Problems for "Quantum Cohomology and Symplectic Resolutions"

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1 Problem Sheet 1

1.1 Problem 1

The normalization $\widetilde{C}$ of C is a the disjoint union of the normalizations of the components of C. Any automorphism of $C \to X$ lifts to an automorphism of $\widetilde{C}$ over X.

- A component which is not crushed by the map has only finitely many automorphisms, since for instance any automorphism of $\widetilde{C} \to f(\widetilde{C})$ induces an automorphism of the extension of function fields, which can be at most the degree of f.

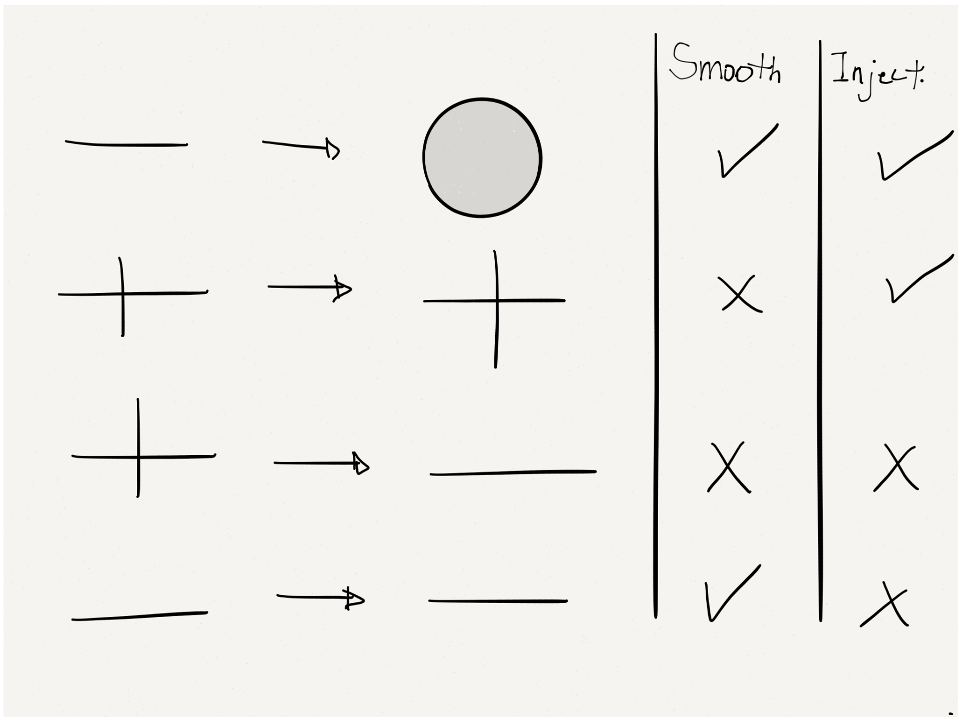
- Any component of $\widetilde{C}$ having genus at least 2 has only finite many automorphisms at all (disregarded the map to X entirely).

- A component of genus 1 has infinitely many abstract automorphisms. However, in C such a component has at least one node since it is joined to some other component, and any automorphism of the map must send nodes to nodes. Now, there are only *finitely* many automorphisms of a smooth genus 1 curve sending a given point to another given point (an elliptic curve has only finitely many automorphisms).

- For the rational componetns of $\widetilde{C}$, apply the same argument as above, noting that there are only finitely many automorphisms of $\mathbb{P}^1$ sending a given triple of points to another given triple.

