Setup: $T = \mathrm{Spec}\, \mathcal{O}_K$ $K = \#\mathrm{fld}$ $S \to T$ smooth

$(\mathcal{M}, \nabla)$ vector bundle w/ flat connection over S/T

i.e. $\nabla: \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_S} \Omega^1_{S/T}$ s.t. $\nabla^2 = 0$

Relation to diff eq'ns (in the special case S = open subsch of $\mathbb{A}^1_{\mathcal{O}_K}$ w/ coord. x since rel dim $S = 1$, then flatness is automatic)

Let $D = x\frac{d}{dx}$

A diff operator $L = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$, $a_i \in K(x)$

the diff eq'n $Ly = 0$ $\Leftrightarrow$ $(\mathcal{O}_S^n, \nabla)$ given by $\nabla(D)\vec{f} = D\vec{f} - \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 0 \quad 1 \\ -a_0 & \cdots & & -a_{n-1} \end{pmatrix}\vec{f}$

y a solution $\in \mathcal{O}_S$ $\Leftrightarrow$ $\begin{pmatrix} y \\ Dy \\ \vdots \\ D^{n-1}y \end{pmatrix} \in \mathcal{O}_S^n$ a horizontal section

Def (classical def'n for diff eq'n; cf. [DGS, p.83-84])

$(\mathcal{O}_S^n, \nabla)$ above or L has regular singularity at $x = 0$ if $\forall i$ a_i doesn't have pole at 0 (i.e. $a_i \in K(x) \cap K[[x]]$)

Regular singularities for $(\mathcal{M}, \nabla)$

Def ([K70, (11.2)] for rel dim $S = 1$ case) at $s \in \overline{S} \setminus S$ ($\overline{S}$ cpt), x = loc coord at s

$(\mathcal{M}, \nabla)$ has reg sing at s $(x = 0)$ if $\exists$ an $\mathcal{O}_{\overline{S}, s}$-lattice $\mathcal{M}_s$ of $\mathcal{M}_\eta$ [η = generic pt, $\mathcal{M}_\eta$ = fiber at η]

s.t. $\nabla(x\frac{d}{dx})(\mathcal{M}_s) \subseteq \mathcal{M}_s$ $\circledast$ (bad notation here, this is not the fiber of the ext. of $\mathcal{M}$ at s)

(To relate to the geom def in cf. [K71, II, III], $\mathcal{M}_s$ gives the extension $\overline{\mathcal{M}}$ of $\mathcal{M}$ over s and $\circledast$ means the connection is given by $\overline{\nabla}: \overline{\mathcal{M}} \to \overline{\mathcal{M}} \otimes \Omega^1_{S/T}(\log s)$ here s means the divisor $[s]$)

Def: for general S, to say $(\mathcal{M}, \nabla)$ has regular singularities,

we mean $\forall$ curve $C \xrightarrow[\pi]{} S$ (sm), $\pi^*(\mathcal{M}, \nabla)$ has regular singularities (at all pts in $\overline{C} \setminus C$)

(this is equ to ext'n + $\nabla: \overline{\mathcal{M}} \to \overline{\mathcal{M}} \otimes \Omega^1_{S/T}(\log(\overline{S} \setminus S))$ def cf. [K71, II, III])

rmk: relation to the classical def'n for diff eq'ns. still use $S \subseteq \mathbb{A}^1_{\mathcal{O}_K}$ as an example

over $K(x)$, any $(\mathcal{M}, \nabla) = \bigoplus$ cyclic v.b. w/ conn.

