EXERCISES

1. Lecture 3

Exercise 1.1. This is a warm-up for something I want to discuss on Friday (so may be tricky to work out now.) Let X be smooth and projective and consider $Y = X \times \mathbb{P}^1$; in class, we will take twisted X-bundles over $\mathbb{P}^1$. Given a section class $\beta' = (\beta, 1)$, consider the moduli space $\overline{\mathcal{M}}_{0,3}(Y, \beta')$. What is its virtual dimension? Let F_0, F_∞ denote the fibers of Y over 0 and ∞ respectively. Consider the locus

$$Z = \mathrm{ev}_1^{-1}(F_0) \cap \mathrm{ev}_2^{-1}(F_0) \cap \mathrm{ev}_3^{-1}(F_\infty)$$

and argue that

$$Z = \bigcup_{\beta_1+\beta_2=\beta} \overline{\mathcal{M}}_{0,3}(X, \beta_1) \times_X \\ (\mathrm{ev}_1^{-1}(F_0) \cap \mathrm{ev}_2^{-1}(F_\infty) \cap \overline{\mathcal{M}}_{0,2}(Y, (\beta_2, 1)).$$

Here, in the second line, ev denotes the evaluation maps for $\overline{\mathcal{M}}_{0,2}(Y, (\beta_2, 1))$. Check that the natural expected dimensions of Z and the terms on the right agree (pulling back a divisor decreases the expected dimension by one.) In fact, this equality becomes an equality of virtual classes (try to write down the correct statement). Similarly, define Z' the same as above except with the condition $\mathrm{ev}_2^{-1}(\infty)$. Since $[0] = [\infty] \in H^2(\mathbb{P}^1)$, deduce an identity for virtual classes. If we instead do everything equivariantly with respect to the $\mathbb{C}^*$ action on $\mathbb{P}^1$, then it is no longer true that $[0] = [\infty]$; instead $[0] - [\infty] = t \in H^2_{\mathbb{C}^*}(\mathbb{P}^1)$ where t is the tangent weight at 0. Modify your identity above in this setting.

Exercise 1.2. Let $(\mathbb{C}^*)^2$ act on $\mathbb{P}^2$ with tangent weights (t_1, t_2) at the fixed point $[0 : 0 : 1]$ in the $x-$ and $y-$ direction. What are the tangent weights at the other fixed points? Set $X = T^*\mathbb{P}^2$; with a $(\mathbb{C}^*)^2 \times \mathbb{C}^*$ action where the last factor scales the cotangent fibers with weight $\hbar$; describe the stable basis element for $[0 : 0 : 1]$, for the one-parameter subgroup where $t_1 >> t_2 >> 0$.