ALGEBRA QUAL PREP: PROBLEMS ON MODULES AND HOMOLOGICAL ALGEBRA

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These are hints/solution sketches for the problems. They are not a model for what to write on the quals.

1. Fall 2010 A4

(i) Take a presentation $A^m \to A^n \to M$. Since $\operatorname{Hom}(-,N)$ is left exact, we get a SES

$$0 \to \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_A(A^n,N) \to \operatorname{Hom}_A(A^m,N)$$

The presentation induces a presentation $B \otimes_A A^m \to B \otimes_A A^n \to B \otimes_A M$. This induces a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}_A(M,N) \otimes_A B & \longrightarrow & \operatorname{Hom}_A(A^n,N) \otimes_A B & \longrightarrow & \operatorname{Hom}_A(A^m,N) \otimes_A B \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{Hom}_B(M \otimes_A B, N \otimes_A B) & \longrightarrow & \operatorname{Hom}_B(A^n \otimes_A B, N \otimes B) & \longrightarrow & \operatorname{Hom}_B(A^m \otimes_A B, N \otimes_A B) \end{array}$$

Using $B \otimes_A A^m \cong B^m$, we can identify the second and third vertical arrows as isomorphisms. Hence the first one is as well, by the 5 Lemma.

(ii) The splitness is equivalent to $\operatorname{Hom}_A(M'',M) \to \operatorname{Hom}_A(M'',M'')$ being surjective (consider a pre-image of $\mathrm{Id} \in \operatorname{Hom}_A(M'',M'')$). This surjectivity can be checked locally, i.e. it is enough to know that $\operatorname{Hom}_A(M'',M)_{\mathfrak{m}} \to \operatorname{Hom}_A(M'',M'')_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$. By (i), we have

$$\operatorname{Hom}_A(M'',M)_{\mathfrak{m}} \cong \operatorname{Hom}_{A_{\mathfrak{m}}}(M''_{\mathfrak{m}}, M_{\mathfrak{m}})$$

and similarly for the other term, so this localized surjectivity is the assumption.

2. Fall 2012 M1

We have

$$0 \to \ker f_A \to \mathbf{Z}^m \to \operatorname{Im}(f_A) \to 0.$$

Since f_A is a submodule of $\mathbf{Z}^n$, it is free. Hence (really only using the projectivity) we have $\mathbf{Z}^m \cong \underbrace{\ker f_A}_{\mathbf{Z}^a} \oplus \underbrace{\operatorname{Im}(f_A)}_{\mathbf{Z}^b}$.

By the normal form for submodules of a module over a PID, the map $\operatorname{Im}(f_A) \hookrightarrow \mathbf{Z}^n$ can be diagonalized, hence $\mathbf{Z}^n \cong \mathbf{Z}^b \oplus \mathbf{Z}^c$ with $\operatorname{Im}(f_A) \cong \mathbf{Z}^b$ mapping diagonally to $\mathbf{Z}^b$. It is clear that the torsion of $\operatorname{coker} f_A$ is the torsion of the cokernel of this map $\mathbf{Z}^b \to \mathbf{Z}^b$, and also clear that the torsion of $\operatorname{coker} f_{A^t}$ is the torsion of the transposed map, which is the same.

3. Fall 2013 A3

(a) Let N be an A-module. If $N \otimes_A B = 0$, then $N \otimes_A B/\mathfrak{m}_B = 0$. But we have $N \otimes_A A/\mathfrak{m} \hookrightarrow N \otimes_A B/\mathfrak{m}$ since $A/\mathfrak{m} \hookrightarrow B/\mathfrak{m}$ and N is flat, so then also $N \otimes_A A/\mathfrak{m} = 0$.

If N were finitely generated, Nakayama's lemma would imply that $N = 0$. If N is not finitely generated, pick a finitely generated submodule $N' \hookrightarrow N$. Then $B \otimes_A N' \hookrightarrow B \otimes_A N$ by flatness. Now the earlier argument implies that $B \otimes_A N' = 0$ for all such N'. But every element of $B \otimes_A N$ is in the image of such a map, so $B \otimes_A N = 0$.