(Def: [K70, (11.4)] (M, ∇) is cyclic if $\exists\, m \in M$ s.t. $m, \nabla(D)m, \cdots, \nabla(D)^i m, \cdots$ generate M over $K(x)$.

wrt basis $\{m, \nabla(D)m, \cdots, \nabla(D)^{n-1}m\}$, the matrix of $\nabla(D)$ is $\begin{pmatrix} 0 & & 0 & * \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & * \\ 0 & & 1 & * \end{pmatrix}$

$n = \mathrm{rk} M$ i.e. $M \simeq \mathcal{O}_{S,\eta}^n$

cyclic $(\mathcal{M}, \nabla)$ $\underset{\text{equ.}}{\Leftrightarrow}$ diff eq'ns (to be precise, one sees the matrix here is $-(\ ,)^t$ (matrix in the $Ly = 0$ example) so dual to each other)

for decomposition into cyclic ones, cf. [DGS, III, Lem 4.1])

and (M, ∇) has reg sing $\Leftrightarrow$ all cyclic pieces have regular singularities

the classical def'n for cyclic one is eqv. to the def'n for (M, ∇) above

by [K70, Thm (11.9) (Fuchs, Turritin, Lutz)].

Local monodromy at $s \in \bar{S} \setminus S$ $\qquad D = x\frac{d}{dx}$ $\quad x$ = local coord. at s

Recall from Sug Woo's talk: for $\nabla(D)f = Df - af$, $a \in K$
$(\mathcal{O}_S, \nabla)$ formal solution / horizontal section f "=" $x^a = e^{a \log x}$

$K \hookrightarrow \mathbb{C}$, local monodromy $\in GL_1(\mathbb{C}) = \mathbb{C}^\times$ is given by $e^{2\pi i \cdot a}$

Indeed, geom loc monodromy is always $= e^{2\pi i \cdot}$ (a matrix can be read out from ∇)
rk n

[Manin's thm / Formal Fuchsian thy: Assume $(\mathcal{M}, \nabla)$ has regular singularity at s.
[K70, Thm (12.0) + Rmk (12.3)] Pick a basis $\vec{e}$ and write
see also [DGS, III.7, III.8] $\qquad \nabla(D)\vec{e} = B\vec{e}$, where $B \in M_{n\times n}(\mathcal{O}_{\bar{S},s})$

1) the eigenvalues of $B \bmod x$ in $K \bmod \mathbb{Z}$ is independent of the choice of $\vec{e}$
(or set $x=0$) $\qquad$ (w/ $\nabla(D)\vec{e} \subseteq \mathcal{O}_{\bar{S},s}\vec{e}$)

2) there exists a basis $\vec{e}$ s.t. all eigenvalues of $B \bmod x$ are NOT differed by any thing in $\mathbb{Z}\setminus\{0\}$
(i.e. if $\alpha - \beta \in \mathbb{Z}$, then $\alpha = \beta$)
eigenvalues of $B \bmod x$

3) over $\mathcal{M} \otimes K[[x]]$, there exists a basis $\vec{e}\,'$ s.t. $\nabla(D)\vec{e}\,' = B_0\vec{e}\,'$
[now assume we got B s.t. $B \bmod x$ satisfy the conclusion of (2)]
$B \bmod x = B_0 \in M_{n\times n}(K)$

formally we can solve this const. coeff. linear sys. of ODEs and see local monody over $\mathbb{C}$ coincide w/ $e^{2\pi i B_0}$

Upshot: we can talk about local monodromy using B_0 now (without referring to the geom. picture)

Warning: the choice of B_0 s.t. (2) holds cannot be dropped. cf. [K70, Example (12.5)]

mod p story

for this part, we consider S s.t. $p\mathcal{O}_S = 0$ still use $(\mathcal{M}, \nabla)$ to denote a v.b. w/ connection.
(we do ~~not~~ need to assume $\dim S = 1$)

sheaf of T-derivations $\mathcal{O}_S \to \mathcal{O}_S$

Note: in char p, D^p is also a T-derivation

p-curvature: $\psi : \mathrm{Der}(S/T) \longrightarrow \mathrm{End}_T(\mathcal{M})$
("char p version of curvature i.e. $\psi = 0 \Leftrightarrow$ full set of horizontal sections) $\quad D \longmapsto (\nabla(D))^p - \nabla(D^p)$

