ choose coordinate (étale map $X \to \mathbb{A}^2$, $\searrow \mathbb{A}^1 \swarrow$) x



Then $D = \frac{d}{ds}$ can be (uniquely) lifted to X, s.t. $\frac{dx}{ds} = 0$, denote as $\tilde{D}$. If $(\mathcal{E}, \nabla_{/T})$ on X, then $\nabla_{\tilde{D}} e_i = \langle \tilde{D}, \alpha^j dx \otimes e_j + \beta^j ds \otimes e_j \rangle$.
basis $\{e_1, \ldots, e_n\}$
$= \beta^j e_j$

while $\nabla_{/S, \tilde{D}} e_i = \langle \tilde{D}, \alpha^j dx \otimes e_j \rangle = 0$.

However, this suggests: look at $\nabla : \mathcal{E} \to \Omega^1_{X/T} \otimes \mathcal{E}$,
in coordinate (x, s), $\{e_1, \ldots, e_n\}$, $e_i \longmapsto \alpha^j dx \otimes e_j + \beta^j ds \otimes e_j$
in coordinate (y, s), $x = f(y, s)$, $e_i \mapsto \alpha^j f_y \, dy \otimes e_j + (\alpha^j f_s + \beta^j) ds \otimes e_j$
So if e_i's are horizontal sections, i.e. $\nabla_{/S} e_i = 0$, then $\alpha^j f_s = 0$, then $\beta^j ds \otimes e_j$ is independent of choice of coordinate.
In other words, there is a canonical map:

$H^0_{dR}(X/S, (\mathcal{E}, \nabla_{/S})) \to \Omega^1_{S/T} \otimes H^0_{dR}(X/S, (\mathcal{E}, \nabla_{/S}))$, which you can verify is a flat connection.

We exepect similar argument applies to H^i_{dR}.

i.e. given $(\mathcal{E}, \nabla_{/T})$, form $H^i_{dR}(X/S, (\mathcal{E}, \nabla_{/S}))$, then the lost horizontal information is (partially?) restored as a canonical connection on H^i_{dR}.

例 $X = \{y^2 = x(x-1)(x-s)\} \subseteq \mathbb{A}^2_S$, $(\mathcal{E}, \nabla_{/T}) = (\mathcal{O}_X, d)$.

$$\begin{array}{l} X \\ \downarrow \\ S = \mathbb{C}[s, \frac{1}{s(1-s)}] \\ \downarrow \\ T = \operatorname{Spec} \mathbb{C} \end{array}$$

As in the example in §2 (where s is fixed), one finds $\{\omega = Ay\,dx + 2B\,dy,\ x\omega\}$ form an $\underline{\underline{\mathcal{O}_S}}$-basis for H^1_{dR}. Note A, B, P depend on s.

Compute: $\begin{cases} d\omega = A\,dy \wedge dx + A_s\,y\,ds \wedge dx + 2B_x\,dx \wedge dy + 2B_s\,ds \wedge dy \\ d(x\omega) = \quad \cdots \quad \text{(we do computation for } \omega\text{, } x\omega \text{ is similar)} \end{cases}$

Note dx, dy, ds are not independent, have relation:

$$2y\,dy = P_x\,dx + P_s\,ds.$$

Using this we can rewrite ~~dx~~ $d\omega$ purely in terms of dx, ds (which form a basis of $\Omega^1_{X/T}$):

$$d\omega = \left(\frac{A \cdot P_s}{2y} + A_s \cdot y + \frac{B_s \cdot P_x}{y} - \frac{B_x \cdot P_s}{y}\right) ds \wedge dx.$$

So the connection we seek: $H^1_{dR}(X/S, (\mathcal{E}, \nabla_{/S})) \to \Omega^1_{S/T} \otimes H^1_{dR}(X/S, (\mathcal{E}, \nabla_{/S}))$ should send ω to $ds \wedge \omega'$, where ω' is the image of $\left(\frac{AP_s}{2y} + A_s \cdot y + \frac{B_s \cdot P_x}{y} - \frac{B_x \cdot P_s}{y}\right) dx$ in H^1_{dR}.

To write this image as an $\mathcal{O}_S$-linear combination of ω and $x\omega$, we do the same computation as in example in §2, treating s as fixed.

Result: $\boxed{\textbf{to be added}}$

⑥

5. Precise def. of Gauss-Manin.

5.1 $0 \to \pi^*\Omega^1_{S/T} \to \Omega^1_{X/T} \to \Omega^1_{X/S} \to 0$

Consider filtration $F^i(\Omega^r_{X/T}) := \mathrm{image}(\Omega^{r-i}_{X/T} \otimes \pi^*\Omega^i_{S/T} \to \Omega^r_{X/T})$.

(r-forms with at least i terms coming from $\Omega^1_{S/T}$).

$gr^i(\Omega^r_{X/T}) =$ " r-forms with exactly i terms from $\Omega^1_{S/T}$ "

$= \Omega^{r-i}_{X/S} \otimes \pi^*\Omega^i_{S/T}$.

The spectral sequence w.r.t. this filtration for $R^{p+q}\pi_*(\Omega^\bullet_{X/T} \otimes \mathcal{E})$ is

$E_1^{p,q} = R^{p+q}\pi_*(gr^p) = R^{p+q}\pi_*(\Omega^{\bullet-p}_{X/S} \otimes \mathcal{E} \otimes \pi^*\Omega^p_{S/T})$

(projection formula) $= \Omega^p_{S/T} \otimes \cancel{H^q_{dR}(X/S)} \, H^q_{dR}(X/S, \Omega^\bullet_{X/S} \otimes \mathcal{E})$

d_1 gives a flat connection on $H^i_{dR}(X/S, \Omega^\bullet_{X/S} \otimes \mathcal{E})$ for each i.