(b) The fiber over $\mathfrak{p} \in \operatorname{Spec} A$ in $\operatorname{Spec} B$ is $\operatorname{Spec}(B \otimes A_{\mathfrak{p}}/\mathfrak{p})$. If this is empty then $B \otimes A_{\mathfrak{p}}/\mathfrak{p} = 0$ while $A_{\mathfrak{p}}/\mathfrak{p} \neq 0$. That proves $\implies$.

For $\impliedby$, consider a module N over A. Pick $\mathfrak{m}$ a maximal ideal of A such that $N_{\mathfrak{m}} \neq 0$, and let $\mathfrak{n} \in \operatorname{Spec} B$ map to $\mathfrak{m}$. Since $B_{\mathfrak{n}} \to A_{\mathfrak{m}}$ is local, it is faithfully flat by (a), hence $N_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} B_{\mathfrak{n}} \cong N \otimes_A B_{\mathfrak{n}} \neq 0$, hence $N \otimes_A B \neq 0$.

(c) The condition $M \subset M'$ is equivalent to $M' = M + M'$. Hence we reduce to checking an equality of submodules of A holds if and only if it holds after tensoring up to B. This follows from applying the definition of faithful flatness to the quotient.

4. Spring 2015 Q2

(i) For (i), we use that flatness can be checked locally. Since Dedekind domains are DVRs locally, the classification of finitely generated modules over a DVR shows that the claim is true for finitely generated modules. A torsion-free R-module is a filtered colimit of finitely generated torsion-free R-modules, and since filtered colimits preserve exactness this shows that torsion-free R-modules are flat.

Consider $I = (x, y) \subset R = \mathbf{C}[x, y]$. We have $I \hookrightarrow R$, but $I \otimes_R I \to I \otimes_R R = I$ is the multiplication map, and we know it's not injective (see the previous homework!).

(ii) Use the short exact sequence

$$0 \to I \to R \to R/I \to 0.$$

Tensoring with R/J, we get

$$0 \to \operatorname{Tor}^1(R/I, R/J) \to I \otimes_R R/J \to R/J \to R/I \otimes R/J \to 0.$$

This shows that $\operatorname{Tor}^1(R/I, R/J) \cong \ker(I/IJ \to R/J)$, which is $I \cap J/IJ$. If $I \cap J = IJ$, then, using that the local rings of a Dedekind domain are DVRs, we find that we must have either I or J is the unit ideal at each localization. This implies $I + J = 1$.

For a counterexample with $R = \mathbf{C}[x, y]$, we can take $I = (x)$ and $J = (y)$. Then $I \cap J = (xy)$ and $IJ = (xy)$, yet $I + J = (x, y)$.

5. Spring 2010 M5

(a) Any complex with a chain homotopy $hd + dh = \mathrm{Id}$ has vanishing homology, since for any cycle x we have $x = hdx + dhx = d(hx)$.

Conversely, suppose (F_*, d) is exact. Since F_0 is free, we can find a section $h_0 \colon F_0 \to F_1$. We proceed by induction to define $h_i \colon F_i \to F_{i+1}$ with the desired property:

$$dh_i(x) = x - h_{i-1}d.$$

Since $d(x - h_{i-1} dx) = 0$, it is in the image of F_i by exactness. Hence we can find h_i with the desired property.

(b) For the complex $\mathrm{Hom}(F_*, M)$ we also have a chain homotopy h_* with these properties.

(c) It suffices to to give a counterexample to (b). Consider

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}/p \to 0$$

as a short exact sequence of $\mathbf{Z}$-modules. Applying $\mathrm{Hom}(-, \mathbf{Z}/p)$ gives

$$0 \to \mathbf{Z}/p \to \mathbf{Z}/p \xrightarrow{0} \mathbf{Z}/p$$

which is not exact.