Indeed, $\psi(D)$ is an $\mathcal{O}_S$-linear horizontal end on $(\mathcal{M}, \nabla)$ (for pfs, see [K70, §5])
and $\psi(gD) = g^p\psi(D)$
So we write $\psi : \mathrm{Der}(S/T) \longrightarrow F_{S*}\mathrm{End}_S(\mathcal{M}, \nabla)$ (a morphism of $\mathcal{O}_S$-sheaves)
where $F_S : S \to S$ the absolute Frob

more to be discussed later

Example: $S = \mathbb{A}^1 \setminus \{0\}$ over $k \leftarrow$ finite fld $\quad D = x\frac{d}{dx}$
$(\mathcal{O}_S, \nabla)$ given by $\nabla(D)f = Df - af$, $a \in k$ $\qquad 1 \in \mathcal{O}_S$.
Given the properties of ψ, to see $\psi \equiv 0$ (0 map), we only need to consider $\psi(D)1$.
Note $\nabla(D)1 = -a$ and $D^p = D$
so $\psi(D)1 = (-a)^p + a$ $\qquad \psi \equiv 0 \Leftrightarrow a \in \mathbb{F}_p$.

An observation for later: consider $S/K \leftarrow$ # fld $(\mathcal{O}_S, \nabla)$ given by $\nabla(D)f = Df - af$
$\mathbb{A}^1_K \setminus \{0\}$ $\qquad a \in K$

then $\psi = 0 \ \forall\forall\, \mathfrak{p} \subseteq \mathcal{O}_K \iff a \in \mathbb{Q}$.

(for each $\mathfrak{p}$, $(\mathcal{O}_S, \nabla)$ mod $\mathfrak{p}$ is over finite fld, apply the p-curv discussion to it)

A generalization of $\psi = 0$ in rk 1 case is

__Def__: for $(\mathcal{M}, \nabla)$ in char p, it is __nilpotent of exponent__ $\le \nu$ if $\forall D_1, \dots, D_\nu$ sections of $\mathrm{Der}(S/T)$ over some opens of S, we have $\psi(D_1) \cdots \psi(D_\nu) = 0$

(Note, another property of ψ is that all $\psi(D_i)$ commute, so order doesn't matter.)

Back to $S/\mathcal{O}_K$, $(\mathcal{M}, \nabla)/S$ (global fld setting)

__Thm__ (Katz, [K70, Thm (13.0)]) (For ease of statement, rel dim $S = 1$ / for general case, just pull back to curves)

If $(\mathcal{M}, \nabla)$ is nilp of exp $\le \nu$ for $\forall\forall\, \mathfrak{p} \subseteq \mathcal{O}_K$, then $(\mathcal{M}, \nabla)$ has regular singularities and the local monodromies are quasi-unipotent of exponents $\le \nu$.

(Recall, B_0 from local monodromy write $B_0 = \underset{\text{semisimple}}{D} + \underset{\text{nilp}}{N}$

quasi-unip of exp $\le \nu$ means all eigenvalues of $D \in \mathbb{Q}$ and $N^\nu = 0$)

__rmks__:

① For $(\mathcal{M}, \nabla) =$ (rel de Rham coh $\mathcal{H}^n$, Gauss-Manin connection), we have $(\mathcal{M}, \nabla)^{\text{mod } p}$ to be nilp of exp $\le \{ i \mid h^{i,n-i} \neq 0 \}$. Thus we can apply the thm above to conclude regular sing + quasi-unip local monodromy.

→ to see this nilp claim, the main claim is that $gr^{\bullet}_{con} H^n_{dR}$ has p-curv $= 0$ (graded piece of conjugate filtration)

__Ogus__: this is a direct conseq. of Mazur's thm (description of Hodge and conjugate filtration on crystalline cohom.)

__Katz/Deligne__ ([K70, §7], ideas attributed to Deligne): use Cartier isomorphism to see that $gr^{\bullet}_{con} H^n_{dR}$ comes from Frob pull back and hence p-curv $= 0$.