5.2 General theory of spectral sequence tells us:

$d_1^{p,q}$ is the connecting homomorphism for

$0 \to gr^{p+1} \to F^p/F^{p+2} \to gr^p \to 0$

Use this we can explicitly compute d_1 ([Katz-Oda]).

The result is as follows:

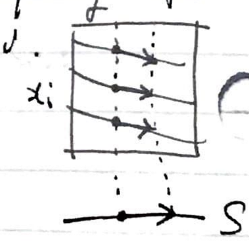
WLOG, S affine. Cover X by affine $\{U_\alpha\}$ such that each U_α is étale over $\mathbb{A}^n_S$, then we may choose local coordinates $\{x_1^\alpha, \dots, x_n^\alpha\}$. Consider Čech for $\mathcal{U} = \{U_\alpha\}$, $\Omega^\bullet_{X/S} \otimes \mathcal{E}$ ($\Omega^\bullet_{X/S}$, NOT $\Omega^\bullet_{X/T}$), and $R^i\pi_*$.

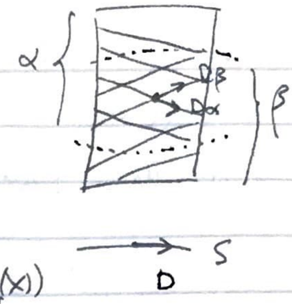
$C^{p,q} = C^p(\mathcal{U}, \Omega^q_{X/S} \otimes \mathcal{E})$.

We define a map $\mathrm{Der}_T(S) \to$ chain endomorphism of $\mathrm{Tot}\, C^{\bullet\bullet}$ as follows: Fix a total ordering of $\{\alpha\}$.

Given $D \in \mathrm{Der}_T(S)$.

- On each U_α, w.r.t. coordinate $\{x_1^\alpha, \dots, x_n^\alpha\}$, can uniquely lift D to $D_\alpha \in \mathrm{Der}_T(X)$, satisfying $\langle D_\alpha, dx_i^\alpha \rangle = 0$, $\forall i$.





• On $U_\alpha \cap U_\beta$, consider $D_{\alpha\beta} := D_\alpha - D_\beta$
The collection $\{D_{\alpha\beta}\}$ clearly forms a 1-cocycle ~~for~~ in $C^1(\mathcal{U}, Der_T(X))$.
Let the corresponding class in $H^1(X, Der_T(X))$ be denoted as $\rho(D)$. (Kodaira-Spencer class)

• Our chain endomorphism of Tot $C^{\cdot,\cdot}$ is then

$$\mathcal{K}(D) := \tilde{D} + \rho(D) \cup (-).$$

where $\tilde{D} : C^{\cdot,\cdot} \to C^{\cdot,\cdot}$, $\underset{(\alpha_0 < \cdots < \alpha_p)}{\sigma_{\alpha_0 \cdots \alpha_p}} \longmapsto \nabla_{D_{\alpha_0}}(\sigma_{\alpha_0 \cdots \alpha_p})$.

· $\rho(D) \cup (-)$ is cup product with $\rho(D) : C^{\cdot,\cdot} \to C^{\cdot+1,\cdot-1}$.
(Detailed formulae see Katz.)

Note:

① The "Zariski" filtration $F_Z^i := \bigoplus_{p \geq i} C^{p, \cdot}$ gives

$$E_1^{p,q} = H^{p+q}(C^p(\mathcal{U}, \Omega^{\cdot}_{X/S} \otimes \mathcal{E})) = C^p(\mathcal{U}, \mathcal{H}^q_{dR}(X/S, (\mathcal{E}, \nabla_{/S'})))$$

...meaning $p + \cdot = p+q$, so $\cdot = q$

$$\Rightarrow H^{p+q}_{dR}(X/S, (\mathcal{E}, \nabla_{/S'})).$$

$\tilde{\mathcal{R}}(D)$ respects this filtration, thus induces

$$H^i_{dR}(X/S, (\mathcal{E}, \nabla_{/S'})) \to H^i_{dR}(X/S, (\mathcal{E}, \nabla_{/S'})).$$

This "is" the Gauss-Manin connection.

② The "Hodge" filtration $F_H^i := \bigoplus_{q \geq i} C^{\cdot, q}$ gives

$$E_1^{p,q} = H^{p+q}(C^{\cdot}(\mathcal{U}, \Omega^p_{X/S} \otimes \mathcal{E})) = H^q(X, \Omega^p_{X/S} \otimes \mathcal{E})$$

...again $p + \cdot = p+q$ so $\cdot = q$

$$\Rightarrow H^{p+q}_{dR}(X/S, (\mathcal{E}, \nabla_{/S'})).$$

$\tilde{\mathcal{R}}(D)$ does not respect this filtration, but

$$\tilde{\mathcal{R}}(D) : F_H^i \to F_H^{i-1}$$

thus inducing $gr^p_H H^i_{dR} \to gr^{p-1}_H H^i_{dR}$.

If spectral seq. degenerates at $E_1^{\cdot\cdot}$, then $\tilde{\mathcal{R}}(D)$ induces $H^q(X, \Omega^p_{X/S} \otimes \mathcal{E}) \to H^{q+1}(X, \Omega^{p-1}_{X/S} \otimes \mathcal{E})$, which is precisely $\rho(D) \cup (-)$.

③ If X affine, ~~then~~ étale over $\mathbb{A}^n_S$, then we need only one U_α. Then $\rho(D) = 0$. $\tilde{\mathcal{R}}(D)$ is induced from $\tilde{D}$. This is consistant with previous discussion in §4