1.2 Problem 2

There are four strata, depicted below



- The open (smooth, injective) stratum consists of an isomorphism from $\mathbb{P}^1$ to a smooth conic in $\mathbb{P}^2$. For a fixed smooth conic, there is obviously one such isomorphism up to reparametrization, so the fibers are points.
- The injective, singular stratum consists of maps from a nodal union of two $\mathbb{P}^1$s to a nodal union of lines in $\mathbb{P}^2$. For a given image curve, there is again only fiber since the map is an isomorphism.
- The singular, non-injective stratum consists of maps from a nodal union of two $\mathbb{P}^1$s to a line in $\mathbb{P}^2$. The moduli of such maps is described by the image of the node, which is evidently $\mathbb{P}^1$.
- The smooth, non-injective stratum consists of 2:1 maps from $\mathbb{P}^1$ to a line in $\mathbb{P}^2$. The moduli of such maps is described by the 2 branch points, which is evidently

$$\mathrm{Sym}^2 \mathbb{P}^1 \setminus \Delta = (\mathbb{P}^1 \times \mathbb{P}^1 - \Delta)/(\mathbb{Z}/2).$$

In particular, the only interesting fibers of $\overline{\mathcal{M}_{0,0}}(\mathbb{P}^2, 2[\text{line}])$ are over the locus of double lines, i.e. the *Veronese surface* in $\mathbb{P}^5$. To see what the fibers are, we compute $\mathbb{P}^1 \times \mathbb{P}^1 -$

$\Delta/(\mathbb{Z}/2)$. Choose affine coordinates t and s on the two copies of $\mathbb{P}^1$. The Veronese embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ is $(s,t) \mapsto (s,t,st)$. Therefore, we must take invariants under $s \leftrightarrow t$ of $k[s,t,st]$ localized away from $s = t$, which is $k[t+s, st]_{s-t}$.

The reducible curves in the third category arise from semistable reduction as the branch points s and t move together. ♠♠♠ TONY: [how to check this rigorously?] Since we have only worked with affine components, it is not so clear a priori which compactification we get for the full fiber. The missing points are obtained by taking $s \to \infty$ or $t \to \infty$. In terms of the coordinates $s+t$ and ts, the slope is $\frac{s+t}{st}$, so it clear that as for finite t, as $s \to \infty$ we get an $\mathbb{A}^1$, which is then completed to a $\mathbb{P}^1$ by allowing $t \to \infty$. So we get one line at infinity (remember that $s \leftrightarrow t$, so we don't have to the consider the case with s and t swapepd), i.e. a compactification of $\mathbb{P}^1$.

As a sanity check, we mention that the space $\overline{\mathcal{M}_{0,0}}(\mathbb{P}^2, 2[\text{line}])$ is supposed to be the blowup of the space of conics $\mathbb{P}^5$ over the Veronese surface. Indeed, we are finding that it maps isomorphically to $\mathbb{P}^5$ except over the Veronese surface, at which point it has fibers $\mathbb{P}^2$, which is what we expect for the blowup.

The cubic case is more complicated, but it seems fairly clear how to proceed.

1.3 Problem 3

Recall that

$$\begin{aligned}\langle 1, \gamma_1, \dots, \gamma_n \rangle_{0,\beta}^X &= \overline{\mathcal{M}_{0,n+1}}(X,\beta)^{\text{vir}} \frown (\text{ev}_1^* 1 \smile \text{ev}_2^* \gamma_2 \smile \dots \smile \text{ev}_n^* \gamma_n) \\ &= f^* \overline{\mathcal{M}_{0,n}}(X,\beta)^{\text{vir}} \frown (\text{ev}_2^* \gamma_2 \smile \dots \smile \text{ev}_n^* \gamma_n).\end{aligned}$$

Now, for since the right hand side is non-zero we must have that $(\text{ev}_2^* \gamma_2 \smile \dots \smile \text{ev}_n^* \gamma_n) \in H^d(\overline{\mathcal{M}_{0,n}}(X,\beta))$ where d is the expected dimension of $\overline{\mathcal{M}_{0,n}}(X,\beta)$. But then $\text{ev}_1^* 1 \smile \text{ev}_2^* \gamma_2 \smile \dots \smile \text{ev}_n^* \gamma_n \in H^d \overline{\mathcal{M}_{0,n+1}}(X,\beta)$ whereas $f^* \overline{\mathcal{M}_{0,n}}(X,\beta)^{\text{vir}}$ is in $H_{d+1}(\overline{\mathcal{M}_{0,n+1}}(X,\beta))$ so the cap product vanishes by formal degree incompatibilities.