(p-curv $= 0 \iff$ being __Frob pull back__ is a result of Cartier which we'll state and sketch a pf,
(to be made precise) (not the Cartier isom)
↑ also used in Ogus's pf)

② to get regular singularity, we only need nilp for infinitely many $\mathfrak{p}$

to get quasi-unipotent, we need nilp at a set of primes over a density 1 set of rat'l primes b/c it is a computation similar to rk 1 case so we use Chebotarev.

__Cartier: p-curv__ $\iff$ full set of horizontal sections over $\mathcal{O}_S \iff$ being Frob pull back

__pf / explanation__: for S = open in $\mathbb{A}^1$

$$\begin{tikzcd} S \arrow[r, "F"] \arrow[rr, bend left, "F_S"] \arrow[dr] & S^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d] \\ & T \arrow[r, "F_T"] & T \end{tikzcd}$$

rel. Frob

__Observe__: $\mathcal{M}^\nabla \in \mathrm{Qcoh}(S^{(p)})$

b/c if $\nabla(m) = 0$, then $\nabla(gm) = (dg^p) m + g^p \nabla(m) = 0$ $\forall g \in \mathcal{O}_S$.

__Key point__: $F^*(\mathcal{M}^\nabla) \xrightarrow{\sim} \mathcal{M}$ if $\psi = 0$ for $(\mathcal{M}, \nabla)$ (for the other direction, if $F^*(\mathcal{M}^\nabla) \xrightarrow{\sim} M$)

Sketch of pf: (we use $\mathcal{M}^D$ to compute p-curv $=0$)

Construction of horizontal sections: $\forall m \in \mathcal{M}$

$$P(m) := m - x\nabla\left(\frac{d}{dx}\right)m + \cdots + \frac{(-x)^i}{i!}\nabla\left(\frac{d}{dx}\right)^i m + \cdots + \frac{(-x)^{p-1}}{(p-1)!}\nabla\left(\frac{d}{dx}\right)^{p-1} m$$

is horizontal

(b/c by Lebniz rule, everything cancels out except

apply $\nabla\left(\frac{d}{dx}\right)$ $\qquad \frac{(-x)^{p-1}}{(p-1)!}\nabla\left(\frac{d}{dx}\right)^p m$

note: $\left(\frac{d}{dx}\right)^p = 0$ so $\psi = 0 \iff \nabla\left(\frac{d}{dx}\right)^p m = 0 \ \forall m$)

Note: $P(m)\big|_{x=0} = m\big|_{x=0}$

m as lin comb. of horizontal sections:

$$m = P(m) + x P\left(\nabla\left(\frac{d}{dx}\right)m\right) + \cdots + \frac{x^i}{i!}P\left(\nabla\left(\frac{d}{dx}\right)^i m\right) + \cdots + \frac{x^{p-1}}{(p-1)!}P\left(\nabla\left(\frac{d}{dx}\right)^{p-1} m\right)$$

(can be checked directly (and use $\nabla\left(\frac{d}{dx}\right)^p m = 0$)

to see why this formula should hold, think about the example of trivial connection $(\mathcal{O}_S, d)$

then the above formula is just the Taylor expansion)

rmk: for higher dim S, there is a similar formula coming from truncating Taylor expansion under $\psi = 0$ condition.

Grothendieck - Katz p-curvature conj:

for $(\mathcal{M}, \nabla)/S$ over $\mathcal{O}_K$, if the p-curv $\psi \equiv 0$ $\forall \forall \mathfrak{p} \subseteq \mathcal{O}_K$

then $(\mathcal{M}, \nabla)$ is isotrivial.

rmks: ① for $(\mathcal{M}, \nabla) = (\mathcal{O}_S, \nabla)$ $\quad \nabla\left(x\frac{d}{dx}\right)f = x\frac{df}{dx} - af$

direct computation before shows $a \in \mathbb{Q}$, sol $= x^a$

② for $(\mathcal{M}, \nabla) = $ (de Rham, GM), conj proved by Katz relating p-curv. to Kodaira - Spencer map.