Fall 2022 Wed NT seminar on Arithmetic of Algebraic Differential Equations

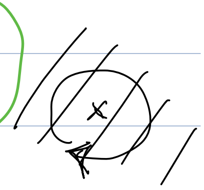
Informal Introduction by Sug Woo Shin 8/31/2022

$\boxed{\text{Intro}}$

DE = Differential Equations

an = analytic, alg = algebraic

$$\begin{tikzcd} \begin{array}{c}\text{alg geom.}\\ \text{alg DE}\end{array} \arrow[r] \arrow[rr, bend left, "\text{Hilbert's 21st Problem: surj?}"] & \begin{array}{c}\text{analytic geom.}\\ \text{an DE}\end{array} \arrow[r] & \begin{array}{c}\text{topology.}\\ \text{monodromy.}\end{array} \end{tikzcd}$$



Ex $U = \mathbb{C} - \{0\}$.

$$\frac{df}{dz} = \alpha \cdot \frac{f}{z}. \quad \alpha \in \mathbb{C}.$$

(viewed as alg or an DE)

$\pi_1(U) \cong \mathbb{Z}\cdot 2\pi i \ni$ gen acts on sol. by multiplying $e^{2\pi i\alpha}$ (monodromy repn).

local sol: $\langle z^\alpha = e^{\alpha \log z} \rangle$.

Rmk p-adic analogue is subtle (Diao-Lan-Liu-Zhu)

$\boxed{\text{Alg vs An}}$ (over $\mathbb{C}$).

Background on comparing the first two bubbles above

$\exists$ analytification functors

$$(\text{alg var}/\mathbb{C}) \xrightarrow{an} (\text{an var}/\mathbb{C})$$

$$X \longmapsto X^{an}$$

$$\mathrm{Coh}(X) \xrightarrow{an} \mathrm{Coh}(X^{an})$$

$$\mathcal{F} \longmapsto \mathcal{F}^{an}$$

$$\mathcal{O}_X,\ \Omega_X \longmapsto \mathcal{O}_{X^{an}},\ \Omega_{X^{an}}$$

Chow's thm $\quad (\text{proj var}/\mathbb{C}) \xrightarrow[\sim]{an} (\text{proj an. var}/\mathbb{C})$

$$X \subseteq \mathbb{P}^n \longmapsto X^{an} \subseteq \mathbb{P}^{n,an}$$

Ex $\quad (\text{sm. proj. curves}/\mathbb{C}) \xrightarrow{\sim} (\text{cpt Riemann surfaces})$

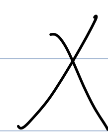
Non-Ex $\quad$ graph $\{w = e^z\} \subseteq \mathbb{C}^2 \quad$ (not proj, not alg.)

GAGA $\quad \mathrm{Coh}(X) \xrightarrow[\sim]{an} \mathrm{Coh}(X^{an})$

X : proj. $\quad$ moreover, $H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(X^{an}, \mathcal{F}^{an})$

$\boxed{\text{Precise Version of intro diagram}}$

$$\underset{\text{sm.}}{U} \overset{\text{open}}{\subseteq} \underset{\text{sm. proj}}{X} \qquad X \setminus U = D. \ \ U(\text{sm. divisors})$$



alg or an.

Def A vector bundle $/U$ $\mathcal{E}$ with a flat connection $(\mathcal{E}, \nabla)$

(VBC) $\quad$ "$\nabla^2 = 0$" $\quad \nabla : \mathcal{E} \to \Omega_U \otimes \mathcal{E}$

Leibniz rule : $\nabla(s \cdot f) = ds \otimes f + s \cdot \nabla(f)$

Slogan $\quad$ DE "=" VBC $\quad$ (alg or an)

sol. of DE = $\ker \nabla$. $\quad$ (Local System)

$VBC(X,D) \supseteq VBC(X)$. similarly for analytic ver.

allowing "log poles" along D.

Let's make the earlier diagram precise.

This is the key "regular singularities" (to be defined in a later talk) condition to impose along D.

$$
\begin{tikzcd}
(\mathcal{E},\nabla) \arrow[r, mapsto] & (\mathcal{E}^{an},\nabla^{an}) \arrow[r, mapsto] & \ker \nabla^{an}. \\
VBC_{reg}(U) \arrow[r, "\sim", "\text{Deligne}"'] & VBC(U^{an}) \arrow[r, "\sim", "\text{standard}"'] & LS(U^{an}) \\
VBC(X,D) \arrow[u] \arrow[r, "\sim", "\text{GAGA}"'] & VBC(X^{an},D^{an}) \arrow[u, "\mathrm{res}"', "\text{key}"] & Rep(\pi_1(U^{an})) \arrow[u, phantom, "\cong"]
\end{tikzcd}
$$

<u>Ex</u> (cont'd) $U = \mathbb{C} - \{0\} \subseteq X = \mathbb{P}^1_{\mathbb{C}}$.

$\mathcal{E} = \mathcal{O}_U$. $\nabla(f) = df - \alpha \frac{f}{z} dz = \left(\frac{df}{dz} - \alpha \frac{f}{z}\right) dz$

(This one is regular in the above sense.)

$\ker \nabla^{an} =$ Sol. of DE "$\frac{df}{dz} = \alpha \frac{f}{z}$".

<u>Non-ex</u> Same U, $\mathcal{E}$, with $\nabla(f) = df - \alpha f \cdot dz$ is <u>not</u> regular. Sol is exp function, which grows "too fast". ∎