For the second equality

$$\langle D, \gamma_1, \dots, \gamma_n \rangle_{0,\beta}^X = (D \cdot \beta) \langle \gamma_1, \dots, \gamma_n \rangle_{0,\beta}^X$$

we make an intuitive argument. The left hand side counts maps from rational curves to X with fundamental class β with $n+1$ marked points $p_1, \dots, p_{n+1}$ passing through the cycles $D, \gamma_2, \dots, \gamma_n$. The right hand side counts maps from rational curves to X with fundamental class β and n marked points $p_2, \dots, p_n$ passing through $\gamma_2, \dots, \gamma_n$. For any such map counted by the right hand side, there are "$\#D \cap \beta$" $= D \cdot \beta$ choices for markings of p_1 to augment it to a map counted by the left hand side.

♠♠♠ TONY: [how to make this rigorous?]

1.4 Problem 4

Noting that $H_k^\vee = H_{n-k}$ and $H_2(\mathbb{P}^n) \cong \mathbb{Z}$ generated by the dual to H_{n-1}, we have by definition of the quantum product

$$\begin{aligned} H_i \bullet H_j &= \sum_\beta q^\beta \langle H_i, H_j, H_k \rangle H_k^\vee \\ &= \sum_n q^n \langle H_i, H_j, H_k \rangle H_{n-k} \end{aligned}$$

Now, by definition $\langle H_i, H_j, H_k \rangle$ is non-zero only when $i + j + k$ equals the dimension of the virtual fundamental class, which is

$$\dim X + (n-3) - K_X \cdot \beta = n + (3-3) + (n+1)\beta = n + (n+1)\beta.$$

Since $0 \leq i, j, k \leq n$, the only possibilities for β are 0 and 1.

1. If $i + j \leq n$, then we must be in the first case. Then we are *constant* maps from $\mathbb{P}^1$ to $\mathbb{P}^n$ with three marked points passing through three linear spaces of complementary dimension. Since that intersection is obviously a single point, there is obviously only one such map (up to reparametrization).

   Therefore, in this case $\langle H_i, H_j, H_k \rangle = 1$, so $H_i \bullet H_j = H_{i+j}$.

2. If $i + j = 2n + 1$, then we are counting the number of *lines* in $\mathbb{P}^n$ with three marked points passing through linear spaces of codimension $i + j + k = 2n + 1$. We can represent a map $\mathbb{P}^1 \to \mathbb{P}^n$ with image a line by $n + 1$ choices of linear polynomial, up to scalar. The condition that the corresponding marked point passes through H_i imposes i homogeneous linear conditions on the coefficients, which are generically independent, so there is in the end only one linear dimension, which collapses to a single map after modding out by scalars.

   Therefore, in this case $\langle H_i, H_j, H_k \rangle = 1$, so $H_i \bullet H_j = H_{i+j-n-1}$.

1.5 Problem 5

The "quantum connection" ∇_λ is flat if

$$\nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda = \nabla_{[\lambda,\mu]}.$$

It is not quite clear to me in what sense this is a connection, but it seems like it should be the case that $[\lambda, \mu] = 0$, since what else could the commutator of two characters be?

It suffices to test this on “monomials” of the form γq^α, which we compute below:

$$\begin{aligned}
\nabla_\lambda \nabla_\mu (q^\alpha \gamma) &= \nabla_\lambda \left[(\alpha \cdot \mu) q^\alpha \gamma - q^\alpha \sum_\beta \sum_\eta q^\beta \langle \mu, \gamma, \eta \rangle_\beta \eta^\vee \right] \\
&= (\alpha \cdot \lambda)(\alpha \cdot \mu) q^\alpha \gamma - q^\alpha \sum_{\beta,\eta} (\lambda \cdot \beta + \lambda \cdot \alpha) q^\beta \langle \mu, \gamma, \eta \rangle_\beta \eta^\vee \\
&\quad - (\alpha \cdot \mu) q^\alpha \sum_{\beta',\eta} q^{\beta'} \langle \lambda, \gamma, \eta \rangle_\beta \eta^\vee + q^\alpha \sum_{\beta,\eta} q^\beta \langle \mu, \gamma, \eta \rangle_\beta \sum_{\beta',\delta} \langle \lambda, \eta^\vee, \delta \rangle_{\beta'} \delta^\vee .
\end{aligned}$$