6. Fall 2011 M5

(i) Use

$$\ldots \to \mathbf{Z}/p^2 \xrightarrow{p} \mathbf{Z}/p^2 \xrightarrow{p} \mathbf{Z}/p^2 \to 0.$$

Applying $\mathrm{Hom}_{\mathbf{Z}/p^2}(-, \mathbf{Z}/p\mathbf{Z})$, we get

$$\mathbf{Z}/p \xrightarrow{0} \mathbf{Z}/p \xrightarrow{0} \ldots$$

so we find $\mathrm{Ext}^i_{\mathbf{Z}/p^2}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z}) = \mathbf{Z}/p\mathbf{Z}$ for all i.

(ii) Use Baer's criterion. We want to show that for any $I \subset R$, the induced map $\mathrm{Hom}_R(R, M) \twoheadrightarrow \mathrm{Hom}_R(I, M)$ is surjective. But this is obvious in our case, with $M = \mathbf{Z}/p^2\mathbf{Z}$.

Why is this enough? In general, we need to show that for any $P \hookrightarrow Q$, any map $P \to M$ can be extended to a map $Q \to M$. Consider a maximal submodule Q' of Q to which it can be extended, say to $f: Q' \to M$. If $Q' \neq Q$, take $x \in Q - Q'$. We have an ideal $I := \{r \in R : rx \in Q'\}$, and a map $g: I \to M$. We can extend this to a $\widetilde{g}: R \to M$, and use to define a map $\widetilde{f}: (Q' + Rx) \to M$ as follows:

$$\widetilde{f}(q + rx) = f(q) + \widetilde{g}(r).$$

An injective resolution for $\mathbf{Z}/p$ over $\mathbf{Z}/p^2$ is

$$\mathbf{Z}/p^2 \xrightarrow{p} \mathbf{Z}/p^2 \xrightarrow{p} \mathbf{Z}/p^2 \to 0.$$

Applying $\mathrm{Hom}_{\mathbf{Z}/p^2}(\mathbf{Z}_p, -)$ we get

$$\mathbf{Z}/p \xrightarrow{0} \mathbf{Z}/p \xrightarrow{0} \ldots$$

as before.

7. Spring 2012 M5

(a) First we establish the result for finitely generated A. In that case we have a short exact sequence

$$0 \to R \to F \to A \to 0$$

where $R \cong \mathbf{Z}^m$, $F \cong \mathbf{Z}^n$. Tensoring with C_*, we get a short exact sequence of complexes (exactness because C_* is free)

$$0 \to R \otimes C_* \to F \otimes C_* \to A \otimes C_* \to 0.$$

The LES in homology then reads

$$H_n(C_* \otimes R) \to H_n(C_* \otimes F) \to H_n(C_* \otimes A) \to H_{n-1}(C_* \otimes R) \to H_{n-1}(C_* \otimes F) \to \dots$$

Since R and F are free, we have $H_n(C_* \otimes R) \cong H_n(C_*) \otimes R$ and $H_n(C_* \otimes F) \cong H_n(C_*) \otimes F$. By right exactness of tensor product,

$$\frac{H_n(C_*) \otimes F}{H_n(C_*) \otimes R} \cong H_n(C_*) \otimes (F/R) \cong H_n(C_*) \otimes A.$$

Also, by the LES of tensoring $0 \to R \to F \to A \to 0$ with $H_{n-1}(C_*)$, we have

$$\ker \colon H_{n-1}(C_*) \to R \to H_{n-1}(C_*) \otimes F = \mathrm{Tor}_1(H_{n-1}(C_*), A).$$

(b) Let $A = \mathbf{Q}/\mathbf{Z}$. By (a) we know $H_n(C_* \otimes A) = 0$ unless $n = 0, 1$. For $n = 0$, it is $H_0(C_*) \otimes_Z \mathbf{Q}/\mathbf{Z} \cong \mathbf{Q}/\mathbf{Z}$. For $n = 1$, it is $\mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/5; \mathbf{Q}/\mathbf{Z}) \cong \mathbf{Z}/5\mathbf{Z}$.