Since most of the terms are symmetric, we see that $\nabla_\lambda \nabla_\mu = \nabla_\mu \nabla_\lambda$ if and only if

$$\sum_\beta (\lambda \cdot \beta) q^\beta \langle \mu, \gamma, \eta \rangle_\beta \eta^\vee$$

is symmetric in λ and β. But by §1.3, this is equal to

$$\sum_\beta q^\beta \langle \lambda, \mu, \gamma, \eta \rangle_\beta \eta^\vee$$

which is manifestly symmetic in λ and μ.

2 Problem Sheet 2

2.1 Problem 1

(1) As we have seen many times in the lectures, $T^*\mathbb{P}^1$ is the blowup of the singular quadric cone in $\mathbb{C}^3$ at the cone point. In particular, if $C \to T^*\mathbb{P}^1$ is proper, then we can compose to obtain a map from C to the (affine) quadric cone, which must be constant. Therefore C maps to a fiber, but the only positive-dimensional fiber is the exceptional fiber, which is isomorphic to $\mathbb{P}^1$. That shows that

$$\overline{\mathcal{M}_{0,0}}(\mathbb{P}^1, d) = \overline{\mathcal{M}_{0,0}}(T^*\mathbb{P}^1, d).$$

Now, we recall the expected dimension formula:

$$\text{expected dim } \overline{\mathcal{M}_{0,n}}(X, \beta) = \dim X + (n-3) - K_X \cdot \beta.$$

Using this on $\overline{\mathcal{M}_{0,0}}(\mathbb{P}^1, d)$, we find that

$$\text{expected dim } \overline{\mathcal{M}_{0,0}}(\mathbb{P}^1, d) = 1 + (-3) - (-2d) = 2d - 1.$$

On the other hand, using it on $\overline{\mathcal{M}_{0,0}}(T^*\mathbb{P}^1, d)$ yields

$$\text{expected dim } \overline{\mathcal{M}_{0,0}}(T^*\mathbb{P}^1, d) = 2 + (-3) - 0 = -1$$

because $T^*\mathbb{P}^1$ has trivial canonical bundle, since $T^*\mathbb{P}^1$ is a symplectic manifold and hence has a non-vanishing two-form.

(2) For $X = \mathbb{P}^1$, we have $T_X = \mathcal{O}(2)$, so $f^*T_X = \mathcal{O}(2d)$. Then $H^1(\mathbb{P}^1, f^*T_X) = H^1(\mathbb{P}^1, \mathcal{O}(2d)) = 0$.

For $X = T^*\mathbb{P}^1$, any map from $\mathbb{P}^1$ to X must compose with the projection $X \to \mathbb{P}^1$ to be a finite morphism of degree d. This reduces us to considering the restriction of TX to its *zero section*. But the normal bundle of the zero section of X in any vector bundle V on X is X itself, so we have

$$f^*T_X \cong f^*(T_{\mathbb{P}^1} \oplus T^*_{\mathbb{P}^1}) \cong \mathcal{O}(-2d) \oplus \mathcal{O}(2d).$$

Then $H^1(\mathbb{P}^1, f^*T_X)$ has dimension $2d - 1$.

The Euler class of the corresponding vector bundle on $\overline{\mathcal{M}_{0,0}}(T^*\mathbb{P}^1, d)$ lives in degree H^{2d-1}.

(3) ♠♠♠ TONY: [????]

2.2 Problem 2

(1) This follows directly from the Atiyah-Hirzebruch localization theorem, which says that ♠♠♠ TONY: [the rest of this doesn't make any sense to me]

3 Problem Sheet 3

♠♠♠ TONY: [to be continued...]