8. Spring 2013 M2

(i) We start building the resolution. The kernel of $\mathbf{Z}[t] \to \mathbf{Z}/2$ is $(2, t)$. So we take $\mathbf{Z}[t]^{\oplus 2} \to \mathbf{Z}[t]$ sending generators to $2, t$. The kernel is then generated by the vector $\begin{pmatrix} t \\ -2 \end{pmatrix}$. Thus we build the resolution

$$0 \to \mathbf{Z}[t] \xrightarrow{\begin{pmatrix} t \\ -2 \end{pmatrix}} \mathbf{Z}[t]^{\oplus 2} \xrightarrow{\begin{pmatrix} 2 & t \end{pmatrix}} \mathbf{Z}[t]$$

(ii) Apply $\mathrm{Hom}_R(-, \mathbf{Z}/4)$. The above becomes

$$\mathbf{Z}/4 \xrightarrow{\begin{pmatrix} 0 \\ -2 \end{pmatrix}} \mathbf{Z}/4^{\oplus 2} \xrightarrow{\begin{pmatrix} 2 & 0 \end{pmatrix}} \mathbf{Z}/4$$

Then we find $\mathrm{Ext}^0 = \mathbf{Z}/2$, $\mathrm{Ext}^1 = \mathbf{Z}/2 \oplus \mathbf{Z}/2$, and $\mathrm{Ext}^2 = \mathbf{Z}/2$.

9. Fall 2015 M2

(a) We argue by induction on i. The result is obvious for $i = 0$. Take a surjection $R^n \to M$, with kernel $M' \subset R^n$. Since R is Noetherian, we also get that M' is finitely generated, hence $\mathrm{Tor}_{i-1}(M', N)$ is finite. By the LES we get $\mathrm{Tor}_{i-1}(M', N) \cong \mathrm{Tor}_i(M, N)$, so we win.

(b) We claim that $\mathrm{Tor}_i^R(M, N)$ is killed by multiplication by $\#M$ and $\#N$. This is clearly sufficient. Multiplication by $n \in \mathbf{Z}$ on M induces a map $[n]_i \colon \mathrm{Tor}_i^R(M, N) \to \mathrm{Tor}_i^R(M, N)$ by functoriality, and we claim that this is multiplication by n. This follows by the fact that Tor_i form a universal family of δ-functors (explicitly prove this by "dimension shifting"). The claim evidently implies what we want.

(c) Take the sequence

$$0 \to I \to R \to R/I \to 0.$$

Tensor with R/I:

$$0 \to \mathrm{Tor}^1_R(R/I, R/I) \to I \otimes_R (R/I) \to R \otimes_R (R/I) \to (R/I) \otimes_R (R/I) \to 0.$$

Since the map $R \otimes_R (R/I) \to (R/I) \otimes_R (R/I)$ is an isomorphism, we find $\operatorname{Tor}_R^1(R/I, R/I) \cong I \otimes_R (R/I) \cong I/I^2$.

For any prime $\mathfrak{p} \supset I$, we have $(I/I^2)_{\mathfrak{p}} \supset I_{\mathfrak{p}}/\mathfrak{p} I_{\mathfrak{p}}$. By Nakayama's Lemma (and the noetherianity of R), we deduce that $I_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in V(I)$. If $\mathfrak{p} \not\supset I$ then obviously $I_{\mathfrak{p}} = 0$. So we find that $\{\mathfrak{p} \in \operatorname{Spec} R : I_{\mathfrak{p}} = 0\} = V(I)$ is closed. On the other hand, the condition that $I_{\mathfrak{q}} = 0$ is open. So $V(I)$ is an open and closed subset of $\operatorname{Spec} R$. If I is non-zero then it is a proper subset, hence $\operatorname{Spec} R$ is disconnected, and then R is not a domain.